UNIQUENESS FOR SHAPE FROM SHADING VIA PHOTOMETRIC STEREO TECHNIQUE

Roberto Mecca

Dipartimento di Matematica "G. Castelnuovo"
Sapienza - Universitá di Roma

ABSTRACT

We deal with the inverse problem of reconstructing a surface with photometric stereo technique, i.e. using two or more pictures of the surface lighted under different light sources. The new model studied in this paper allows us to extend previous results [1, 2] obtaining a uniqueness result and to solve the classical *convex/concave ambiguity* of the Shape from Shading (SfS) problem. Finally, we propose an approximation scheme for the solution of the problem testing it on real and synthetic images.

***Index Terms*—** Shape from shading, Photometric Stereo, Nonlinear Systems, Hamilton-Jacobi Equations, Semi-Lagrangian Schemes.

1. INTRODUCTION

The SfS problem is a classical problem in image processing and computer vision. The main goal is to reconstruct a surface using the variation of brightness (shading) of its image. We will make the following assumptions:

1. The light sources are located at infinity;
2. The surface is Lambertian;
3. There are no self-reflections on the surface;
4. There are no black shadows in the images (i.e. there are no points $(x, y) \in \Omega$ such that $I(x, y) = 0$);
5. The optical center is sufficiently far from the surface so that perspective deformations can be neglected;
6. The albedo $\alpha(x, y)$ is known.

Let us denote by the unit vector $\omega = (\omega_1, \omega_2, \omega_3) = (\widetilde{\omega}, \omega_3)$ the light source direction ($\omega_3 > 0$) and by $I : \overline{\Omega} \to (0, 1]$ a function representing the image (where $\Omega \subset \mathbb{R}^2$). The orthographic SfS problem consists in determining the function $u : \overline{\Omega} \to \mathbb{R}$ which satisfies the following brightness equation

$$\alpha(x, y)(n(x, y) \cdot \omega) = I(x, y), \quad \forall (x, y) \in \Omega \tag{1}$$

that is

$$\frac{-\nabla u(x, y) \cdot \widetilde{\omega} + \omega_3}{\sqrt{1 + |\nabla u(x, y)|^2}} = I(x, y), \quad \forall (x, y) \in \Omega \tag{2}$$

where $n(x, y)$ is the outgoing normal to the surface of height u, the albedo is assumed equal to one all over Ω and the boundary condition $u(x, y) = g(x, y)$ on $\partial\Omega$ is given as well as the light vector ω and the greylevel I.

In the classical SfS problem those are standard assumption and a single image of the surface is used (we refer the interested reader to [3] for extensions of this classical model to perspective view and to [4,5] for non-Lambertian surfaces). It is important to note that this problem is in general ill-posed since a single image is not sufficient to determine the surface due the *convex/concave ambiguity*.

We refer to the classical monography by Horn and Brooks [6] for a mathematical formulation of the problem and to the more recent surveys [7, 8]. In order to recognize the surface without ambiguities, we want to study SfS using Photometric Stereo techniques (SfS-PS), where we use more than one image of the same surface, i.e. we increase the information about the surface in a natural way adding new images associated to different light source directions.

In particular, we consider two images I_1 and I_2 taken from the same point of view, but using two different light sources (respectively located at the directions ω' and ω''). That is, we consider the following system of Hamilton-Jacobi equations with the same Dirichlet boundary condition:

$$\begin{cases} \dfrac{-\nabla u(x, y) \cdot \widetilde{\omega}' + \omega_3'}{\sqrt{1 + |\nabla u(x, y)|^2}} = I_1(x, y), & \forall (x, y) \in \Omega; \\ \dfrac{-\nabla u(x, y) \cdot \widetilde{\omega}'' + \omega_3''}{\sqrt{1 + |\nabla u(x, y)|^2}} = I_2(x, y), & \forall (x, y) \in \Omega; \\ u(x, y) = g(x, y), & \forall (x, y) \in \partial\Omega. \end{cases} \tag{3}$$

Onn, Bruckstein [2] and Kozera [1] have studied this problem as a system of non-linear PDEs looking for smooth solution. More precisely, it is proved uniqueness for functions u of class $C^2(\Omega)$ (i.e. C^k is the set of the functions l where the derivative $l', l'', \ldots, l^{(k)}$ exist and are continuous), the images I_1 and I_2 have to be of class $C^1(\Omega)$, which is not realistic. Our goal here is to extend these results to less regular solutions, obtain uniqueness and to construct an algorithm for their approximation.

It is important to note that the SfS-PS problem (3) is well-posed. This is a very important issue for a mathematician problem since it guarantees the uniqueness of solution. Furthermore, it has also the merit to have the solution in a very weak function space (that is Lipschitz functions, i.e. the set of the functions almost everywhere differentiable).

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This is important specially for the real applications, in fact the regularity of the function space determines the smoothness of the object's surface reconstructed, that is, as the solution is regular, the admissible surfaces must be smooth.

2. THE NEW DIFFERENTIAL APPROACH

Our new approach is based on the elimination of the non-linearity of the problem considering it from the first equation of (3) as follow

$$\sqrt{1+|\nabla u(x,y)|^2} = \frac{-\nabla u(x,y)\cdot\widetilde{\omega}'+\omega_3'}{I_1(x,y)} \tag{4}$$

and replacing it on the second equation. We obtain the linear hyperbolic problem:

$$\begin{cases} b(x,y)\cdot\nabla u(x,y) = f(x,y), & a.e.(x,y)\in\Omega; \\ u(x,y) = g(x,y) & \forall(x,y)\in\partial\Omega. \end{cases} \tag{5}$$

where $b(x,y) = (I_2(x,y)\omega_1' - I_1(x,y)\omega_1'', I_2(x,y)\omega_2' - I_1(x,y)\omega_2'')$ and $f(x,y) = I_2(x,y)\omega_3' - I_1(x,y)\omega_3''$.

The condition of the complete illumination of the surface from the light source ω is mathematically guaranteed by the following inequality

$$u((x,y)+t\widetilde{\omega}) < u(x,y)+t\omega_3 \quad \forall t>0. \tag{6}$$

Let us emphasize the fact that we can identify the points where the surface u is not differentiable with the points where the functions b and f are discontinuous (i.e. where I_1 and I_2 are discontinuous) [9].

If we define the family of piecewise continuous curves $(\gamma_1(s),\ldots,\gamma_k(s))$ (where s is the argument of the parametric representation) like the curves of discontinuity for b and f, it is possible to prove the following theorem:

Theorem 2.1 *Let us suppose that* $(\gamma_1(s),\ldots,\gamma_k(s))$*, the family of discontinuous curves for* $b(x,y)$ *and* $f(x,y)$ *are not characteristic curves (with respect to the problem (5)). Then the problem (5) admits a unique Lipschitz solution* $u(x,y)$.

The proof of this theorem follow the characteristics method used taking into account, in particular, the discontinuity of the vector field $b(x,y)$. In particular it consists in:

1. prove that the vector field b does not admit singular points (i.e. there are no points $(x,y)\in\Omega$ such that $||b(x,y)||=0$);
2. verify that, if from one side with respect to a discontinuity curve the vector field b is incoming, then from the other it is outgoing (and viceversa);
3. construct the Lipschitz solution using the characteristics method starting from the boundary point across all the domain.

A complete proof of this theorem is contained in [9]. This theorem, beyond the goal to obtain a weaker solution, is very important because its proof gives us the possibility to construct a numerical scheme to approximate the unique weak solution of the problem. In particular we use the semi-lagrangian scheme where the driving force of this method is the flow of information in the model equation [10]. At the numerical level, the semi-lagrangian approximation mimics the method of characteristics looking for the point of the boundary where the characteristic curve passes and following this curve for all the image domain Ω.

3. NUMERICAL APPROXIMATION

For the numerical part we use synthetic and real images. For the synthetic ones we consider the square domain $[-1,1]^2$ with uniform discretization space step $\Delta = 2/n$ where n represents the number of intervals that divide the square (that is $x_i = -1+i\Delta$, $y_j = -1+j\Delta$ with $i,j = 0,\ldots,n$). We call Ω_d the set that contains the internal (x_i,y_j) and $\partial\Omega_d$ the set of points (x_i,y_j) that discretize the boundary.

3.1. Semi-Lagrangian Schemes

The semi-Lagrangian method to the resolution of equation (5) considers the idea of integrating the solution along the characteristics [10]. We have then to consider the following equivalent equation obtained dividing by the norm of $b(x,y)$:

$$\nabla_{\rho(x,y)}u(x,y) = \frac{f(x,y)}{||b(x,y)||}, \quad \forall(x,y)\in\Omega \tag{7}$$

with $\rho(x,y) = \frac{b(x,y)}{||b(x,y)||}$. We can divide without problem because we are using the first point of the sketch of the proof (i.e. the one where $||b(x,y)||\neq 0$). Now, considering the definition of directional derivative, we can write the following approximation

$$\frac{u(x+h\rho_1(x,y), y+h\rho_2(x,y)) - u(x,y)}{h} \simeq \frac{f(x,y)}{||b(x,y)||}. \tag{8}$$

We introduce two discrete functions $u^n(x_i,y_j) = u^n_{i,j}$ and $u^{n+1}(x_i,y_j) = u^{n+1}_{i,j}$ defined only on the grid nodes. Therefore, (8) brings us to the following semi-Lagrangian scheme:

$$u^{n+1}_{i,j} = u^n(x_i+h\rho_1(x_i,y_j), y_j+h\rho_2(x_i,y_j)) - \frac{f(x_i,y_j)}{||b(x_i,y_j)||}h \tag{9}$$

$\forall(x_i,y_j)\in\Omega_d$ and we intend $h>0$. It consists in a fix point operator where the convergence of the sequence of functions, in the Lipschitz function space, has been proved [9].

If we assign an initial function $u^0_{i,j}$, that is $u^0(x_i,y_j) = g(x_i,y_j)$ $\forall(x_i,y_j)\in\partial\Omega_d$ (and for example $u^0_{i,j} = 0$

$\forall (x_i, y_j) \in \Omega_d$), it is possible to start the iterations computing the successive approximations $u_{i,j}^{n+1}$ as described in (9). For every point (x_i, y_j) and every iteration n it will be necessary to compute $u^n(x_i + h\rho_1(x_i, y_j), y_j + h\rho_2(x_i, y_j))$ and thus to approximate with an interpolation method the function u^n at a point not belonging to the grid $\Omega_d \cup \partial\Omega_d$. The algorithm will stop when:

$$\max_{(x_i,y_j)\in\Omega_d} |u^n(x_i, y_j) - u^{n+1}(x_i, y_j)| < \varepsilon \qquad (10)$$

where ε is a small parameter.

It is clear that the main gap between the theoretical approach and the real application is the absence of the starting data $g(x, y)$ not obtainable only taking two pictures of the surface. With the aim to approximate the boundary condition we use our new approach to the SfS-PS problem with three images (i.e. we add the image function $I_3(x, y)$ obtained with the same reflectance equation (1) using the light source ω'''). In particular it is possible to prove that, considering our new differential approach, we can compute the gradient value of u on the boundary using only the pixels of the third image in $\partial\Omega_d$ [9].

Definition 3.1 *Three image functions I_1, I_2 and I_3 obtained from the SfS orthographic model, are linearly independent if they are generated by three non coplanar light vectors.*

According with other existing approach to the photometric stereo [11], the uniqueness of the boundary condition can be obtained if the starting data are three linear independent images [9]. Another advance using this new approach consists to have not the necessity to all the pixel of the three images, but, relatively to the third one, we need only its value on the boundary. This means that even if there are some sets of black shadows in the internal part of the third image, we can also approximate the surface.

4. NUMERICAL TESTS

In this section we present some examples where we use both cases explained before, that is two synthetic surfaces (with a given boundary condition) and one real object (with three images).

4.1. Synthetic Images

The numerical scheme previously described is tested on the two kinds of surfaces of Fig.1.

The first one (v_{smo}) is very smooth and its peculiarity is in the hight slope between the maximum and the minimum point. The other one (v_{Lip}) has just the required regularity for our scheme. The procedure followed is resumed below:

1. we consider two light sources expressed in the polar coordinates ($\omega = (\sin(\varphi)\cos(\theta), \sin(\theta)\sin(\varphi), \cos(\varphi))$) in order to obtain no black shadow using (6);
2. we construct the images from the analytical formula of the surface. Let us call its height $v(x, y)$;

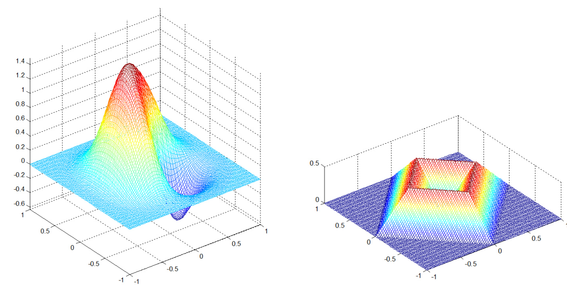

Fig. 1. *On the left a smooth surface (v_{smo}), on the right an only Lipschitz surface (v_{Lip}).*

Table 1. Comparison of different results.

Surface	$\overline{n}$	err
v_{smo}	113	6.514×10^{-2}
v_{Lip}	114	3.487×10^{-2}

3. we set the initial guess such that $u^0(x_i, y_j) = v(x_i, y_j)$ $\forall (x_i, y_j) \in \partial\Omega_d$ and equal to zero in Ω_d;
4. let us start the iteration of (9) till (10) with $\varepsilon = 10^{-7}$ calling $\overline{n}$ the last iteration;
5. the error in the $L^\infty(\Omega_d)$ norm is calculated $err = \max_{(x_i,y_j)\in\Omega_d} |u^{\overline{n}}(x_i, y_j) - v(x_i, y_j)|$.

We set a uniform integration step $\Delta = 0.02$ and a semi-lagrangian step h equal to Δ that allows to obtain the best error exploiting estimate concerning the consistency error [9]. For both surfaces we use the same couple of light sources (i.e. the same angles of their polar coordinates), that is $\varphi_1 = 0.1$, $\theta_1 = \frac{\pi}{3} + 0.2$ for ω', $\varphi_2 = -0.1$ and $\theta_2 = \frac{\pi}{3} + 0.3$ for ω''. The images, related to these light sources, are shown in Fig.2 and in Fig. 3. In Table 1 are shown the number of iterations and the errors for these two synthetic cases.

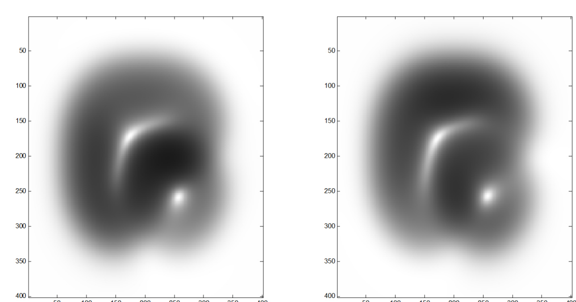

Fig. 2. *On the left the image I_2, on the right the image I_1 of the smooth surface.*

4.2. Real Images

In order to use the Lambertian characteristics of the surface to reconstruct, we use a half-length statue of Beethoven. Then, considering three pictures of 210×77 pixels (Fig.4), we obtain the 3D surface of Fig. 5. In particular we can note that the

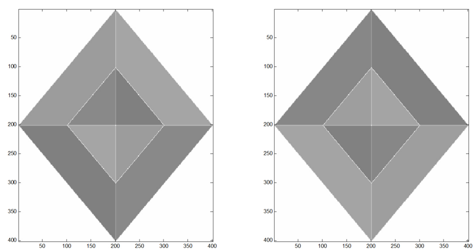

Fig. 3. *On the left the image I_2, on the right the image I_1 of the Lipschitz surface.*

model is robust with respect some little sets of black shadows. In fact, in the second and in the third images there are some black pixels that however don't influence the reconstruction of the surface.

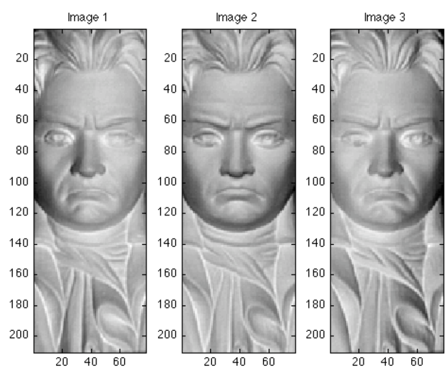

Fig. 4. *From left to right: I_1 obtained by $\varphi_1 = 0.263$, $\theta_1 = -0.305$, I_2 by $\varphi_2 = 0.200$, $\theta_2 = 1.655$ and I_3 by $\varphi_3 = 0.281$, $\theta_3 = 3.502$.*

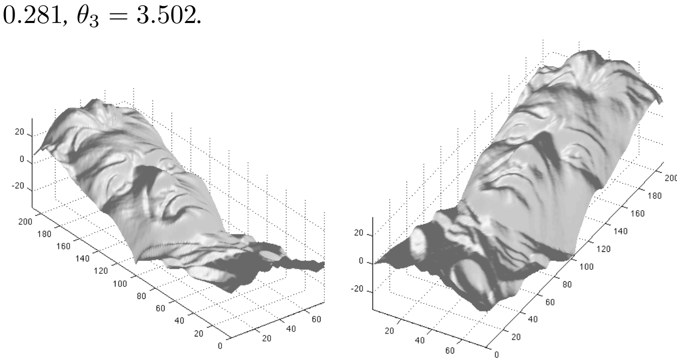

Fig. 5. *Two different points of view of the statue's reconstruction.*

5. CONCLUSION AND PERSPECTIVE

This PDE approach to the orthographic SfS-PS is a step forward that permits to arrive to obtain the weakest type of solution for this kind of formulation of the problem like presented for the images of Fig. 3. In fact it is also possible to prove that we can not obtain uniqueness of solution if we consider discontinuous function space [9]. A further step for the study of these kind of problems is related to weaken the starting hypotheses (for instance adding some specular-reflection effects) in particular considering the perspective model (PSfS-PS) as in [12].

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