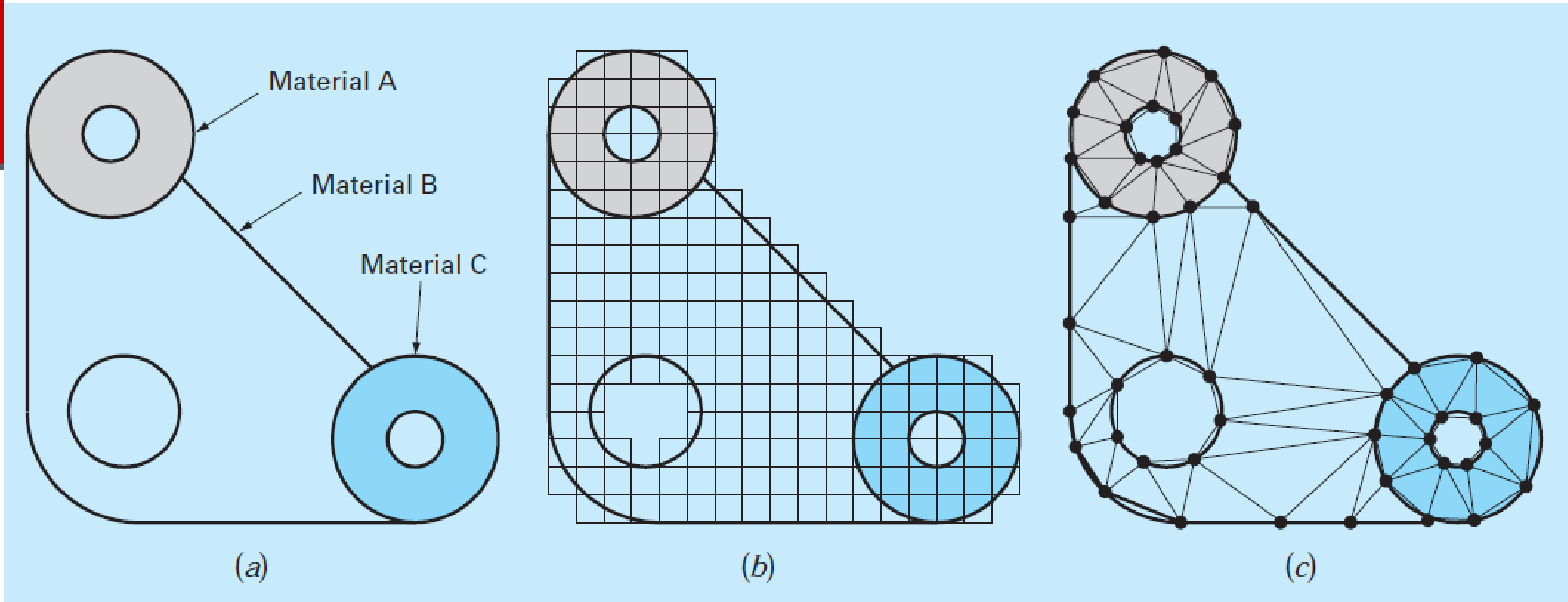
Numerical Methods for Partial Differential Equations (PDE) (5)

Finite Element Methods (FEM)



(a) Domain with irregular geometry and nonhomogeneous composition.

(*b*) very difficult to model with a **finite-difference** approach. This is due to the fact that complicated approximations are required at the boundaries of the system and at the boundaries between regions of differing composition. (*c*) A **finite-element** discretization is much better suited for such systems.

Richiami di analisi funzionale

Definizione

Dato uno spazio funzionale V si dice ***funzionale*** *su V un operatore che associa ad ogni elemento di V un numero reale :*

F:V->R

Si indica F(v)=<F,v>

Funzionale **lineare**

$$F\left(\lambda u+\mu v\right)=\lambda F\left(u\right)+\mu F\left(v\right) \qquad \forall\lambda,\mu\in\Re, \quad \forall u,v\in V$$

è **limitato** se

$$\exists C>0 \qquad \left|F(v)\right|\leq C\left\|v\right\|_V \qquad \forall v\in V$$

Spazio delle funzioni a quadrato sommabile

Sia Ω un aperto di R^n .

Consideriamo lo spazio delle funzioni a quadrato sommabile su Ω :

$$L^2(\Omega) \equiv \left\{ v : \Omega \to R \; t.c. \int_\Omega v(x)^2 \, d\Omega < \infty \right\}$$

Esso è uno spazio di Hilbert il cui prodotto scalare è:

$$(\mathrm{v},\mathrm{u})_{L^2(\Omega)} \equiv \int_\Omega v(x) u(x) \, d\Omega$$

da cui la norma $\quad \| v \|_{L^2(\Omega)} \equiv \sqrt{(\mathrm{v},\mathrm{v})_{L^2(\Omega)}}$

Spazio delle funzioni a quadrato sommabile

Sia Ω un aperto di R^n .

Disuguaglianza di Cauchy-Schwartz:

$$\left| \int_\Omega v(x)u(x)d\Omega \right| \leq \|v\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}$$

$$\|\bullet\|_{L^2(\Omega)} \equiv \sqrt{\int_\Omega |\bullet| d\Omega}$$

Spazi di Sobolev

Sia Ω un aperto di R^n e k un intero positivo.
Lo spazio di Sobolev di ordine k su Ω è formato dalla totalità delle funzioni di $\mathrm{L}^2(\Omega)$ aventi tutte le derivate (distribuzionali) fino all'ordine k appartenenti ad $\mathrm{L}^2(\Omega)$:

$$H^k(\Omega) \equiv \left\{ v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega),\ \forall \alpha : |\alpha| \leq \mathrm{k} \right\}$$

Gli spazi di Sobolev risultano essere spazi di Hilbert rispetto al prodotto scalare seguente:

$$(\mathrm{v},\mathrm{u})_k = \sum_{|\alpha| \leq \mathrm{k}} \int_\Omega (D^\alpha v)(D^\alpha u)\, d\Omega$$

da cui le norme $\quad \|v\|_k = \|v\|_{H^k(\Omega)} = \sqrt{(\mathrm{v},\mathrm{v})_k} = \sqrt{\sum_{|\alpha| \leq \mathrm{k}} \int_\Omega (D^\alpha v)^2\, d\Omega}$

Forma

Dato uno V uno spazio di Hilbert con norma $||.||_V$, si dice **forma** un'applicazione **a** che associa ad ogni coppia di elementi di V un numero reale

$$a : V \times V \longmapsto \Re,$$

Bilineare se è lineare rispetto ad entrambi i suoi argomenti

$$a(\lambda u+\mu w,v)=\lambda a(u,v)+\mu a(w,v) \quad \forall \lambda,\mu \in \Re,\ \forall u,v,w \in V$$
$$a(u,\lambda w+\mu v)=\lambda a(u,v)+\mu a(u,w) \quad \forall \lambda,\mu \in \Re,\ \forall u,v,w \in V$$

Continua

$$se \quad \exists M>0 \quad \textit{tale che} \quad \left|a(u,v)\right|<M\left\|u\right\|_V \left\|v\right\|_V$$

Coerciva $$se \quad \exists \alpha>0 \quad \textit{tale che} \quad a(u,u)\geq \alpha \left\|u\right\|_V^2$$

PDE - Definitions

PDE of order **m**

Strong (classical) solutions: functions ***u(t,x,y,…)*** that are continuously differentiable of order **m** at each point of the domain of the PDE (are C^m(**D**)) and that satisfy the PDE at each point of **D**.

Weak solutions: less regular functions **u(t,x,y,...)** (that is, are not C^m(**D**)) that do not satisfy the PDE everywhere in **D**. They are characterized by an integral formulation (called **variational formulation**), associated with the original PDE, that involves partial derivatives of order less than **m** defined in the sense of distributions.

Poisson’s Equation

Let $\Omega \subset \Re^d$ be open, limited and connected (**domain**) and let $\partial\Omega$ its boundary. Consider the problem:

$$-\Delta u = f \quad \text{on} \quad \Omega$$

where ***f=f(x)*** is a given function.

This is an **elliptic, second order**, **linear**, **non homogeneous** (when $f \neq 0$).

Physically, u may represent the vertical displacement of an elastic membrane due to the application of a specific force equal to f, or it can be the distribution of electric potential due to a density of electric charge f.

Boundary Conditions

To have a unique solution, the equation must be added the appropriate boundary conditions. You can for example assign the value of u on the boundary (**Dirichlet BC**)

$$u = g \qquad \text{on} \quad \partial\Omega$$

g is a given function. When ***g=0*** we have **homogeneous BC**.

Alternatively, impose the value of the normal derivative of u

$$\nabla u \bullet n = \frac{\partial u}{\partial n} = h \qquad \text{on} \quad \partial\Omega$$

with ***n outgoing normal*** from **Ω** and **h** a given function (**Neumann BC**).

In the case of the membrane, this corresponds to have imposed the traction on the boundary. In case of heat diffusion: free boundary through which heat cannot flow.

It may finally assign conditions of different type, for example (**Mixed BC**)

$$\begin{cases} u = g & \text{on} \quad \Gamma_D \\ \dfrac{\partial u}{\partial n} = \mathrm{h} & \text{on} \quad \Gamma n \end{cases}$$

On the space of solutions

$$-\Delta u = 1 \quad \forall x \in (0,1)\times(0,1)$$

It might not make sense to look for a solution $u \in C^2(\overline{\Omega})$
while you are more likely to find a solution

$$u \in C^2(\Omega) \cap C^0(\overline{\Omega})$$

A space larger than $C^2(\overline{\Omega})$

It seeks an alternative formulation to the strong one because it does not allow to treat some cases physically significant.

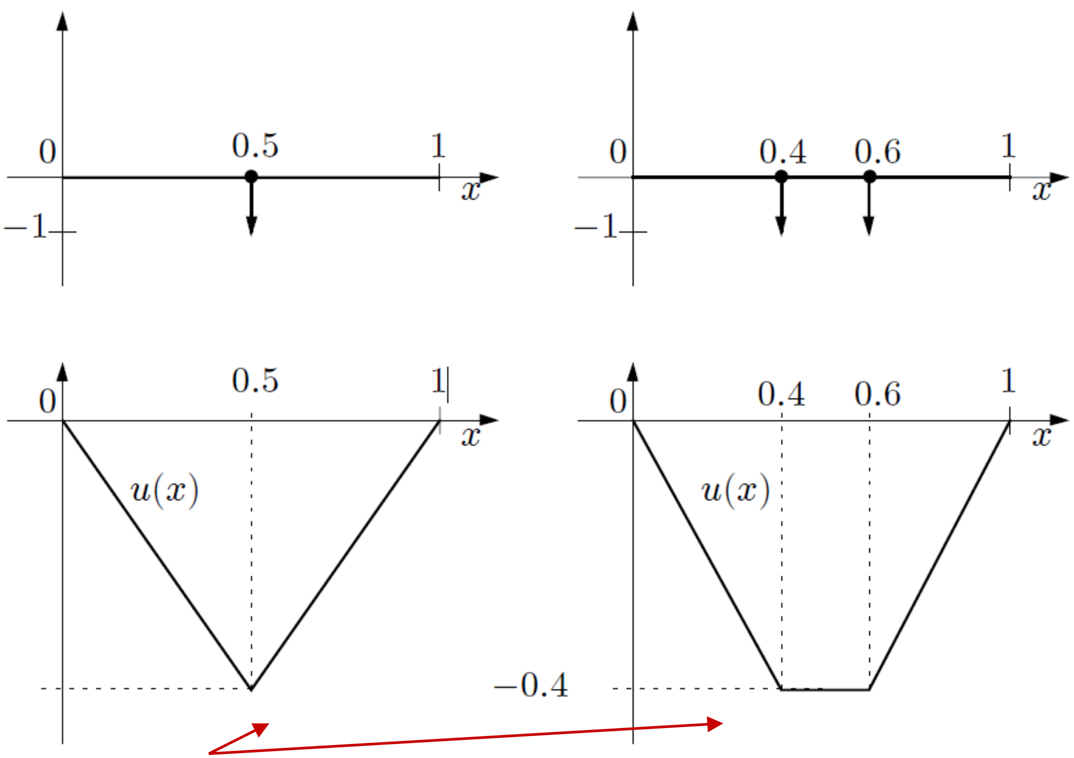
In fact, in the presence of not smooth data, the physical solution may not even belong to the space

$$u \in C^1(\Omega) \cap C^0(\overline{\Omega})$$

Poisson's Equation : 1D

$$\begin{cases} -u''(x) = f(x), & 0 < x < 1, \\ u(0) = 0, & u(1) = 0 \end{cases}$$

Equilibrium condition of an elastic thread with a strain equal to one, fixed at the ends, for small displacements and subject to a transversal force f.



f concentrated loads

u transversal displacement with respect to rest position.

Physical solution, continuous but not differentiable,
Analytical solution continuous until the second derivative

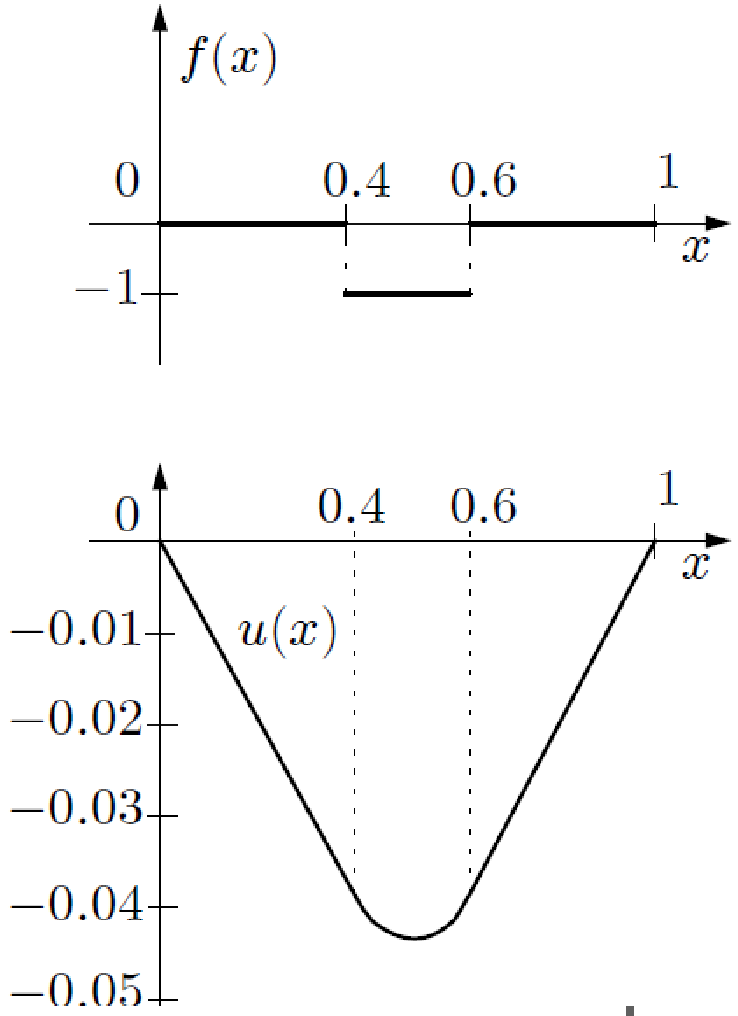
Strong formulation not adequate

Poisson's Equation: 1D

f piecewise constant

$$u(x) = \begin{cases} -\dfrac{1}{10}x & \text{per } x \in [0, 0.4], \\ \dfrac{1}{2}x^2 - \dfrac{1}{2}x + \dfrac{2}{25} & \text{per } x \in [0.4, 0.6], \\ -\dfrac{1}{10}(1-x) & \text{per } x \in [0.6, 1]. \end{cases}$$

Solution only C[1]([0,1])



We need an alternative to the strong formulation of the problem in order to reduce the order of derivation required on the unknown solution u.

We will convert a second-order differential problem to one in integral form of the first order.

This problem is the **weak formulation** of the differential problem.

Homogeneous Dirichlet Problem: Weak formulation

$$\begin{cases} -u''(x) = f(x), & 0 < x < 1, \\ u(0) = 0, & u(1) = 0 \end{cases}$$

Multiply by a test function ***v*** (for now, arbitrary) and integrate on the interval ***(0,1)***

$$-u''v = fv \quad \Rightarrow \quad -\int_0^1 u''v\,dx = \int_0^1 fv\,dx$$

Integration by parts

$$-\int_0^1 u''v\,dx = \int_0^1 u'v'\,dx - \left[u'v\right]_0^1$$

Since u vanishes at the extremes, we impose that the test functions ***v*** vanish at the extremes of the interval (satisfy the homogeneous Dirichlet boundary conditions) -> the boundary term vanishes

Homogeneous Dirichlet Problem

$$\int_0^1 u' v' \, dx = \int_0^1 f v \, dx$$

Choose the space V:

1) Space of the test functions V $\quad If \quad v \in V \text{ then } v(0) = v(1) = 0$

2) If u,v belong to C¹([0,1]) then $u', v' \in C^0([0,1])$

and then the integral to the first member would make sense. In reality the physical solutions can not be continuously differentiable. Moreover, even when $f \in C^0([0,1])$ there is no certainty that the problem admits solution in space

$$V = \{ v \in C^1([0,1]) \ : v(0) = v(1) = 0 \}$$

Homogeneous Dirichlet Problem

$$V = \{v \in C^1([0,1]) \ : v(0) = v(1) = 0\}$$

Is not a complete vector space with the scalar product $(u,v)_1 = \int_0^1 u'(x) v'(x) dx$

Definition:

Space of functions with p-th power integrable (Lebesgue)

$$L^p(0,1) \equiv \left\{ v : (0,1) \mapsto \Re \ \ t.c. \ \|v\|_{L^p(0,1)} \equiv \left(\int_0^1 |v(x)|^p dx \right)^{1/p} < +\infty \right\}$$

$$for \quad 1 \leq p < \infty$$

So that if v'u' is in $L^1(0,1)$ the integral is well defined

$$\boxed{\text{Property: } \ If \quad u,v \in L^2(0,1) \quad then \ \ u'v' \in L^1(0,1)}$$

Square integrable Functions with square integrable derivatives

Definition: Sobolev Space

$$H^1(0,1) \equiv \left\{ v \in L^2(0,1) : v' \in L^2(0,1) \right\}$$

We choose as the space V the subspace of $H^1(0,1)$

$$H_0^1(0,1) \equiv \left\{ v \in H^1(0,1) : v(0) = v(1) = 0 \right\}$$

$$H^1(0,1) \not\subset C^1([0,1])$$

The functions belonging to H^1 are not differentiable in the classical sense. Piecewise continuous functions with corner junctions belong to H^1 but not to C^1. Therefore, solutions <u>continuous but not differentiable</u> are also covered.

Homogeneous Dirichlet Problem: weak formulation

Problem (strong formulation)

(1)
$$\begin{cases} -u''(x) = f(x), & 0 < x < 1, \\ u(0) = 0, & u(1) = 0 \end{cases}$$

is reduced to find the solution to the problem (weak formulation)

(2)
$$u \in V: \quad \int_0^1 u'v'dx = \int_0^1 fvdx \quad \forall v \in V, \; V = H_0^1(0,1)$$

The weak problem (2) is equivalent to a variational problem thanks to the following result:

Variational Formulation of Homogeneous Dirichlet problem

Theorem The **variational** problem

$$\textbf{(3)} \quad \begin{cases} Find \quad u \in V: \; J(u) = \min\limits_{v \in V} J(v) \quad \text{where} \\ \\ J(v) \equiv \dfrac{1}{2} \displaystyle\int_0^1 (v')^2 dx - \int_0^1 fv dx \end{cases}$$

is equivalent to the problem

$$\textbf{(2)} \quad Find \quad u \in V: \quad \int_0^1 u'v'dx = \int_0^1 fvdx \quad \forall v \in V, \; V = H_0^1(0,1)$$

This means that u is solution of (2) if and only if u is solution of (3)

Principle of virtual work of mechanics.

Variational Formulation of Homogeneous Dirichlet problem

$$\begin{cases} Find \quad u \in V: \; J(u) = \min_{v \in V} J(v) \quad \text{where} \\ J(v) \equiv \frac{1}{2}\int_0^1 (v')^2 dx - \int_0^1 fv dx \end{cases}$$

J(v) expresses the global potential energy corresponding to the configuration v of the system.

The **principle of virtual work** establishes that between eligible shifts of the elastic thread, the one that corresponds to the solution is the one that minimizes the potential energy.

In this sense, the theorem states that the weak solution is also the one that minimizes the potential energy.

Principle of minimum of the potential energy

Neumann (natural) homogeneous BC

$$\begin{cases} -u''(x) + \sigma u = f(x), & 0 < x < 1, \\ u'(0) = 0, & u'(1) = 0 \end{cases}$$

σ **Positive function**

Multiply by a test function v (for now arbitrary) and integrate over the interval (0,1). Then we integrate by parts

$$\int_0^1 u' v' \, dx + \int_0^1 \sigma u v \, dx - \left[u' v\right]_0^1 = \int_0^1 f v \, dx$$

Suppose that $f \in L^2(0,1)$ *and* $\sigma \in L^\infty(0,1)$ is a function that is limited almost everywhere on (0,1).

The BC is identically zero by the Neumann conditions imposed on u so no need to require that v vanishes.

Neumann homogeneous BC

$$\begin{cases} -u''(x)+\sigma u = f(x), & 0<x<1, \\ u'(0)=0, & u'(1)=0 \end{cases}$$

Weak Formulation

$$Find \quad u \in H^1(0,1):$$

$$\int_0^1 u'v'dx + \int_0^1 \sigma uvdx = \int_0^1 fvdx \quad \forall v \in V \qquad \text{with} \quad V = H^1(0,1)$$

Mixed homogeneous BC

$$\begin{cases} -u''(x) + \sigma u = f(x), & 0 < x < 1, \\ u(0) = 0, & u'(1) = 0 \end{cases}$$

Dirichlet **Neumann**

$$\int_0^1 u' \mathrm{v}' \, dx + \int_0^1 \sigma u \mathrm{v} \, dx - [u' v]_0^1 = \int_0^1 f v \, dx$$

The term on boundary vanishes only at x = 1, so you have to impose that the test functions are null in x = 0.

Placing $\Gamma_D = \{0\}$ and defining

$$H^1_{\Gamma_D}(0,1) \equiv \{v \in H^1(0,1) : v(0) = 0\}$$

Mixed homogeneous BC

$$\begin{cases} -u''(x)+\sigma u = f(x), & 0 < x < 1, \\ u(0)=0, & u'(1)=0 \end{cases}$$

Weak Formulation

$$Find \quad u \in H^1_{\Gamma_D}(0,1):$$

$$\int_0^1 u'v'dx + \int_0^1 \sigma uvdx = \int_0^1 fvdx \quad \forall v \in V \quad \text{with} \quad V = H^1_{\Gamma_D}(0,1)$$

$$f \in L^2(0,1) \quad \text{e} \quad \sigma \in L^\infty(0,1)$$

Poisson 2D: Homogeneous Dirichlet Problem

Find u such that

$$\begin{cases} -\Delta u = f & in \quad \Omega \\ u = 0 & su \quad \partial\Omega \end{cases} \qquad \Omega \subset \Re^2$$

Multplying by test function v and integrating on Ω.

$$-\int_\Omega \Delta\, uv \; d\Omega = \int_\Omega fv \; d\Omega$$

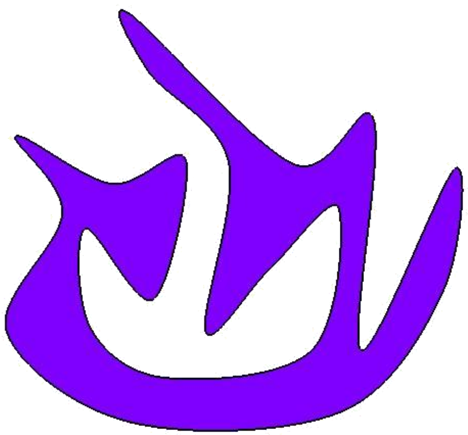
Divergence Theorem. Gauss-Green's formula.

Divergence Theorem

Given a vector field $a(x)=(a_1(x),a_2(x))$ defined on the domain Ω, the area integral $\nabla\cdot\mathbf{a}$ on Ω
Is equal to the curvilinear integral on $\partial\Omega$

$$\int_{\Omega} div(a)\, d\Omega = \int_{\partial\Omega} a \bullet n\, d\gamma$$

Ω



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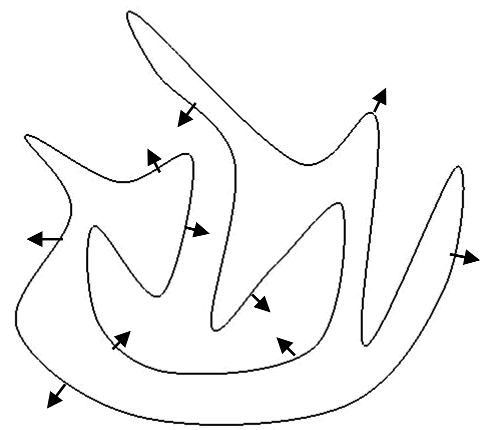

$$div(a) = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2}$$

$\mathbf{n}(x)=(n_1(x),n_2(x))$ **is the outward normal**

Poisson 2D: Homogeneous Dirichlet Problem

$$-\int_\Omega \Delta u\, v\, d\Omega = \int_\Omega f v\, d\Omega$$

$$\nabla \bullet (v\nabla u) = \nabla v \bullet \nabla u + v\Delta u$$

Product rule

$$\int_\Omega \nabla \bullet (v\nabla u) = \int_\Omega \nabla v \bullet \nabla u + \int_\Omega v\Delta u$$

$$\int_{\partial\Omega} v\nabla u \bullet n = \int_\Omega \nabla v \bullet \nabla u + \int_\Omega v\Delta u$$

Divergence Theorem

$$\nabla u \bullet n = \frac{\partial u}{\partial x_1} n_1 + \frac{\partial u}{\partial x_2} n_2 = \frac{\partial u}{\partial n}$$

Green's Formula

$$-\int_\Omega v\, \Delta u\, d\Omega = \int_\Omega \nabla v \bullet \nabla u \quad d\Omega - \cancel{\int_{\partial\Omega} v \frac{\partial u}{\partial n}\, d\gamma} \quad , \quad \nabla v = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2} \right)^T$$

Poisson 2D: Homogeneous Dirichlet Problem

Weak Formulation

$$Find \quad u \in H_0^1(\Omega):$$

$$\int_\Omega \nabla u \bullet \nabla v \; d\Omega = \int_\Omega fv \; d\Omega \qquad \forall v \in H_0^1(\Omega)$$

$$f \in L^2(\Omega)$$

$$H^1(\Omega) \equiv \left\{ v : \Omega \mapsto \Re \;\; t.c. \;\; v \in L^2(\Omega) : \frac{\partial v}{\partial x_i} \in L^2(\Omega), \;\; i = 1,2 \right\}$$

Test functions vanish at the boundary

$$H_0^1(0,1) \equiv \left\{ v \in H^1(0,1) : v = 0 \quad \text{on} \quad \partial\Omega \right\}$$

Poisson 2D: Homogeneous Dirichlet Problem

$$Find \quad u \in H_0^1(\Omega):$$

$$\int_\Omega \nabla u \bullet \nabla v \; d\Omega = \int_\Omega fv \; d\Omega \qquad \forall v \in H_0^1(\Omega)$$

$$f \in L^2(\Omega)$$

is equivalent to variational problem

$$Find \quad u \in V: \qquad J(u) = \inf_{v \in V} J(v) \qquad \text{where}$$

$$J(v) \equiv \frac{1}{2} \int_\Omega |\nabla v|^2 \; d\Omega - \int_\Omega fv \; d\Omega$$

$$V = H_0^1(\Omega)$$

A more Compact Form

Bilinear Form

$$a : V \times V \to \Re, \quad a(u,v) \equiv \int_{\Omega} \nabla u \bullet \nabla v \; d\Omega$$

Linear Functional

$$F : V \to \Re, \quad F(v) \equiv \int_{\Omega} fv \; d\Omega$$

$$\boxed{\begin{array}{ll} Find \quad u \in V : & \\ a(u,v) = F(v) & \forall v \in V \end{array}}$$

Existence and Uniqueness

Theorem (Lax-Milgram)

Let V a Hilbert space $\boldsymbol{a : V \times V \mapsto \Re,}$ a coercive and continuos bilinear form, let $F : V \to \Re,$ a linear continuos functional.
Then it exists a unique solution of the problem

$$\boxed{\begin{aligned} &Find \quad u \in V : \\ &a(u,v) = F(v) \qquad \forall v \in V \end{aligned}}$$

(2)

Moreover, if a(.,.) is also symmetric:

$$\boldsymbol{a(u,v) = a(v,u) \qquad \mathbf{u, v \in V}}$$

Then problem (2) is equivalent to the variational problem

J(v): total energy
a(v,v): internal energy
F(v) : external enegy forces

$$\boxed{\begin{aligned} &Find \quad u \in V : \qquad J(u) \geq J(v) \qquad \forall \mathrm{v} \in \mathrm{V} \\ &\text{where J() is the quadratic functional} \\ &J(v) \equiv \frac{1}{2} a(v,v) - F(v) \end{aligned}}$$

(3)

Proof

Let u be the solution of (**). Setting

$$v = u + \delta w, \quad \delta \in R \quad \Rightarrow \quad J(v) > J(u) \quad \forall \mathrm{v} \in \mathrm{V}, \mathrm{v} \neq \mathrm{u}$$

$$J\left(u+\delta w\right) = \frac{1}{2}\int_{\Omega} |\nabla(u+\delta w)|^2 \, d\Omega - \int_{\Omega} f(u+\delta w) d\Omega$$

$$J\left(u+\delta w\right) = \frac{1}{2}\int_{\Omega} |\nabla u+\delta\nabla w|^2 \, d\Omega - \int_{\Omega} f(u+\delta w) d\Omega$$

$$J\left(u+\delta w\right) = \frac{1}{2}\int_{\Omega} (\nabla u)^2 + \delta^2(\nabla w)^2 + 2\delta\nabla u\nabla w d\Omega - \int_{\Omega} f(u+\delta w) d\Omega$$

$$J\left(u+\delta w\right) = \frac{1}{2}[a(u,u) + 2\delta a(u,w) + \delta^2 a(w,w)] - (f,u) - \delta(f,w)$$

Proof

$$J\left(u+\delta w\right)=\frac{1}{2}[a(u,u)+2\delta a(u,w)+\delta^2 a(w,w)]-[(f,u)+\delta(f,w)]=$$

$$=J(u)+\frac{1}{2}[\delta^2 a(w,w)+2\delta a(u,w)]-\delta(f,w)=$$

$$\frac{J\left(u+\delta w\right)-J(u)}{\delta}=\frac{1}{2}[\delta a(w,w)+2a(u,w)]-(f,w)$$

$$\lim_{\delta\to 0}\frac{J\left(u+w\right)-J(u)}{\delta}=0$$

$\Rightarrow \quad a(u,w)-(f,w)=0 \quad \forall w\in V$, that is u satisfies (*).

Numerical approximation for elliptic problems

GALERKIN's Method

For elliptic problems

↓

Finite Element Method

Variational Method

General form of an elliptic problem

$$\begin{cases} -\Delta y = f & \text{on} \quad \Omega \\ y = g & \text{on} \quad \Gamma \end{cases}$$

boundary conditions

The solution y is searched in an appropriate functional space V, usually of **infinite dimension**.
One can introduce a discrete problem choosing a space V_n of **finite dimension** n, in which is fixed a particular basis:

Base $$V_n := \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$$

Variational Idea

The space V_n can be polynomial space, (Lagrange, Legendre, Chebyshev), or spline space.

The idea is, therefore, to seek an approximation of the solution in the form **y**

$$y(x) \approx y_n := c_1\varphi_1 + c_2\varphi_2 + \ldots + c_n\varphi_n$$

Functions **φ_i** are called the *shape functions*, and the parameters c_i are the degree of freedom.
Thus the problem is reduced to determine the parameters c_i.
According to the space chosen, we end up with a different method.

Variational Idea

- In reality, the boundary problems derived, in large part, by the application of **variational principles**, such as the minimum of a energy function and the principle of virtual work.
- Then the discrete solution can be obtained by imposing the same variational principle on the space V_n, rather than on the continuous space V.

Finite Element Method

V_n piecewise polynomial space

Spectral Method

V_n polynomial space, algebric or trigonometric, on Ω.