# Virtual Reality (VR)
# Augmented Reality (AR)

## Coordinate Systems and Projective Geometry

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May 2012, Università di Bologna

Coordinate Systems and Projective Geometry

Literature:

1. G. Albrecht, Géométrie projective, Encyclopédie Les Techniques de l’Ingénieur, AF 206 1-14,4 2008.
2. G. Klinker, Class notes, TU München, 2010.

# Coordinate Systems and Projective Geometry

1. Scene graph
2. Coordinate systems and transformations
3. Homogeneous coordinates and projective geometry

# Coordinate Systems and Projective Geometry

## World:

- People (hands, eyes, …)
- Objects (subparts, tracked targets, …)
- Trackers

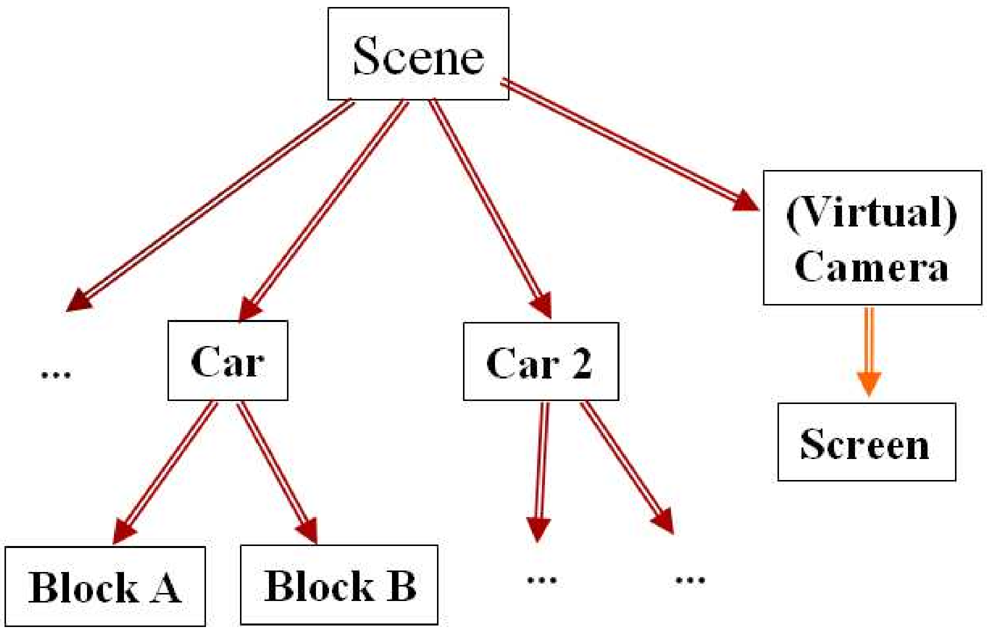
## Representation of the world:

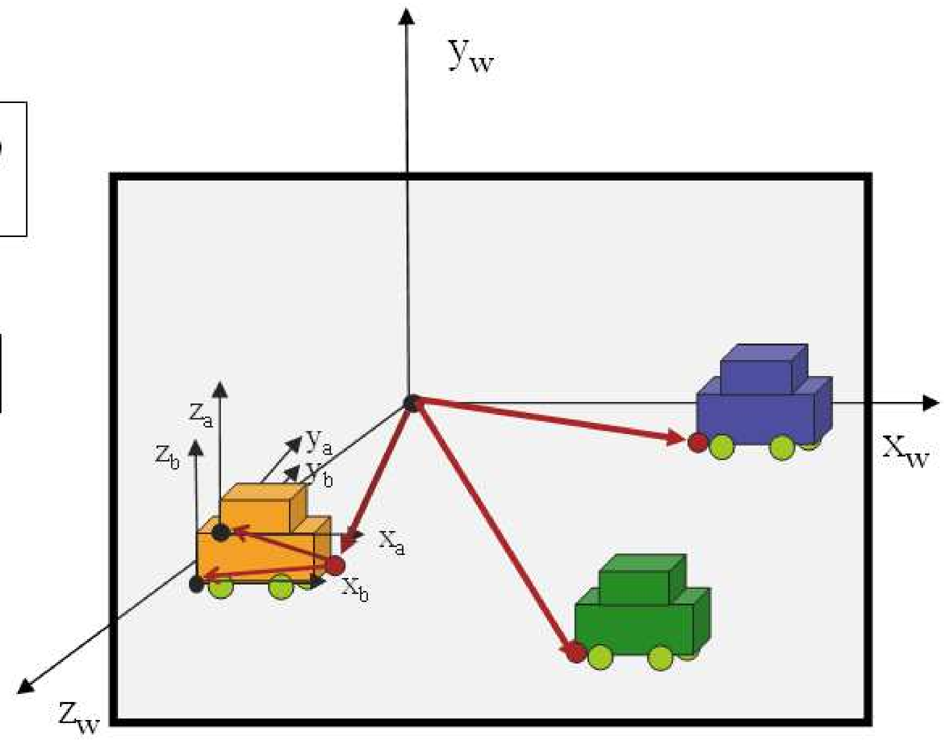
- Individual description of objects (independently of their current position in the world)
- Replication of objects (or parts) without having to (re)describe all geometric details
- Determination of an object’s current position with respect to different reference frames
  - » A tracker
  - » The world
  - » The user
  - » A display

$\Rightarrow$ Describe the world as an interrelated system of coordinate systems

# Coordinate Systems and Projective Geometry

## Scene graph



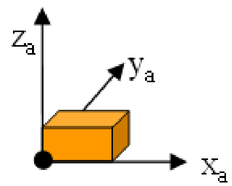




Source:[2]

# Coordinate Systems and Projective Geometry

Scene graph

Block A

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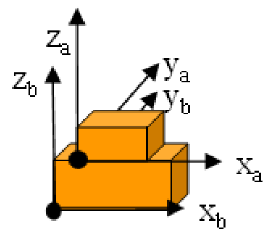
# Coordinate Systems and Projective Geometry

## Scene graph

Block A

Block B

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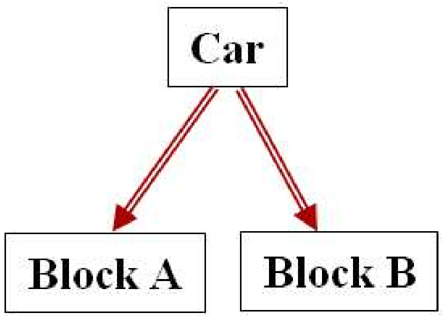
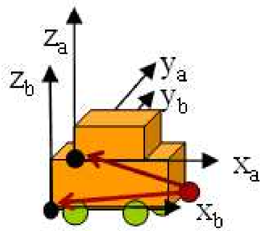
# Coordinate Systems and Projective Geometry

Scene graph

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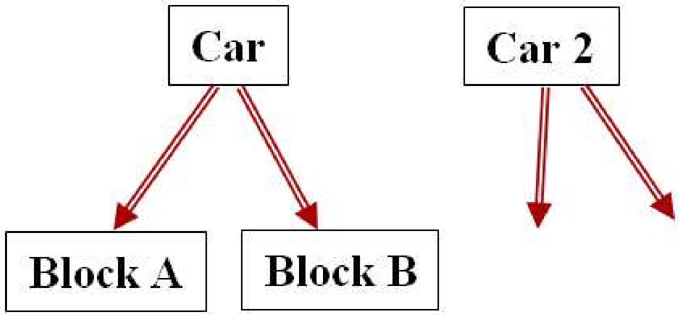
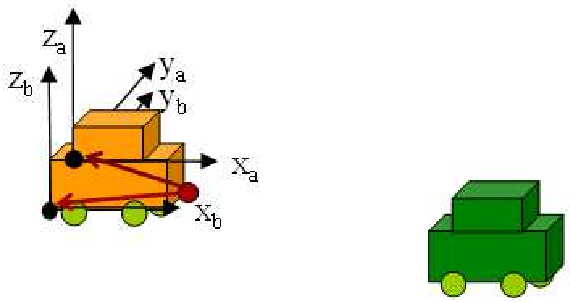
# Coordinate Systems and Projective Geometry

Scene graph



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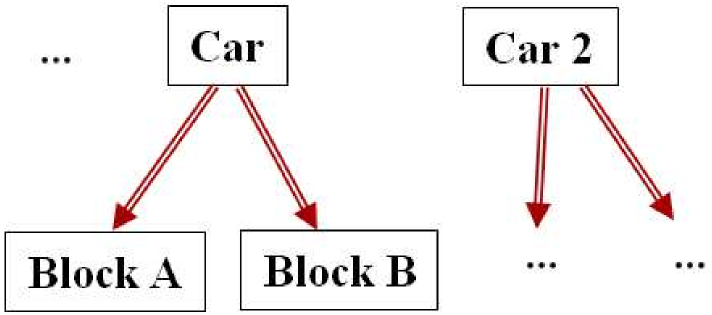
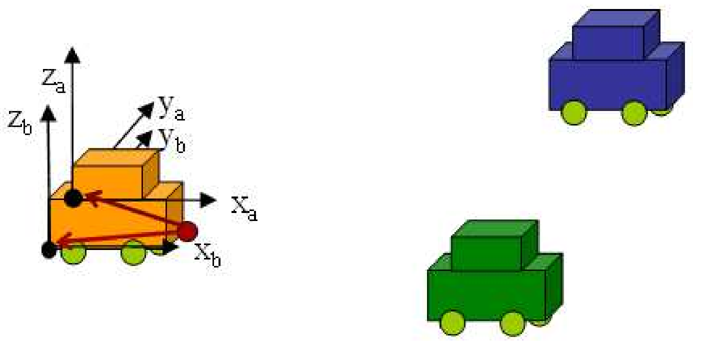
# Coordinate Systems and Projective Geometry

Scene graph

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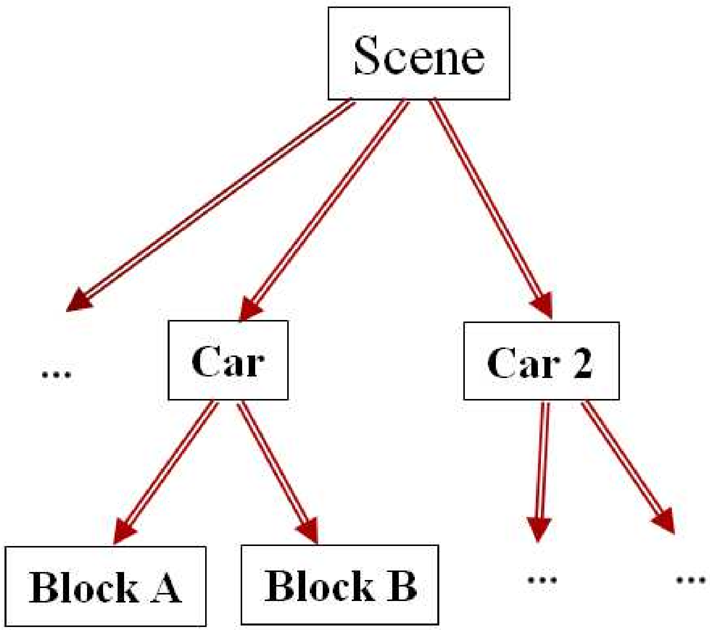
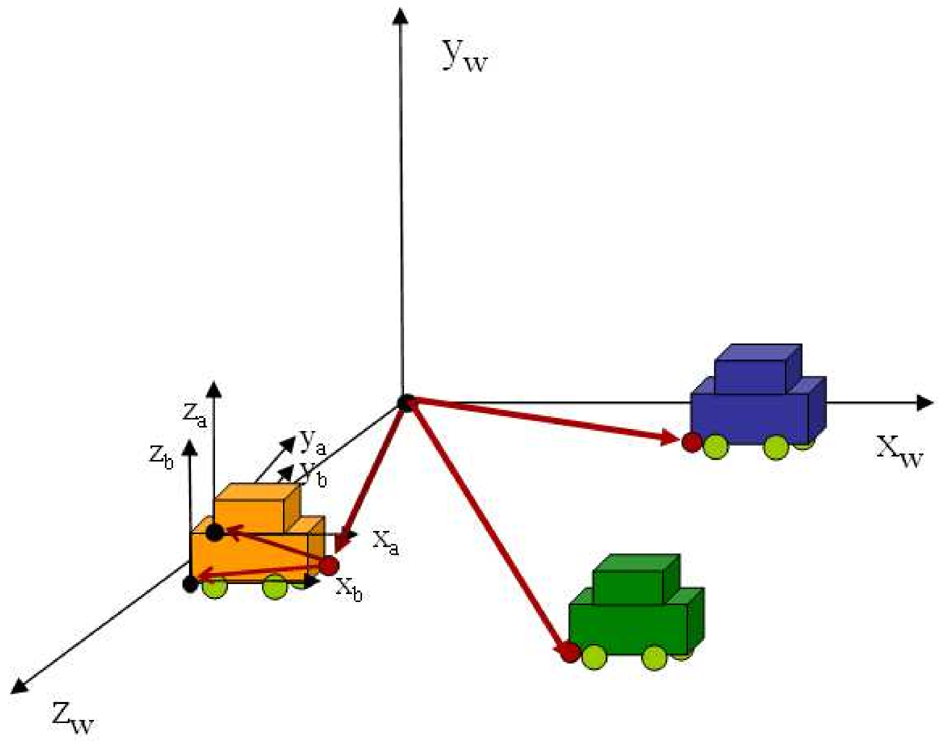
# Coordinate Systems and Projective Geometry

Scene graph

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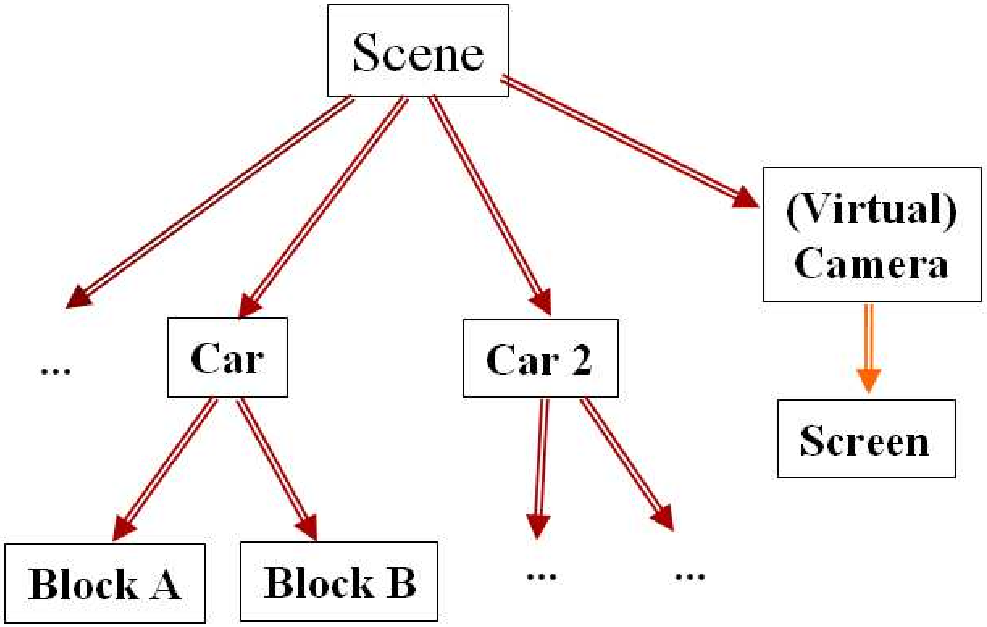
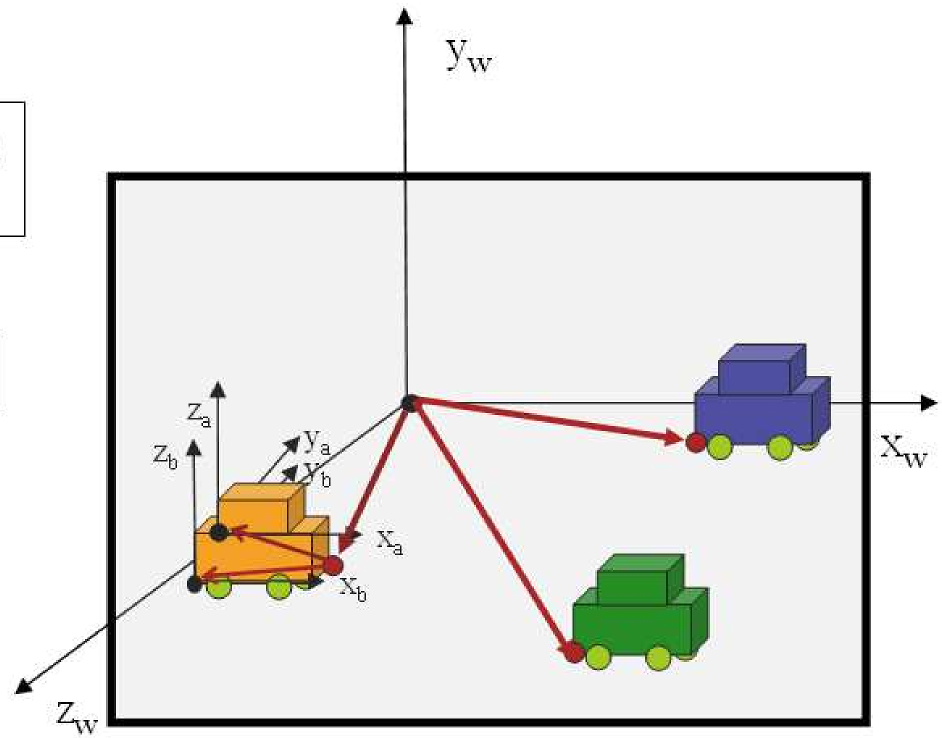
# Coordinate Systems and Projective Geometry

## Scene graph

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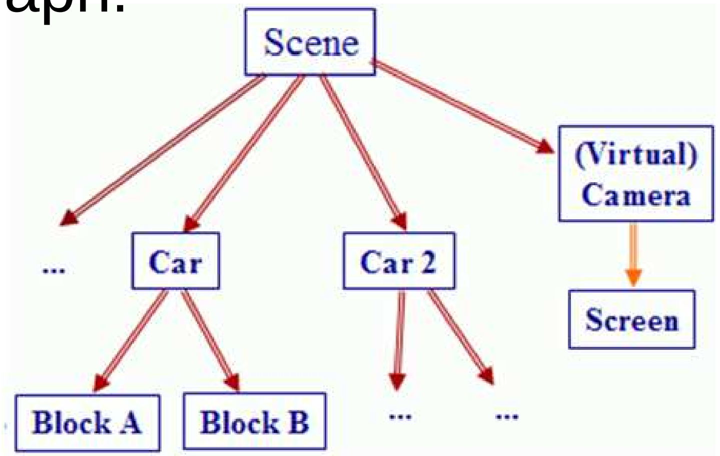
# Coordinate Systems and Projective Geometry

Most important components of a scene graph:

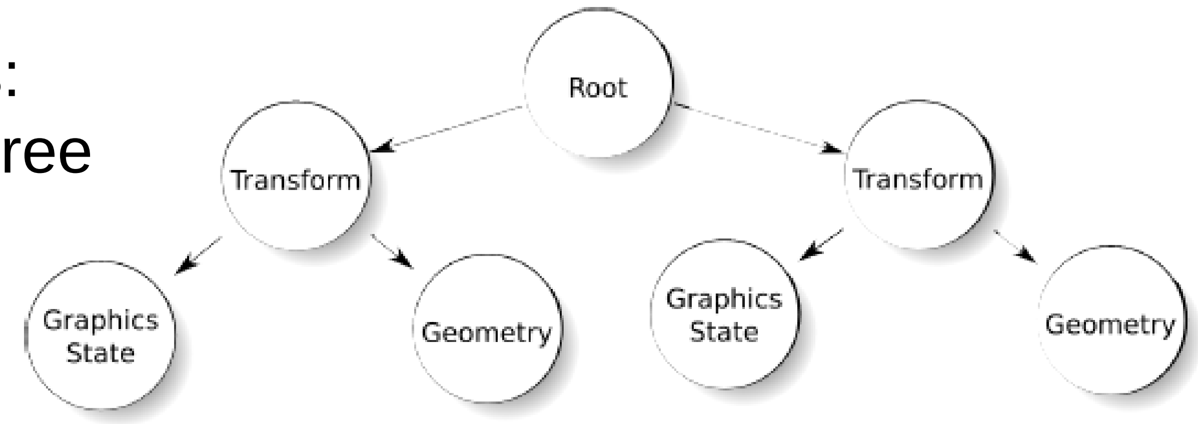



- Nodes: coordinate systems of
  - object parts
  - groups of objects
  - scene
  - camera (eye)
- Directed edges: geometric transformations such as changes in
  - position
  - orientation
  - scale
  - perspective

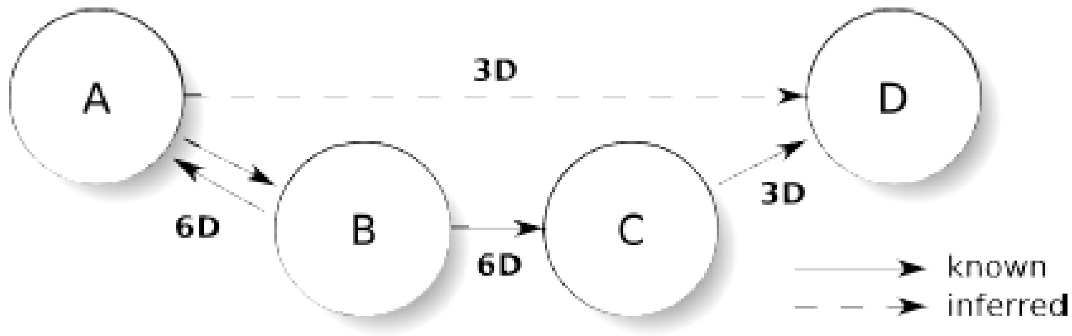
  } of a node relative to its predecessor

# Coordinate Systems and Projective Geometry

Scene graph in graphics:
typically a tree




Scene graph in AR:
can be a true graph, Spatial Relationship Graph (SRG)

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# Coordinate Systems and Projective Geometry

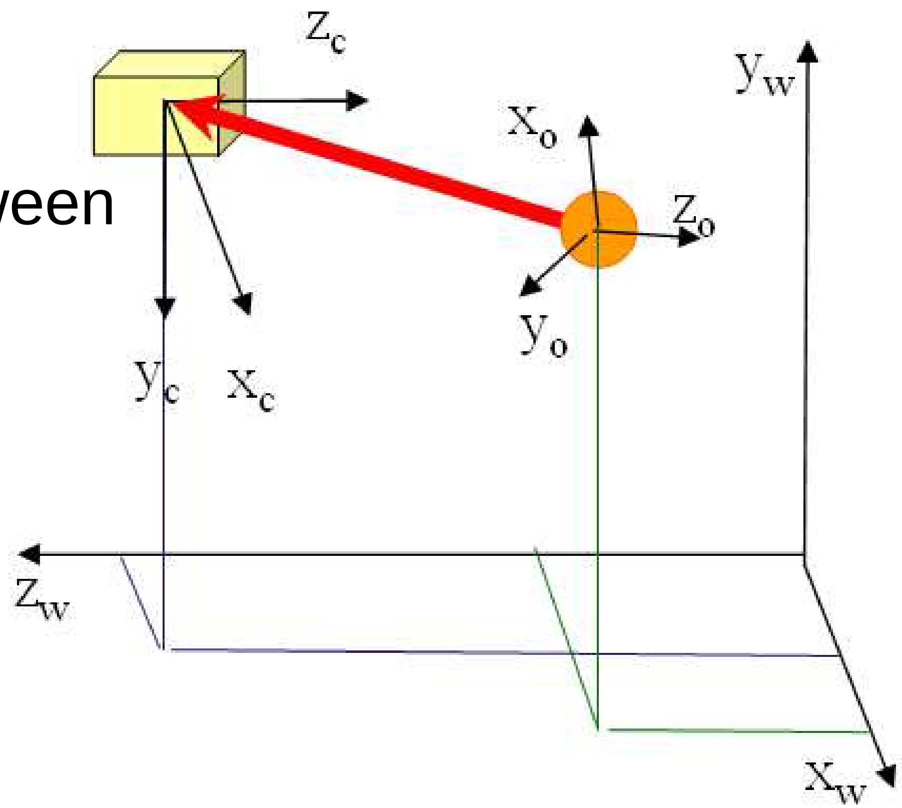
Rendering the scene

= traversing the corresponding scene graph

In practice:

realizing 3D transformations between

- object coordinates
- world coordinates
- camera coordinates

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# Coordinate Systems and Projective Geometry

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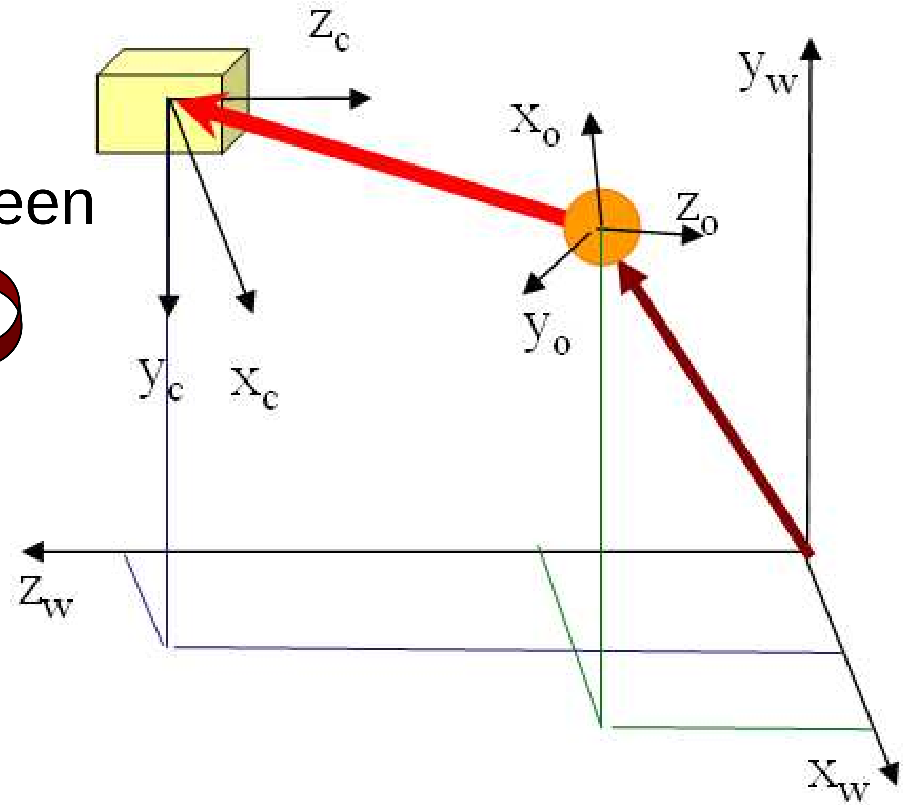


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# Coordinate Systems and Projective Geometry

Rendering the scene

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In practice:

realizing 3D transformations between

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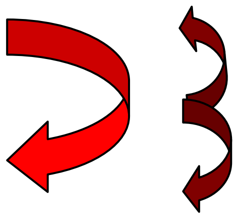

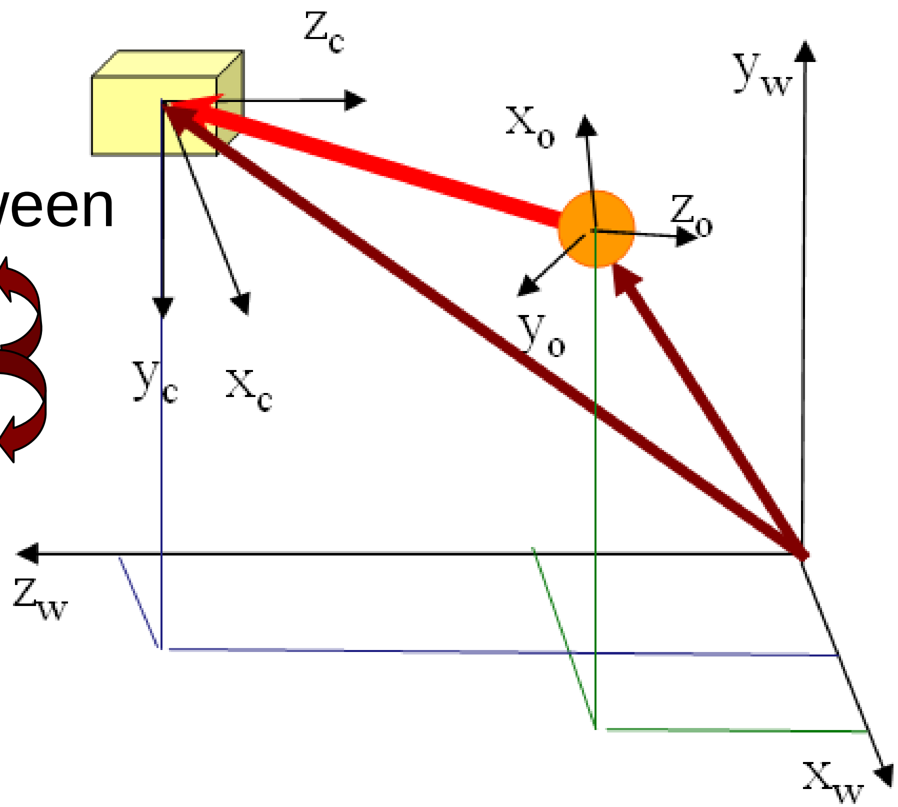


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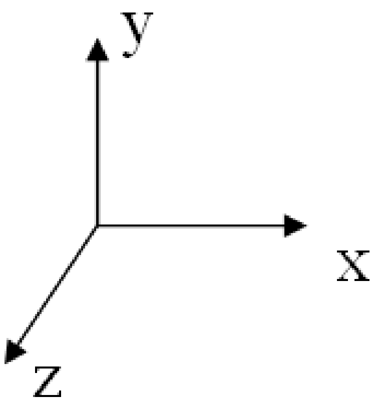
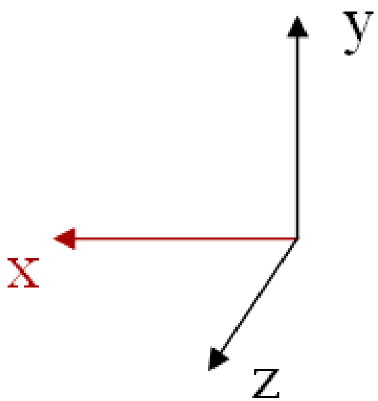
# Coordinate Systems and Projective Geometry

Coordinate systems:

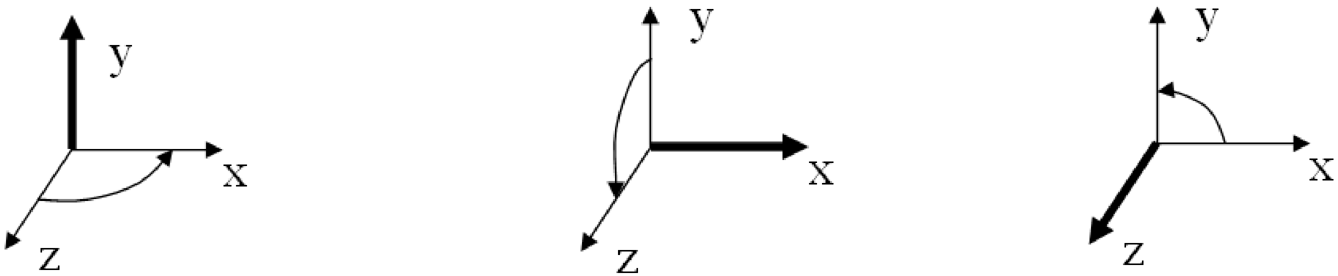



Right-handed




Left-handed

Right-handed rotations:




Counterclockwise rotation around respective axis

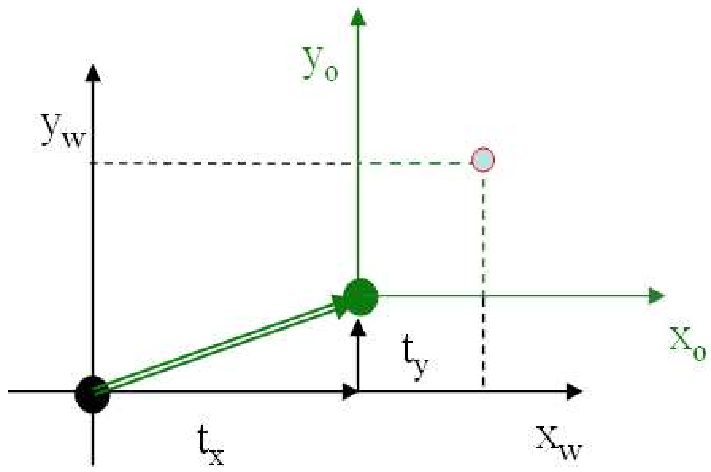
Source:[2]

# Coordinate Systems and Projective Geometry

## 2D transformations:

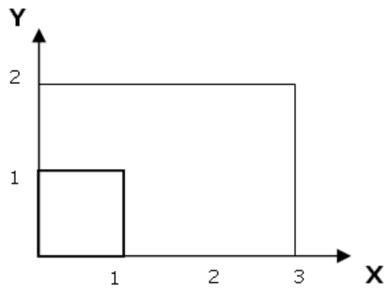
- Translation:

$$x_w = x_o + t_x$$
$$y_w = y_o + t_y$$



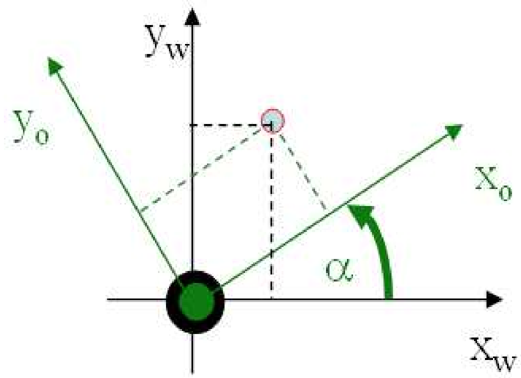

- Scaling:

$$x_w = s_x * x_o$$
$$y_w = s_y * y_o$$




- Rotation:

$$x_w = \cos\alpha * x_o - \sin\alpha * y_o$$
$$y_w = \sin\alpha * x_o + \cos\alpha * y_o$$




Source:[2]

# Coordinate Systems and Projective Geometry

No uniform matrix representation of these transformations in affine coordinates

In homogeneous coordinates: simple matrix multiplication

2D translation:

$$\begin{pmatrix} x_w \\ y_w \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_o \\ y_o \\ 1 \end{pmatrix}$$

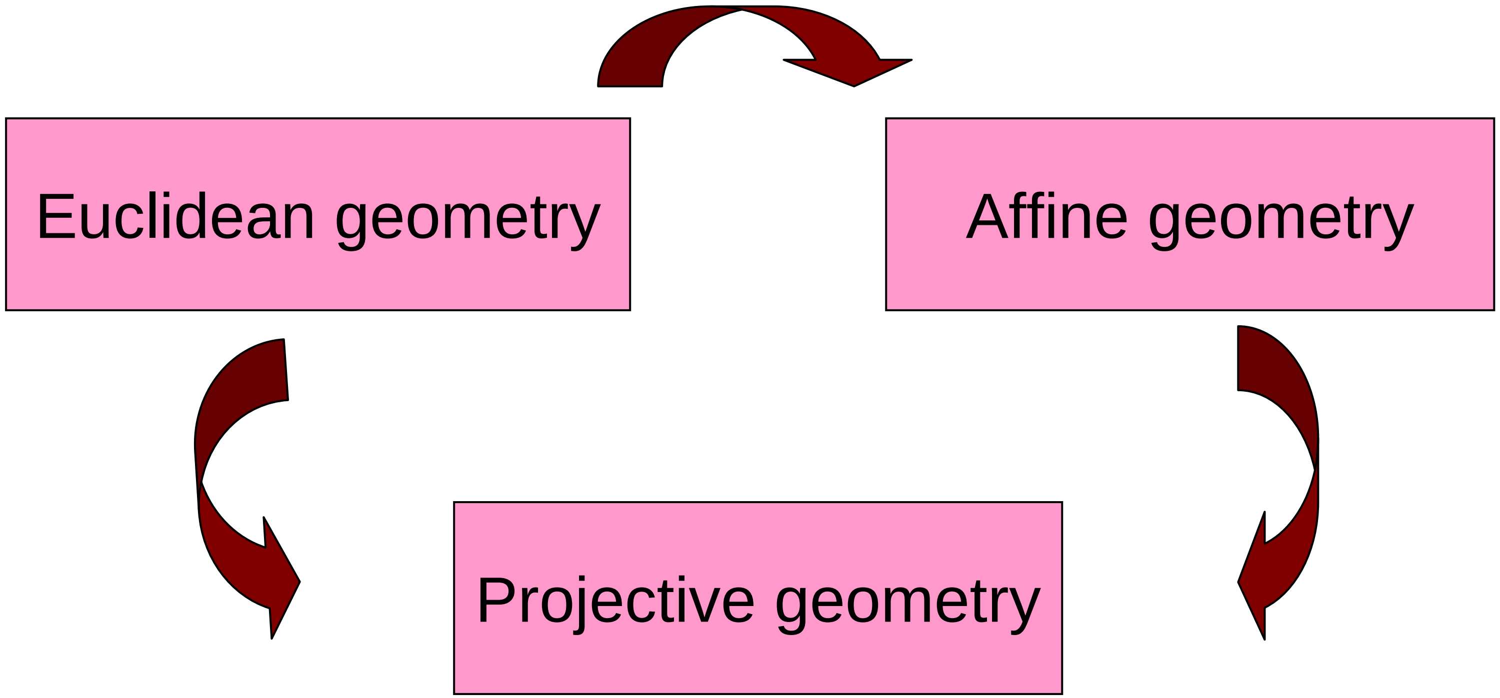
2D scaling:

$$\begin{pmatrix} x_w \\ y_w \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_o \\ y_o \\ 1 \end{pmatrix}$$

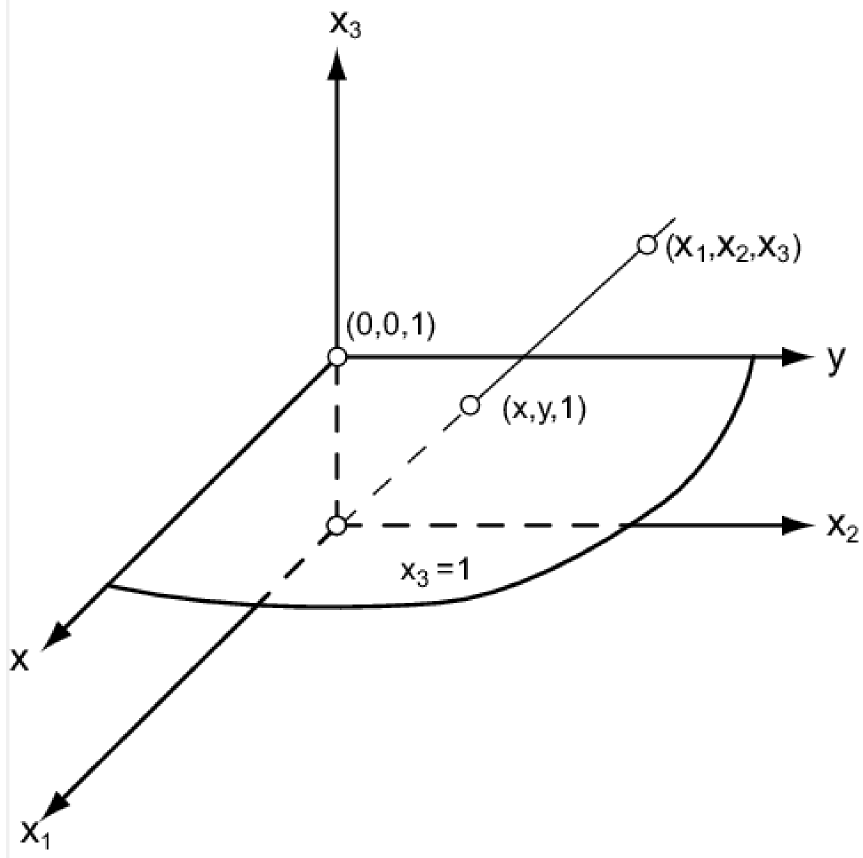
2D rotation:

$$\begin{pmatrix} x_w \\ y_w \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_o \\ y_o \\ 1 \end{pmatrix}$$

# Coordinate Systems and Projective Geometry



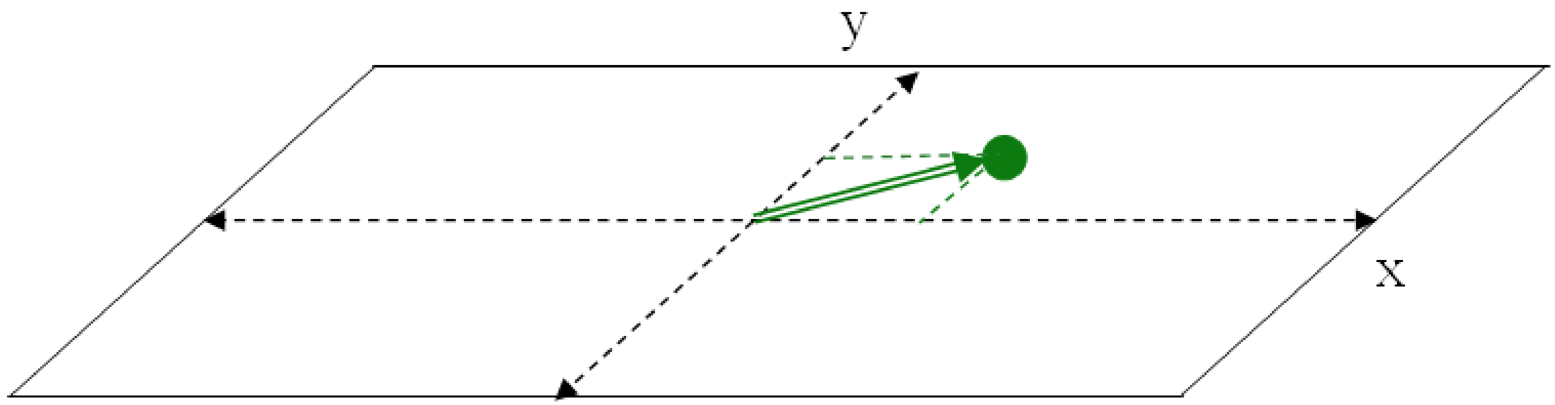
# Coordinate Systems and Projective Geometry




Projective plane $P^2$ = set of one-dimensional vector spaces of $R^3$

Source:[1]

# Coordinate Systems and Projective Geometry

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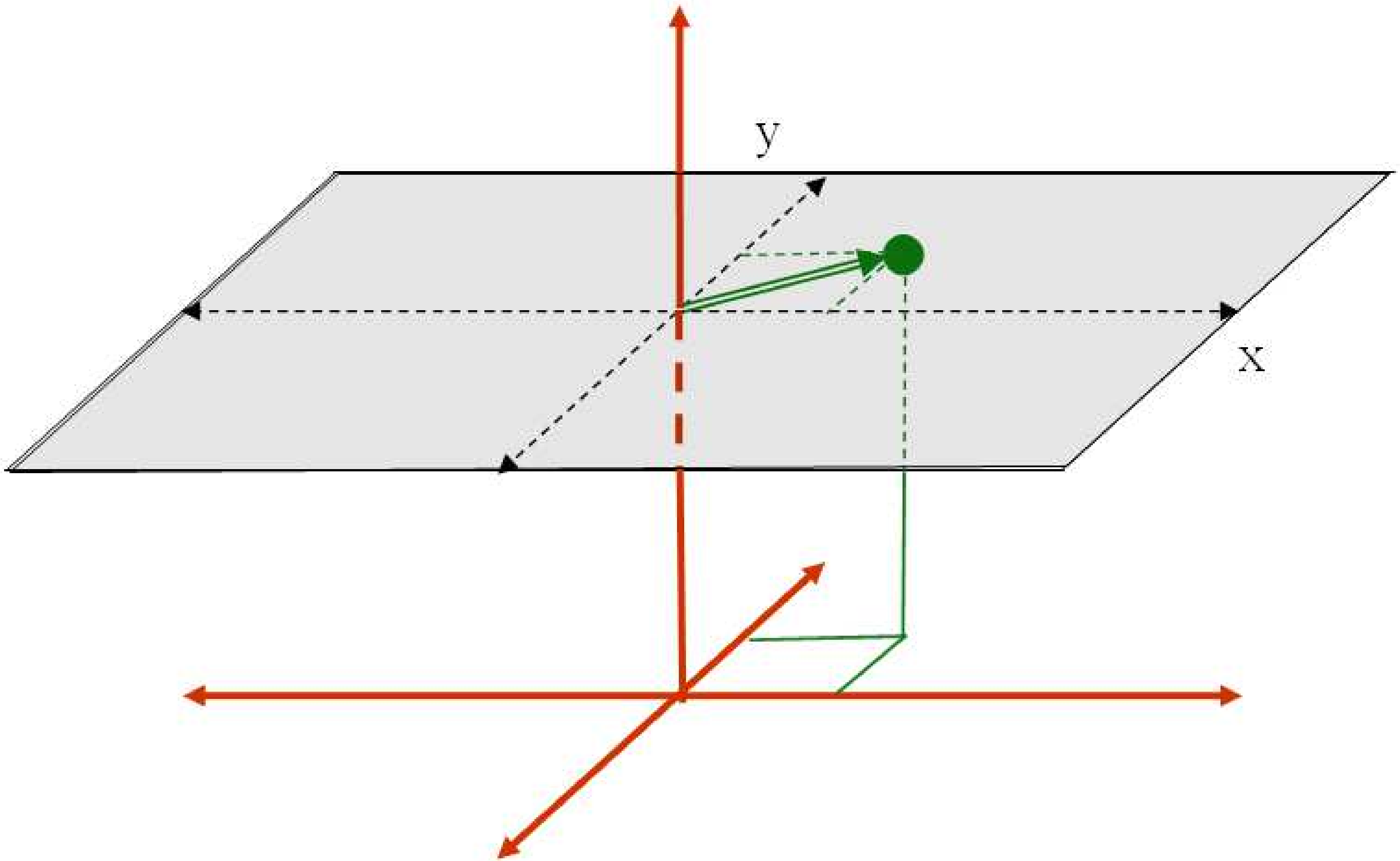
# Coordinate Systems and Projective Geometry

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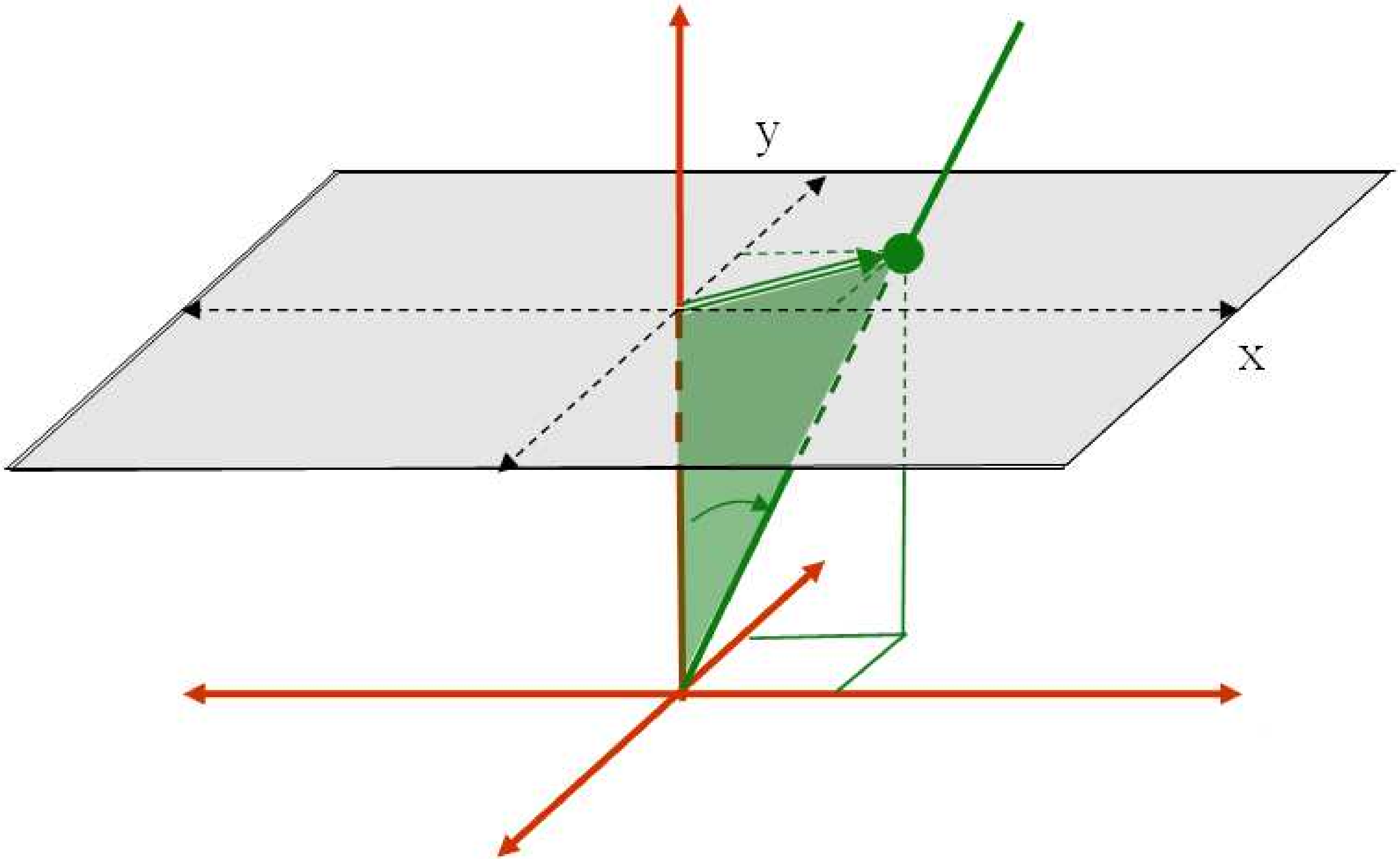
# Coordinate Systems and Projective Geometry



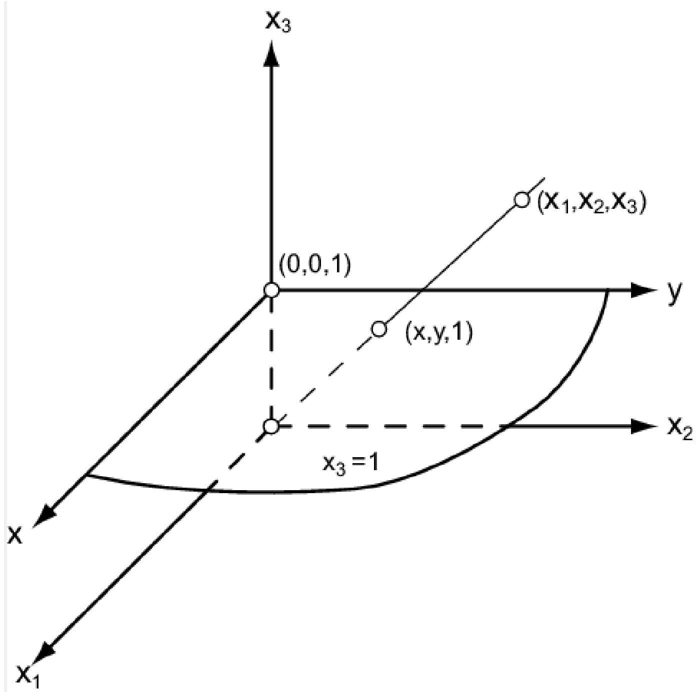

Source:[2]

# Coordinate Systems and Projective Geometry

$(x, y) \ldots \text{affine coordinates}$

$(x_1, x_2, x_3) \ldots \text{projective or homogeneous coordinates}$

$$\boxed{x = \frac{x_1}{x_3}, y = \frac{x_2}{x_3}}$$




$x_3 \neq 0 \ldots \text{proper points}$

$x_3 = 0 \ldots \text{points at infinity}$

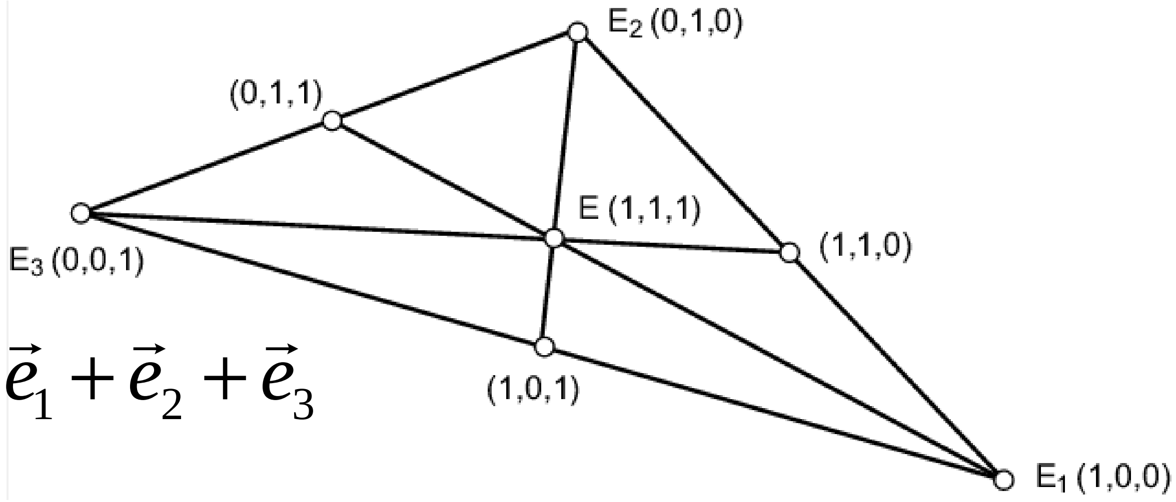
forming the line at infinity

affine plane = projective plane \ line at infinity

# Coordinate Systems and Projective Geometry

Projective coordinate system:

$$\{E_1, E_2, E_3; E\}$$




where $E_i(\vec{e}_i), E(\vec{e}), \vec{e} = \vec{e}_1 + \vec{e}_2 + \vec{e}_3$

Points $E_1, E_2, E_3, E$ have to be in **general position**, i.e., not any three of them have to be collinear

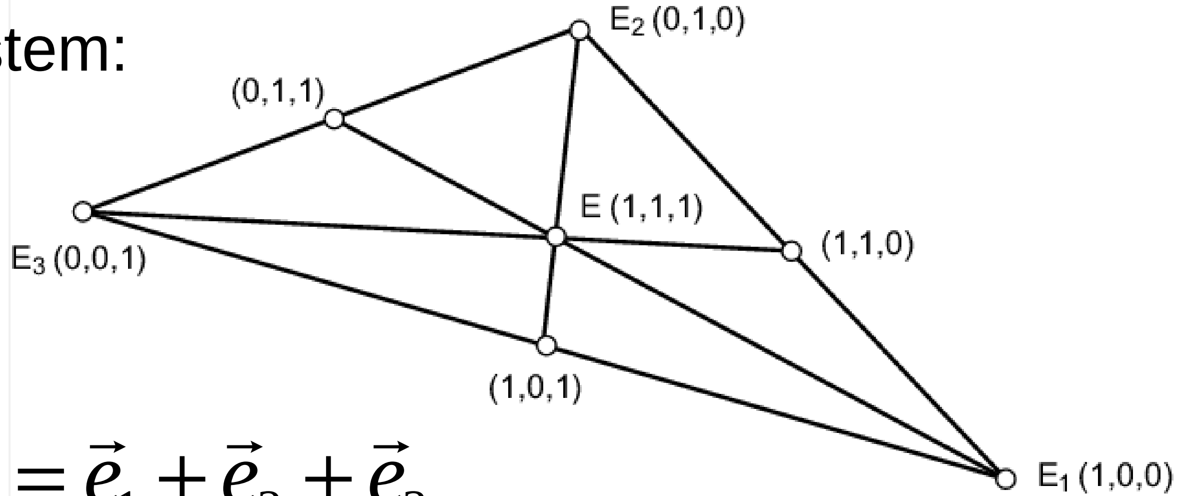
Role of the point E:

Normalization of representing vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$

Source:[1]

# Coordinate Systems and Projective Geometry

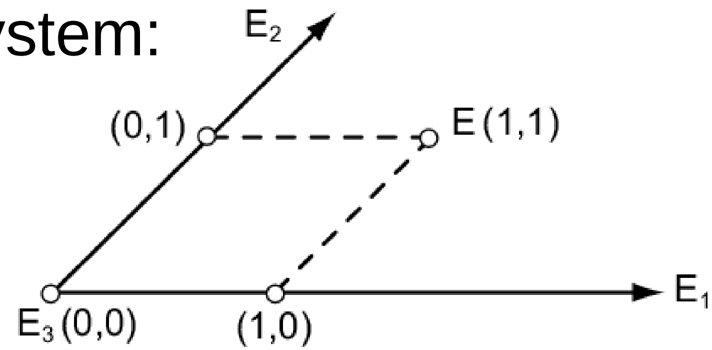
Projective coordinate system:




$$\{E_1, E_2, E_3; E\}$$

$$\text{where } E_i(\vec{e}_i), E(\vec{e}), \vec{e} = \vec{e}_1 + \vec{e}_2 + \vec{e}_3$$

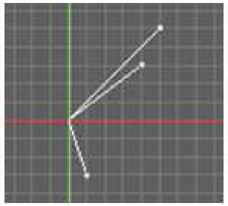
Corresponding affine coordinate system:

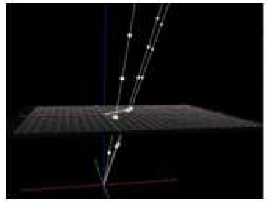



Source:[1]

# Coordinate Systems and Projective Geometry

Examples for points in homogeneous coordinates:





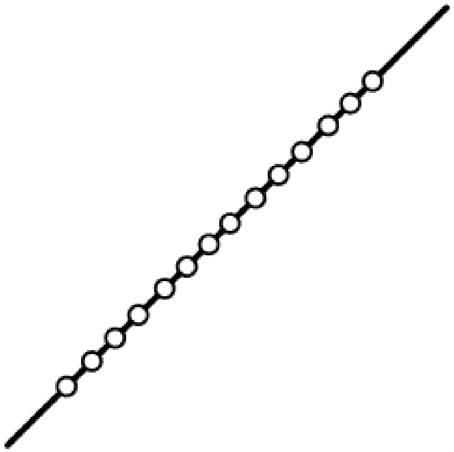
| $\mathbb{R}^2$ | $\mathbb{P}^2$ |
|---|---|
| $P_1 = (0.4, \quad 0.3)^T$ | $\mathbf{x_1} = (0.4w, \quad 0.3w, \quad w)^T$ |
| | $= (0.4, \quad 0.3, \quad 1.0)^T$ |
| | $= (0.8, \quad 0.6, \quad 2.0)^T$ |
| $P_2 = (0.1, \quad -0.3)^T$ | $\mathbf{x_2} = (0.3, \quad -0.9, \quad 3.0)^T$ |
| $P_3 = (0.5, \quad 0.5)^T$ | $\mathbf{x_3} = (0.5, \quad 0.5, \quad 1.0)^T$ |

Source:[2]

# Coordinate Systems and Projective Geometry

Equation of a straight line in homogeneous coordinates:

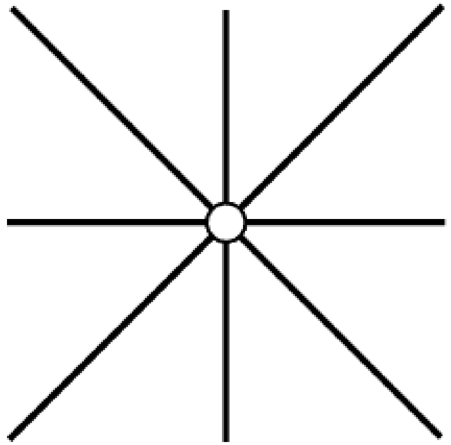
$$\boxed{u_1 x_1 + u_2 x_2 + u_3 x_3 = 0}$$



| | |
|---|---|
| $(x_1, x_2, x_3)$ | variable point coordinates |
| $(u_1, u_2, u_3)$ | constant |

→ A line can be identified by $(u_1, u_2, u_3)$, its line coordinates.

**Dual interpretation:**



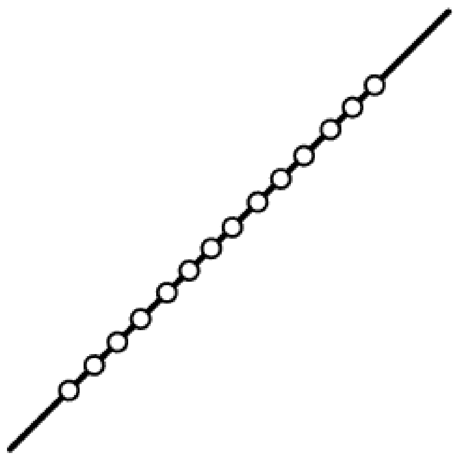
| | |
|---|---|
| $(u_1, u_2, u_3)$ | variable line coordinates |
| $(x_1, x_2, x_3)$ | constant |

→ A point can be identified by $(x_1, x_2, x_3)$, its point coordinates.

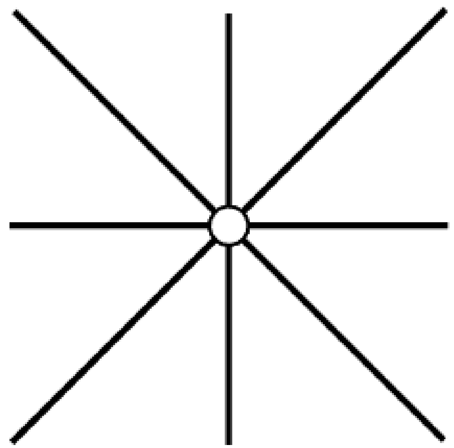
# Coordinate Systems and Projective Geometry

Prinicple of duality of projective geometry:

$$\boxed{u_1x_1 + u_2x_2 + u_3x_3 = 0}$$



line=junction of points



point=intersection of lines

## Coordinate Systems and Projective Geometry

Line joining the points P and Q:

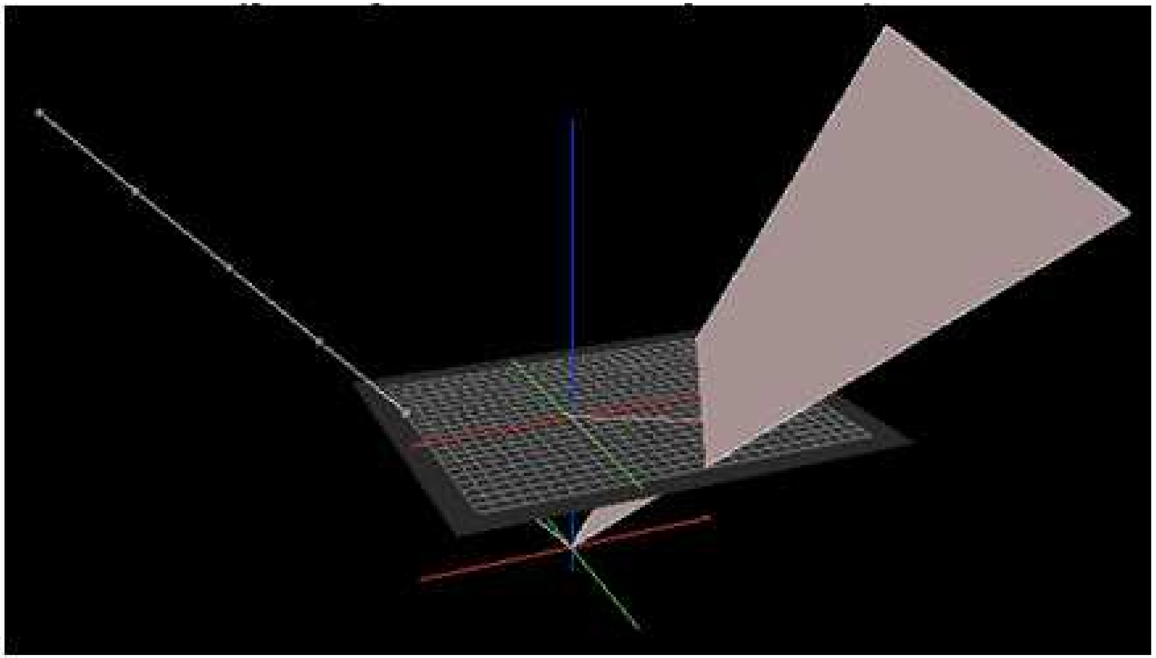
$$\text{Points } P(p_1, p_2, p_3), Q(q_1, q_2, q_3)$$
$$\rightarrow \text{line } h = P \wedge Q : h_1 x_1 + h_2 x_2 + h_3 x_3 = 0 :$$
$$(h_1, h_2, h_3)^T = (p_1, p_2, p_3)^T \wedge (q_1, q_2, q_3)^T$$

Point at the intersection of the lines g and h:

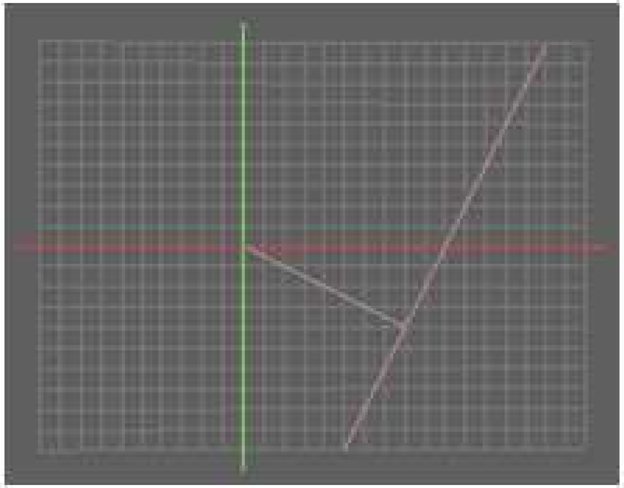
$$\text{Lines } g(g_1, g_2, g_3), h(h_1, h_2, h_3)$$
$$\rightarrow \text{point } S(s_1, s_2, s_3) = g \wedge h :$$
$$(s_1, s_2, s_3)^T = (g_1, g_2, g_3)^T \wedge (h_1, h_2, h_3)^T$$

# Coordinate Systems and Projective Geometry

Interpretation in 3-space:



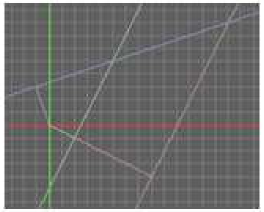
Situation in 2D:

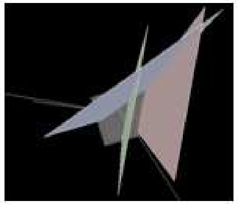


Source:[2]

# Coordinate Systems and Projective Geometry

Examples for lines in homogeneous coordinates:



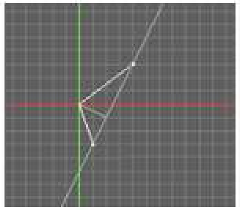


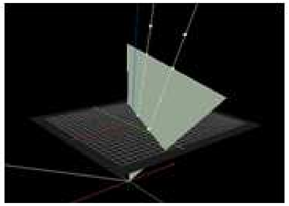
| $\mathrm{R}^2$ | $\mathrm{P}^2$ |
|---|---|
| $l_1 : 2x - y - 2 = 0$ | $\mathbf{l}_1 = (2, \quad -1, \quad -2)^{\mathrm{T}}$ |
| | $= (-1, \quad 0.5, \quad 1.0)^{\mathrm{T}}$ |
| $l_2 : 2x - y - 0.5 = 0$ | $\mathbf{l}_2 = (2, \quad -1, \quad -0.5)^{\mathrm{T}}$ |
| | $= (-4, \quad 2, \quad 1)^{\mathrm{T}}$ |
| $l_3 : x - 3y + 1 = 0$ | $\mathbf{l}_3 = (1, \quad -3, \quad 1)^{\mathrm{T}}$ |

Source:[2]

# Coordinate Systems and Projective Geometry

Line joining two points:





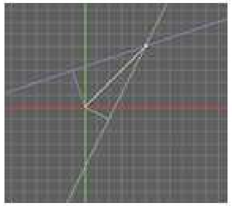
| $\mathbb{R}^2$ | $\mathbb{P}^2$ |
|---|---|
| $P_1 = (0.4, \quad 0.3)^T$ | $\mathbf{x}_1 = (0.4, \quad 0.3, \quad 1.0)^T$ |
| $P_2 = (0.1, \quad -0.3)^T$ | $\mathbf{x}_2 = (0.1, \quad -0.3, \quad 1.0)^T$ |
| $l_2 : 2x - y - 0.5 = 0$ | $\mathbf{l}_2 = (2.0, \quad -1.0, \quad -0.5)^T$ |

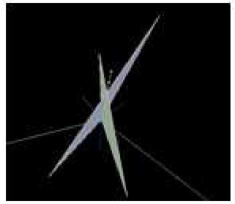
$$\begin{pmatrix} 0.4 \\ 0.3 \\ 1.0 \end{pmatrix} \times \begin{pmatrix} 0.1 \\ -0.3 \\ 1.0 \end{pmatrix} = \begin{pmatrix} 0.6 \\ -0.3 \\ -0.15 \end{pmatrix} = -6.66 \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}$$

Source:[2]

# Coordinate Systems and Projective Geometry

Intersection of two lines:





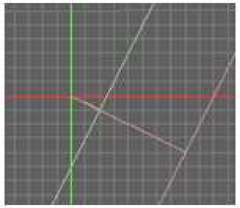
| $R^2$ | $P^2$ |
|---|---|
| $l_2 : 2x - y - 0.5 = 0$ | $\mathbf{l_2} = \left(-4.0,\quad 2.0,\quad 1.0\right)^T$ |
| $l_3 : x - 3y + 1 = 0$ | $\mathbf{l_3} = \left(1.0,\quad -3.0,\quad 1.0\right)^T$ |
| $P_3 = \left(0.5 \quad 0.5\right)^T$ | $\mathbf{x_3} = \left(0.5,\quad 0.5,\quad 1.0\right)^T$ |

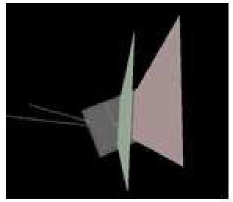
$$\begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 10 \end{pmatrix}$$

Source:[2]

# Coordinate Systems and Projective Geometry

Intersection of two parallel lines:





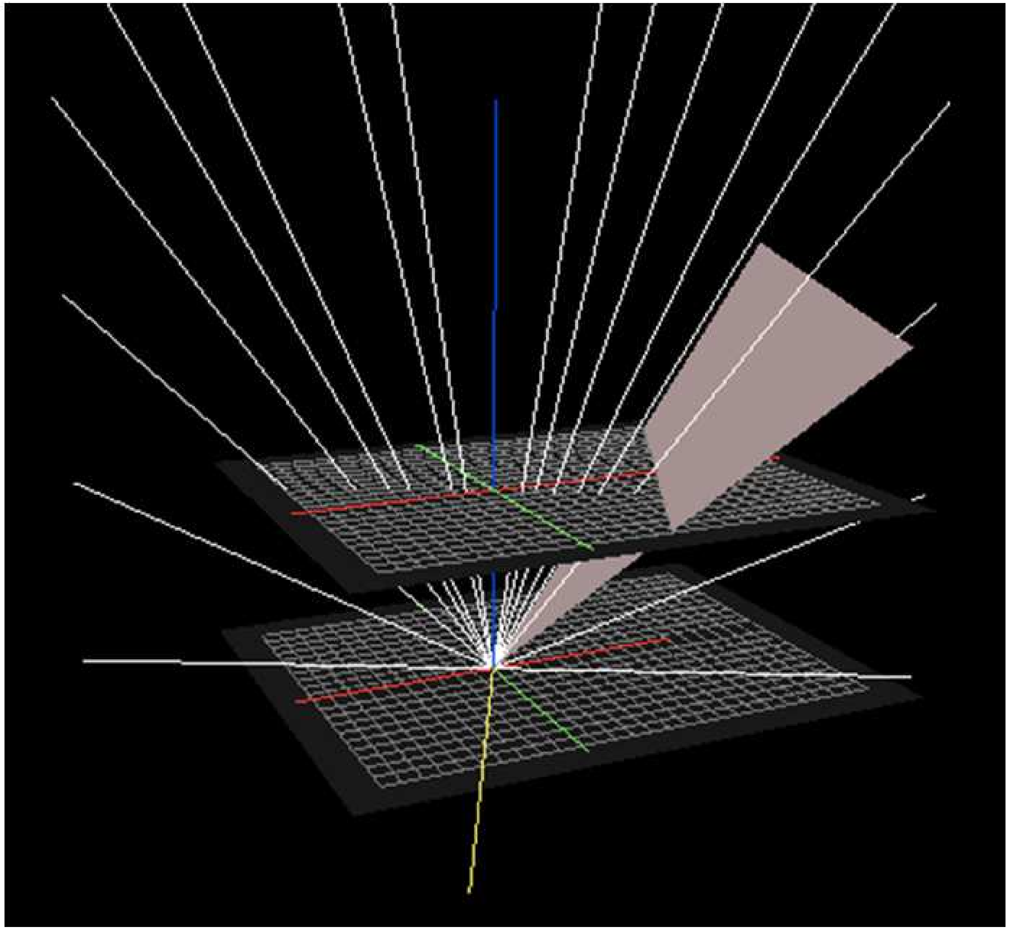
| $\mathrm{R}^2$ | $\mathrm{P}^2$ |
|---|---|
| $l_1 : 2x - y - 2 = 0$ | $\mathbf{l_1} = (2, \quad -1, \quad -2)^{\mathrm{T}}$ |
| $l_2 : 2x - y - 0.5 = 0$ | $\mathbf{l_2} = (2, \quad -1, \quad -0.5)^{\mathrm{T}}$ |

$$\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ -0.5 \end{pmatrix} = \begin{pmatrix} -1.5 \\ -3 \\ 0 \end{pmatrix} = 1.5 \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$$

Source:[2]

# Coordinate Systems and Projective Geometry

Intersection of parallel lines:



Source:[2]

# Coordinate Systems and Projective Geometry

Projective map:

$$\pi : P^2 \to P^2$$

$$X(x_1, x_2, x_3) \mapsto Y(y_1, y_2, y_3)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

$T = (t_{ij}) \in R^{3\times 3}$, regular,

determined up to a constant factor

$$\Updownarrow$$

$$y_3 = t_{31}x_1 + t_{32}x_2 + t_{33}x_3$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t_{13} \\ t_{23} \end{pmatrix} x_3$$

Consider:
all those projective maps that do not change the line at infinity, i.e., $x_3 = 0 \Leftrightarrow y_3 = 0$

$$\Rightarrow t_{31} = t_{32} = 0$$

$$\Rightarrow T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix}$$

$$affine: x_3 = 1, y_3 = 1 \Rightarrow t_{33} = 1$$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t_{13} \\ t_{23} \end{pmatrix}$$

Affine map

# Coordinate Systems and Projective Geometry

**Generalization to 3D:**

$(x, y, z) \ldots \text{affine coordinates}$

$(x_1, x_2, x_3, x_4) \ldots \text{projective or homogeneous coordinates}$

$$\boxed{x = \frac{x_1}{x_4}, y = \frac{x_2}{x_4}, z = \frac{x_3}{x_4}}$$

$x_4 \neq 0 \ldots \text{proper points}$

$x_4 = 0 \ldots \text{points at infinity}$

forming the plane at infinity

affine 3-space = projective 3-space \ plane at infinity

# Coordinate Systems and Projective Geometry

## 3D transformations in homogeneous coordiantes:

OpenGL

3D translation:

$$\begin{pmatrix} x_w \\ y_w \\ z_w \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_o \\ y_o \\ z_o \\ 1 \end{pmatrix}$$

$$\boxed{\text{glTranslate}\ (t_x, t_y, t_z)}$$

3D scaling:

$$\begin{pmatrix} x_w \\ y_w \\ z_w \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_o \\ y_o \\ z_o \\ 1 \end{pmatrix}$$

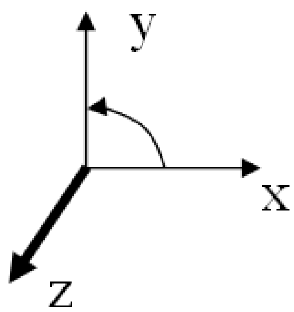
$$\boxed{\text{glScale}\ (s_x, s_y, s_z)}$$

# Coordinate Systems and Projective Geometry

## 3D transformations in homogeneous coordiantes:

3D rotation:

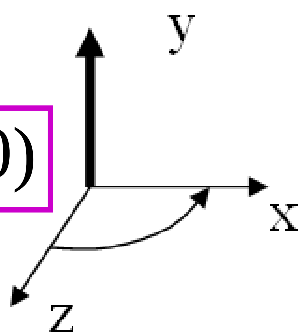
OpenGL




$$\begin{pmatrix} x_w \\ y_w \\ z_w \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_o \\ y_o \\ z_o \\ 1 \end{pmatrix}$$

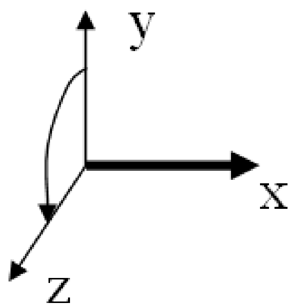
$\boxed{\text{glRotate } (\alpha,0,0,1)}$

$\boxed{\text{glRotate } (\alpha,0,1,0)}$




$$\begin{pmatrix} x_w \\ y_w \\ z_w \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_o \\ y_o \\ z_o \\ 1 \end{pmatrix}$$

$\boxed{\text{glRotate } (\alpha,1,0,0)}$

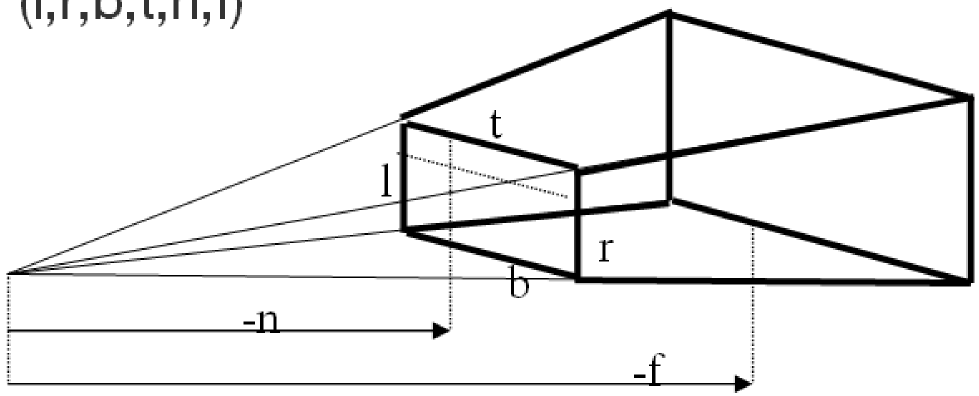



$$\begin{pmatrix} x_w \\ y_w \\ z_w \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_o \\ y_o \\ z_o \\ 1 \end{pmatrix}$$

# Coordinate Systems and Projective Geometry

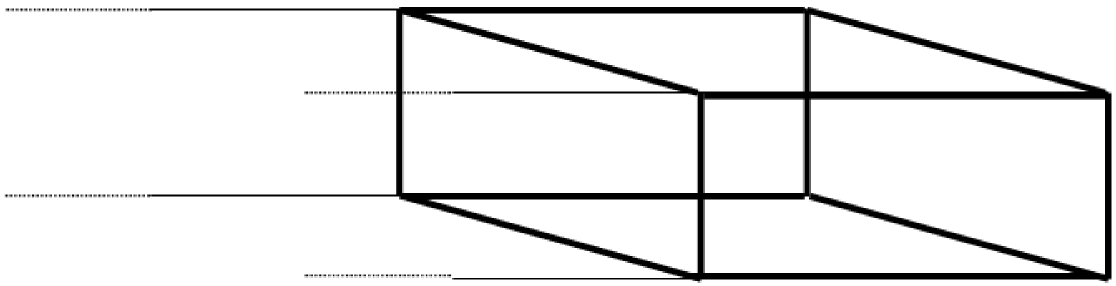
- Perspective Projection: glFrustum* (l,r,b,t,n,f)

$$\begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$




- Orthographic Projection: glOrtho* (l,r,b,t,n,f)

$$\begin{bmatrix} \frac{2n}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & -\frac{2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

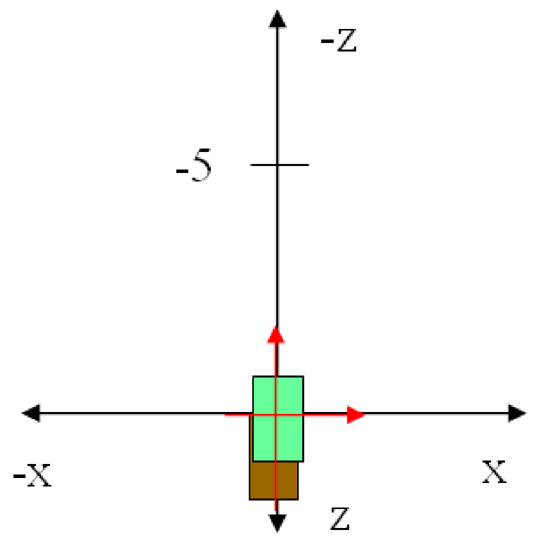


Source:[2]

# Coordinate Systems and Projective Geometry

OpenGL

```
glTranslatef (0.0, 0.0, -5.0);     // along z-axis
glRotatef (45.0, 0.0, 1.0, 0.0); // around y-axis
```

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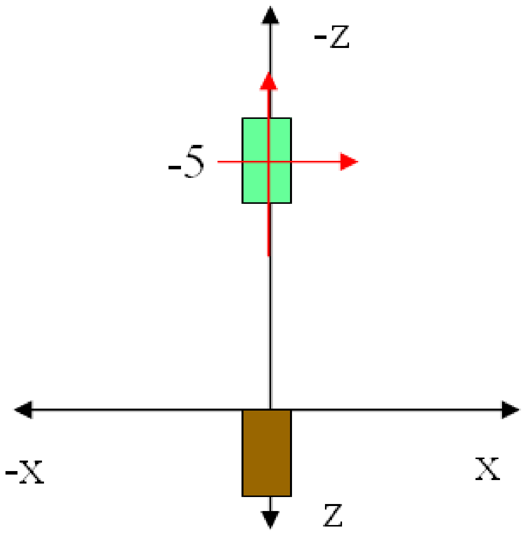
# Coordinate Systems and Projective Geometry

## OpenGL

**glTranslatef (0.0, 0.0, -5.0); // along z-axis**

glRotatef (45.0, 0.0, 1.0, 0.0); // around y-axis

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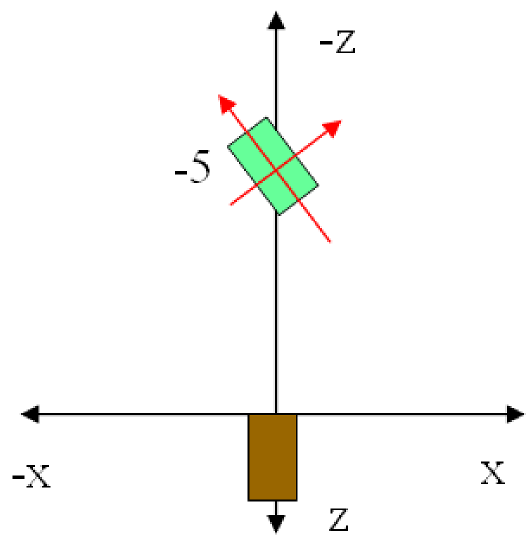
# Coordinate Systems and Projective Geometry

## OpenGL

glTranslatef (0.0, 0.0, -5.0);     // along z-axis
**glRotatef (45.0, 0.0, 1.0, 0.0); // around y-axis**

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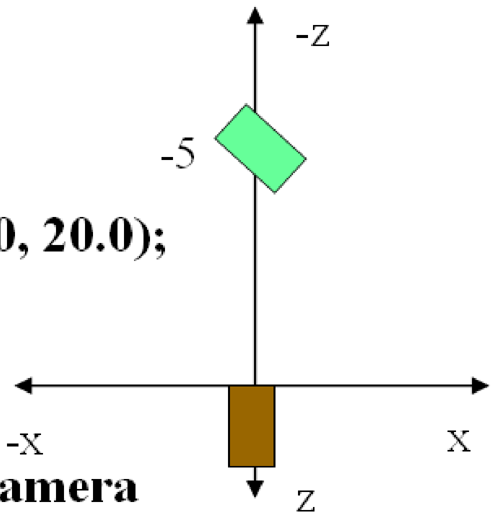
# Coordinate Systems and Projective Geometry

## A sample OpenGL program

```
// Viewport Transformations
glViewport (0.0, 0.0, w, h);
// Projection Transformations
glMatrixMode (GL_PROJECTION);
glLoadIdentity ();
glFrustum (l,r,t,b,n,f);   or gluPerspective (60.0, w/h, 1.0, 20.0);
// Viewing Transformations
glMatrixMode (GL_MODELVIEW);
glLoadIdentity ();
glTranslatef (0.0, 0.0, -5.0);  // move world away from camera
glRotatef (45.0, 0.0, 1.0, 0.0);  // rotate world in front of the camera around y-axis
...
// Modeling Transformations
...
// Draw Object
...
```

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