Bézier and B–Spline curves

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Contents

Bézier curve

Definition

Let $u_0 < u_1 < \ldots < u_k$ be a partition of the interval $[u_0, u_k]$. We call **Bézier curve of degree n** a curve $\overrightarrow{x}(u)$ that is composed of Bézier segments of degree n on every subinterval $[u_l, u_{l+1}]$ $(l = 0, \ldots, k-1)$:

$$\overrightarrow{x}(u)|_{u\in[u_l,u_{l+1}]} = \overrightarrow{x}_l(u) = \sum_{i=0}^{n} \overrightarrow{b}_{nl+i} B_i^n\left(\frac{u-u_l}{u_{l+1}-u_l}\right).$$

Bézier curve

Theorem (C^r continuity of Bézier segments)

Let

$$\vec{x}_l(u) = \sum_{i=0}^{n} \vec{b}_{nl+i} B_i^n\left(\frac{u-u_l}{u_{l+1}-u_l}\right);\; u \in [u_l, u_{l+1}]$$

$$\vec{x}_{l+1}(u) = \sum_{i=0}^{n} \vec{b}_{n(l+1)+i} B_i^n\left(\frac{u-u_{l+1}}{u_{l+2}-u_{l+1}}\right);\; u \in [u_{l+1}, u_{l+2}]$$

be two consecutive Bézier segments and $\delta_k := u_{k+1} - u_k$. *Then, the segments* $\vec{x}_l(u)$ *and* $\vec{x}_{l+1}(u)$ *join with* C^r *continuity if and only if*

$$\frac{1}{\delta_l^i}\Delta^i \vec{b}_{n(l+1)-i} = \frac{1}{\delta_{l+1}^i}\Delta^i \vec{b}_{n(l+1)};\; i = 0, \ldots, r$$

Bézier curve

Proof.

Calculate derivatives by considering $t = \frac{u-u_l}{u_{l+1}-u_l}$ respectively $t = \frac{u-u_{l+1}}{u_{l+2}-u_{l+1}}$. □

Bézier curve

C^1 continuity

C^1 continuity in $u = u_{l+1}$:

$$\Delta^1 \overrightarrow{b}_{n(l+1)-1} = \frac{\delta_l}{\delta_{l+1}} \Delta^1 \overrightarrow{b}_{n(l+1)}$$

Bézier curve

C^1 continuity

C^1 continuity in $u = u_{l+1}$:

$$\Delta^1 \overrightarrow{b}_{n(l+1)-1} = \frac{\delta_l}{\delta_{l+1}} \Delta^1 \overrightarrow{b}_{n(l+1)}$$

$$\overrightarrow{b}_{n(l+1)} - \overrightarrow{b}_{n(l+1)-1} = \frac{\delta_l}{\delta_{l+1}} (\overrightarrow{b}_{n(l+1)+1} - \overrightarrow{b}_{n(l+1)})$$

Bézier curve

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C^1 continuity in $u = u_{l+1}$:

$$\begin{aligned}\Delta^1 \vec{b}_{n(l+1)-1} &= \frac{\delta_l}{\delta_{l+1}} \Delta^1 \vec{b}_{n(l+1)} \\ \vec{b}_{n(l+1)} - \vec{b}_{n(l+1)-1} &= \frac{\delta_l}{\delta_{l+1}} \left(\vec{b}_{n(l+1)+1} - \vec{b}_{n(l+1)} \right)\end{aligned}$$

Illustration

Bézier curve

C^2 continuity

C^2 continuity in $u = u_{l+1}$:

$$\Delta^1 \overrightarrow{b}_{n(l+1)-1} = \frac{\delta_l}{\delta_{l+1}} \Delta^1 \overrightarrow{b}_{n(l+1)} \quad (C^1) \text{ and}$$

$$\Delta^2 \overrightarrow{b}_{n(l+1)-2} = \frac{\delta_l^2}{\delta_{l+1}^2} \Delta^2 \overrightarrow{b}_{n(l+1)} \quad (C^2)$$

Bézier curve

C^2 continuity

C^2 continuity in $u = u_{l+1}$:

$$\Delta^1 \overrightarrow{b}_{n(l+1)-1} = \frac{\delta_l}{\delta_{l+1}} \Delta^1 \overrightarrow{b}_{n(l+1)} \quad (C^1) \text{ and}$$

$$\Delta^2 \overrightarrow{b}_{n(l+1)-2} = \frac{\delta_l^2}{\delta_{l+1}^2} \Delta^2 \overrightarrow{b}_{n(l+1)} \quad (C^2)$$

$$(C^2) \quad \Leftrightarrow \quad \Delta^1 \overrightarrow{b}_{n(l+1)-1} - \Delta^1 \overrightarrow{b}_{n(l+1)-2} = \frac{\delta_l^2}{\delta_{l+1}^2} (\Delta^1 \overrightarrow{b}_{n(l+1)+1} - \Delta^1 \overrightarrow{b}_{n(l+1)})$$

Bézier curve

C^2 continuity

C^2 continuity in $u = u_{l+1}$, continued:

$$\Leftrightarrow \quad \Delta^1 \overrightarrow{b}_{n(l+1)-1} - \Delta^1 \overrightarrow{b}_{n(l+1)-2}$$
$$= \frac{\delta_l^2}{\delta_{l+1}^2} \left(\Delta^1 \overrightarrow{b}_{n(l+1)+1} - \Delta^1 \overrightarrow{b}_{n(l+1)} \right)$$

Bézier curve

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C^2 continuity in $u = u_{l+1}$, continued:

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Bézier curve

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C^2 continuity in $u = u_{l+1}$, continued:

$$\Leftrightarrow \quad \Delta^1 \overrightarrow{b}_{n(l+1)-1} - \Delta^1 \overrightarrow{b}_{n(l+1)-2} = \frac{\delta_l^2}{\delta_{l+1}^2}(\Delta^1 \overrightarrow{b}_{n(l+1)+1} - \Delta^1 \overrightarrow{b}_{n(l+1)}) \qquad | \, (C^1) \Rightarrow$$

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Bézier curve

C^2 continuity

C^2 continuity in $u = u_{l+1}$, continued:

$$\Leftrightarrow \quad \Delta^1 \vec{b}_{n(l+1)-1} - \Delta^1 \vec{b}_{n(l+1)-2}$$

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$$\Leftrightarrow \quad \frac{\delta_l}{\delta_{l+1}} \Delta^1 \vec{b}_{n(l+1)-1} - \Delta^1 \vec{b}_{n(l+1)-2}$$

$$= \frac{\delta_l^2}{\delta_{l+1}^2} \Delta^1 \vec{b}_{n(l+1)+1} - \frac{\delta_l}{\delta_{l+1}} \Delta^1 \vec{b}_{n(l+1)}$$

Bézier curve

C^2 continuity

C^2 continuity in $u = u_{l+1}$, continued:

$$\Leftrightarrow \quad \Delta^1 \vec{b}_{n(l+1)-1} - \Delta^1 \vec{b}_{n(l+1)-2}$$
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$$\Leftrightarrow \quad \frac{\delta_l}{\delta_{l+1}} \Delta^1 \vec{b}_{n(l+1)} - \Delta^1 \vec{b}_{n(l+1)-2}$$
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Bézier curve

C^2 continuity

C^2 continuity in $u = u_{l+1}$, continued:

$$(C^2) \quad \Leftrightarrow \quad \vec{b}_{n(l+1)} - \vec{b}_{n(l+1)-1} - \frac{\delta_{l+1}}{\delta_l}\Delta^1\vec{b}_{n(l+1)-2}$$
$$= \frac{\delta_l}{\delta_{l+1}}\Delta^1\vec{b}_{n(l+1)+1} - \vec{b}_{n(l+1)+1} + \vec{b}_{n(l+1)}$$

Bézier curve

C^2 continuity

C^2 continuity in $u = u_{l+1}$, continued:

$$(C^2) \quad \Leftrightarrow \quad \vec{b}_{n(l+1)} - \vec{b}_{n(l+1)-1} - \frac{\delta_{l+1}}{\delta_l}\Delta^1 \vec{b}_{n(l+1)-2}$$
$$= \frac{\delta_l}{\delta_{l+1}}\Delta^1 \vec{b}_{n(l+1)+1} - \vec{b}_{n(l+1)+1} + \vec{b}_{n(l+1)} \qquad | * (-1)$$

Bézier curve

C^2 continuity

C^2 continuity in $u = u_{l+1}$, continued:

$$
\begin{aligned}
(C^2) \quad \Leftrightarrow \quad & \vec{b}_{n(l+1)} - \vec{b}_{n(l+1)-1} - \frac{\delta_{l+1}}{\delta_l} \Delta^1 \vec{b}_{n(l+1)-2} \\
&= \frac{\delta_l}{\delta_{l+1}} \Delta^1 \vec{b}_{n(l+1)+1} - \vec{b}_{n(l+1)+1} + \vec{b}_{n(l+1)} \qquad | * (-1) \\
\Leftrightarrow \quad & \vec{b}_{n(l+1)-1} + \frac{\delta_{l+1}}{\delta_l} \Delta^1 \vec{b}_{n(l+1)-2} \\
&= \vec{b}_{n(l+1)+1} - \frac{\delta_l}{\delta_{l+1}} \Delta^1 \vec{b}_{n(l+1)+1}
\end{aligned}
$$

Bézier curve

C^2 continuity

C^2 continuity in $u = u_{l+1}$, continued:

$$
\begin{aligned}
(C^2) \quad \Leftrightarrow \quad & \vec{b}_{n(l+1)} - \vec{b}_{n(l+1)-1} - \frac{\delta_{l+1}}{\delta_l}\Delta^1 \vec{b}_{n(l+1)-2} \\
&= \frac{\delta_l}{\delta_{l+1}}\Delta^1 \vec{b}_{n(l+1)+1} - \vec{b}_{n(l+1)+1} + \vec{b}_{n(l+1)} \qquad | * (-1) \\
\Leftrightarrow \quad & \vec{b}_{n(l+1)-1} + \frac{\delta_{l+1}}{\delta_l}\Delta^1 \vec{b}_{n(l+1)-2} \\
&= \vec{b}_{n(l+1)+1} - \frac{\delta_l}{\delta_{l+1}}\Delta^1 \vec{b}_{n(l+1)+1} =: \vec{d}
\end{aligned}
$$

Bézier curve

C^2 continuity

C^2 continuity in $u = u_{l+1}$, continued:

$$\begin{aligned} \Rightarrow \overrightarrow{d} - \overrightarrow{b}_{n(l+1)-1} &= \frac{\delta_{l+1}}{\delta_l}\Delta^1 \overrightarrow{b}_{n(l+1)-2} \\ \text{et} \quad \overrightarrow{b}_{n(l+1)+1} - \overrightarrow{d} &= \frac{\delta_l}{\delta_{l+1}}\Delta^1 \overrightarrow{b}_{n(l+1)+1} \end{aligned}$$

Bézier curve

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Illustration

Bézier curve

Remark

The notion of C^r continuity is not invariant with respect to parameter transformations, i.e., it is not a geometric property of the curve.

Bézier curve

Example:

$$\vec{x}(u) = \begin{pmatrix} u \\ 1 \end{pmatrix}, \; u \in [1,2]$$

$$\vec{y}(v) = \begin{pmatrix} v-1 \\ 1 \end{pmatrix}, \; v \in [3,4]$$

$$\vec{x}_0 = \vec{x}(2) = \vec{y}(3)$$

$$\frac{d}{du}\vec{x}(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{d}{dv}\vec{y}(3)$$

Bézier curve

Example, continued:

Paramter transformation: $u = f(t) = t^2$

$$\vec{x}(u) = \begin{pmatrix} u \\ 1 \end{pmatrix} = \begin{pmatrix} t^2 \\ 1 \end{pmatrix} = \vec{\bar{x}}(t)\,;\; t \in [1, \sqrt{2}]$$

$$\vec{x}_0 = \vec{\bar{x}}(\sqrt{2}) = \vec{y}(3)$$

Bézier curve

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$$\vec{x}_0 = \vec{\bar{x}}(\sqrt{2}) = \vec{y}(3)$$

$$\text{mais: } \frac{d}{dt}\vec{\bar{x}}(\sqrt{2}) = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{d}{dv}\vec{y}(3)$$

B–Splines curves

Parametric polynomial curves:

$$\vec{x}(u) = \sum \vec{a}_i \alpha_i(u)$$

$\alpha_i(u) \ldots$ "blending functions"

B–Spline curves

Definition (normalized B–Spline)

Let $\pi = \{u_i | u_i \leq u_{i+1}\}_{i=-\infty}^{\infty}$ be a partition. Then, we define by recurrence (de Boor/Cox/Mansfield, 1972):

$$N_i^0(u) = \begin{cases} 1\,, & u \in [u_i, u_{i+1}) \\ 0\,, & else \end{cases}$$

$$N_i^n(u) = \frac{u - u_i}{u_{i+n} - u_i} N_i^{n-1}(u) + \frac{u_{i+n+1} - u}{u_{i+n+1} - u_{i+1}} N_{i+1}^{n-1}(u)$$

where "$\frac{0}{0} = 0$".
The function $N_i^n(u)$ is called normalized B–Spline of degree n (of order $n+1$) with the support $[u_i, u_{i+n+1}]$. If $u_i \in \mathbf{Z}$ and ($u_{i+1} = u_i$ or $u_{i+1} = u_i + 1$) we call $N_i^n(u)$ uniform B–Spline.

B–Spline curves

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Scheme

B–Spline curves

Example

$\pi = \{u_0 = 0, u_1 = 1, u_2 = 2, u_3 = 3\}, n = 2$

$$N_0^0(u) = \begin{cases} 1\,, & u \in [0,1) \\ 0\,, & else \end{cases}$$

$$N_1^0(u) = \begin{cases} 1\,, & u \in [1,2) \\ 0\,, & else \end{cases}$$

$$N_2^0(u) = \begin{cases} 1\,, & u \in [2,3) \\ 0\,, & else \end{cases}$$

B–Spline curves

Example, continued

$\pi = \{u_0 = 0, u_1 = 1, u_2 = 2, u_3 = 3\}$, $n = 2$

$$N_0^1(u) =$$

$$N_1^1(u) =$$

B–Spline curves

Example, continued

$\pi = \{u_0 = 0, u_1 = 1, u_2 = 2, u_3 = 3\}, n = 2$

$$N_0^1(u) = \begin{cases} u\,, & u \in [0,1) \\ 2-u\,, & u \in [1,2) \\ 0\,, & else \end{cases}$$

$$N_1^1(u) = \begin{cases} u-1\,, & u \in [1,2) \\ 3-u\,, & u \in [2,3) \\ 0\,, & else \end{cases}$$

B–Spline curves

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$$N_0^2(u) = \begin{cases} \frac{u^2}{2}\,, & u \in [0,1) \\ -u^2+3u-\frac{3}{2}\,, & u \in [1,2) \\ \frac{(3-u)^2}{2}\,, & u \in [2,3) \\ 0\,, & else \end{cases}$$

B–Spline curves

Theorem (Properties of normalized B–Splines)

The normalized B–Splines $N_i^n(u)$ *that are defined with respect to the partition* $\pi = \{u_i\}_{i=-\infty}^{\infty}$ *satisfy:*

B–Spline curves

Theorem (Properties of normalized B–Splines)

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N1) *$N_i^n(u)$ is a piecewise polynomial of degree n.*

B–Spline curves

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N2) $N_i^n(u) \begin{cases} > 0\,, & u \in (u_i, u_{i+n+1}) \\ = 0\,, & \textit{else} \end{cases} \quad \Rightarrow$ *"minimal support": $[u_i, u_{i+n+1}]$*

B–Spline curves

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N3) $N_i^n(u)|_{[u_l,u_{l+1}]} \neq 0$ *for* $i = l-n, \ldots, l$

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N5) *Let l_j be the multiplicity of the knot u_j* $\Rightarrow N_i^n(u) \in C^{n-l_j}$

B–Spline curves

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If there are only simple knots $\Rightarrow N_i^n(u) \in C^{n-1}$

B–Spline curves

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Proof.

see literature □

B–Spline curves

Remarks

1. Different notations for B–Splines in literature, for example:
   - $N_i^n(u)$ with support $[u_{i-1}, u_{i+n}]$ (Farin)
   - $N_{i,\mu}(u)$ with support $[u_i, u_{i+\mu}]$, where $\mu = n + 1 =$ order (Hoschek/Lasser)

B–Spline curves

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2. Other possibilities for defining B–Splines: among others by
   - truncated power functions
   - intersection volume of an n–dimensional simplex with a hyperplane.

B–Spline curves

Definition (B–Spline curve)

Let $m \geq n \in \mathbb{N}$, $\pi_f = \{u_0 \leq u_1 \leq \ldots \leq u_{m+n+1}\}$ be a finite partition, and $D_0(\vec{d}_0), \ldots, D_m(\vec{d}_m) \in E^d (d \in \{2,3\})$. Then, the curve

$$\vec{s}(u) = \sum_{i=0}^{m} \vec{d}_i N_i^n(u)\,;\; u \in [u_0, u_{m+n+1})$$

is called B–Spline curve (of degree n with respect to the partition π_f) with the de Boor points or control points $D_0, \ldots, D_m$.
short:

$$\vec{s}(u) = \sum_{i} \vec{d}_i N_i^n(u)\,;\; \pi = \{u_i\}_{i=-\infty}^{\infty}$$

B–Spline curves

Theorem (Properties of a B–Spline curve)

C1) *Affine invariance:*

$$D_0, \ldots, D_m \leftrightarrow \overrightarrow{s}(u) \text{ affinely invariant}$$

B–Spline curves

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B–Spline curves

Theorem (Properties of a B–Spline curve)

C1) *Affine invariance:*

$$D_0, \ldots, D_m \leftrightarrow \vec{s}(u) \text{ affinely invariant}$$

C2) *Moving a control point $D_l \rightarrow$ modification of the curve $\vec{s}(u)$ only for $u \in [u_l, u_{l+n+1})$*

C3)

$$\vec{s}(u)|_{u \in [u_l, u_{l+1})} = \sum_{i=l-n}^{l} \vec{d}_i N_i^n(u)$$

B–Spline curves

Theorem (Properties of a B–Spline curve)

C1) *Affine invariance:*

$$D_0, \dots, D_m \leftrightarrow \vec{s}(u) \text{ affinely invariant}$$

C2) *Moving a control point $D_l \rightarrow$ modification of the curve $\vec{s}(u)$ only for $u \in [u_l, u_{l+n+1})$*

C3)

$$\vec{s}(u)|_{u \in [u_l, u_{l+1})} = \sum_{i=l-n}^{l} \vec{d}_i N_i^n(u)$$

C4) *Convex hull property:*

$$\vec{s}(u)|_{u \in [u_l, u_{l+1})} \subset H(D_{l-n}, \dots, D_l)$$

B–Spline curves

Theorem (Properties of a B–Spline curve, continued)

C5) *Variation Diminishing Property:*
The number of intersection points of the curve $\vec{s}(u)$ with an arbitrary plane is $\leq$ than the number of intersection points of the control polygon with the same plane.

B–Spline curves

Theorem (Properties of a B–Spline curve, continued)

C5) *Variation Diminishing Property:*
The number of intersection points of the curve $\vec{s}(u)$ with an arbitrary plane is $\leq$ than the number of intersection points of the control polygon with the same plane.

C6) *If k control points $D_{l-k} = D_{l-k+1} = \ldots = D_{l-1}$ coincide and if the multiplicity of the knot u_l is between $n-k+1$ and n, then the B–Spline curve $\vec{s}(u)$ contains the point D_{l-1}: $\vec{s}(u_l) = \vec{d}_{l-1}$.*

B–Spline curves

Theorem (Properties of a B–Spline curve, continued)

C5) *Variation Diminishing Property:*
The number of intersection points of the curve $\vec{s}(u)$ with an arbitrary plane is $\leq$ than the number of intersection points of the control polygon with the same plane.

C6) *If k control points $D_{l-k} = D_{l-k+1} = \ldots = D_{l-1}$ coincide and if the multiplicity of the knot u_l is between $n-k+1$ and n, then the B–Spline curve $\vec{s}(u)$ contains the point D_{l-1}: $\vec{s}(u_l) = \vec{d}_{l-1}$.*

C7) *If $n+1$ control points $D_{l-n}, \ldots D_l$ are collinear, then the B–Spline curve $\vec{s}(u)$ contains part of the straight line (different from a point if $u_l \neq u_{l+1}$) that passes through $D_{l-n}, \ldots D_l$.*

B–Spline curves

Theorem (Properties of a B–Spline curve, continued)

C5) *Variation Diminishing Property:*
The number of intersection points of the curve $\vec{s}(u)$ with an arbitrary plane is $\leq$ than the number of intersection points of the control polygon with the same plane.

C6) *If k control points $D_{l-k} = D_{l-k+1} = \ldots = D_{l-1}$ coincide and if the multiplicity of the knot u_l is between $n-k+1$ and n, then the B–Spline curve $\vec{s}(u)$ contains the point D_{l-1}: $\vec{s}(u_l) = \vec{d}_{l-1}$.*

C7) *If $n+1$ control points $D_{l-n}, \ldots D_l$ are collinear, then the B–Spline curve $\vec{s}(u)$ contains part of the straight line (different from a point if $u_l \neq u_{l+1}$) that passes through $D_{l-n}, \ldots D_l$.*

C8) *If n control points $D_{l-n}, \ldots D_{l-1}$ are collinear and the multiplicity of the knot $u_l \leq n-1$, then the straight line that passes through $D_{l-n}, \ldots D_{l-1}$ is tangent to the B–Spline curve $\vec{s}(u)$ in $\vec{s}(u_l)$.*

B–Spline curves

Theorem (Properties of a B–Spline curve, continued)

C9) *First derivative of $\vec{s}(u)$:*

$$\frac{d}{du}\vec{s}(u) = \sum_{i=1}^{m} \vec{d}_i^{(1)} N_i^{n-1}(u)\,, \text{ where } \vec{d}_i^{(1)} = n\frac{\vec{d}_i - \vec{d}_{i-1}}{u_{i+n} - u_i}$$

B–Spline curves

Theorem (Properties of a B–Spline curve, continued)

C9) *First derivative of $\vec{s}(u)$:*

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Démonstration

C1) – C4): using the properties of normalized B–Splines, C5): later

B–Spline curves

Demonstration, continued

C6) $D_{l-k} = \ldots = D_{l-1}$ and $n - k + 1 \leq$ multiplicity of the knot $u_l \leq n$

B–Spline curves

Demonstration, continued

C6) $D_{l-k} = \ldots = D_{l-1}$ and $n - k + 1 \leq$ multiplicity of the knot $u_l \leq n$

$$\Rightarrow \vec{s}(u_l) = \sum_{i=l-n}^{l} \vec{d}_i N_i^n(u_l)$$

B–Spline curves

Demonstration, continued

C6) $D_{l-k} = \ldots = D_{l-1}$ and $n - k + 1 \leq$ multiplicity of the knot $u_l \leq n$

$$\Rightarrow \vec{s}(u_l) = \sum_{i=l-n}^{l} \vec{d}_i N_i^n(u_l) = \sum_{i=l-n}^{l-1} \vec{d}_i N_i^n(u_l)$$

B–Spline curves

Demonstration, continued

C6) $D_{l-k} = \ldots = D_{l-1}$ and $n - k + 1 \leq$ multiplicity of the knot $u_l \leq n$

$$\begin{aligned} \Rightarrow \vec{s}(u_l) &= \sum_{i=l-n}^{l} \vec{d}_i N_i^n(u_l) \\ &= \sum_{i=l-n}^{l-1} \vec{d}_i N_i^n(u_l) \\ &= \sum_{i=l-n}^{l-k-1} \vec{d}_i \underbrace{N_i^n(u_l)} + \sum_{i=l-k}^{l-1} \vec{d}_i N_i^n(u_l) \end{aligned}$$

B–Spline curves

Demonstration, continued

C6) $D_{l-k} = \ldots = D_{l-1}$ and $n-k+1 \leq$ multiplicity of the knot $u_l \leq n$

$$\Rightarrow \vec{s}(u_l) = \sum_{i=l-n}^{l} \vec{d}_i N_i^n(u_l)$$

$$= \sum_{i=l-n}^{l-1} \vec{d}_i N_i^n(u_l)$$

$$= \sum_{i=l-n}^{l-k-1} \vec{d}_i \underbrace{N_i^n(u_l)}_{=0} + \sum_{i=l-k}^{l-1} \vec{d}_i N_i^n(u_l)$$

B–Spline curves

Demonstration, continued

C6) $D_{l-k} = \ldots = D_{l-1}$ and $n - k + 1 \leq$ multiplicity of the knot $u_l \leq n$

$$
\begin{aligned}
\Rightarrow \vec{s}(u_l) &= \sum_{i=l-n}^{l} \vec{d}_i N_i^n(u_l) \\
&= \sum_{i=l-n}^{l-1} \vec{d}_i N_i^n(u_l) \\
&= \sum_{i=l-n}^{l-k-1} \vec{d}_i \underbrace{N_i^n(u_l)}_{=0} + \sum_{i=l-k}^{l-1} \vec{d}_i N_i^n(u_l) \\
&= \vec{d}_{l-1} \underbrace{\sum_{i=l-k}^{l-1} N_i^n(u_l)}
\end{aligned}
$$

B–Spline curves

Demonstration, continued

C6) $D_{l-k} = \ldots = D_{l-1}$ and $n-k+1 \leq$ multiplicity of the knot $u_l \leq n$

$$\begin{aligned}
\Rightarrow \vec{s}(u_l) &= \sum_{i=l-n}^{l} \vec{d}_i N_i^n(u_l) \\
&= \sum_{i=l-n}^{l-1} \vec{d}_i N_i^n(u_l) \\
&= \sum_{i=l-n}^{l-k-1} \vec{d}_i \underbrace{N_i^n(u_l)}_{=0} + \sum_{i=l-k}^{l-1} \vec{d}_i N_i^n(u_l) \\
&= \vec{d}_{l-1} \underbrace{\sum_{i=l-k}^{l-1} N_i^n(u_l)}_{=1}
\end{aligned}$$

B–Spline curves

Demonstration, continued

C6) $D_{l-k} = \ldots = D_{l-1}$ and $n - k + 1 \leq$ multiplicity of the knot $u_l \leq n$

$$
\begin{aligned}
\Rightarrow \vec{s}(u_l) &= \sum_{i=l-n}^{l} \vec{d}_i N_i^n(u_l) \\
&= \sum_{i=l-n}^{l-1} \vec{d}_i N_i^n(u_l) \\
&= \sum_{i=l-n}^{l-k-1} \vec{d}_i \underbrace{N_i^n(u_l)}_{=0} + \sum_{i=l-k}^{l-1} \vec{d}_i N_i^n(u_l) \\
&= \vec{d}_{l-1} \underbrace{\sum_{i=l-k}^{l-1} N_i^n(u_l)}_{=1} = \vec{d}_{l-1}
\end{aligned}
$$

B–Spline curves

Demonstration, continued

B–Spline curves

Demonstration, continued

C7) follows with C4)

B–Spline curves

Demonstration, continued

C7) follows with C4)

C8) $D_{l-n}, \ldots, D_{l-1}$ collinear

B–Spline curves

Demonstration, continued

C7) follows with C4)

C8) $D_{l-n}, \ldots, D_{l-1}$ collinear

$$\Rightarrow \overrightarrow{d}_i = \overrightarrow{d}_{l-1} + t_i(\overrightarrow{d}_{l-n} - \overrightarrow{d}_{l-1}); \; i = l-n, \ldots, l-1 \quad (*)$$

B–Spline curves

Demonstration, continued

C7) follows with C4)

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$$\Rightarrow \vec{d}_i = \vec{d}_{l-1} + t_i(\vec{d}_{l-n} - \vec{d}_{l-1})\,;\; i = l-n, \ldots, l-1 \quad (*)$$

$$\Rightarrow \vec{s}(u_l) = \sum_{i=l-n}^{l} \vec{d}_i N_i^n(u_l)$$

B–Spline curves

Demonstration, continued

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B–Spline curves

Demonstration, continued

C7) follows with C4)

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$$\Rightarrow \vec{d}_i = \vec{d}_{l-1} + t_i(\vec{d}_{l-n} - \vec{d}_{l-1})\,;\; i = l-n, \ldots, l-1 \quad (*)$$

$$\Rightarrow \vec{s}(u_l) = \sum_{i=l-n}^{l} \vec{d}_i N_i^n(u_l) \quad |(*) \Rightarrow$$

$$= \sum_{i=l-n}^{l} (\vec{d}_{l-1} + t_i(\vec{d}_{l-n} - \vec{d}_{l-1}))N_i^n(u_l)$$

B–Spline curves

Demonstration, continued

C7) follows with C4)

C8) $D_{l-n}, \ldots, D_{l-1}$ collinear

$$\Rightarrow \vec{d}_i = \vec{d}_{l-1} + t_i(\vec{d}_{l-n} - \vec{d}_{l-1});\ i = l-n, \ldots, l-1 \quad (*)$$

$$\begin{aligned}
\Rightarrow \vec{s}(u_l) &= \sum_{i=l-n}^{l} \vec{d}_i N_i^n(u_l) \quad |(*) \Rightarrow \\
&= \sum_{i=l-n}^{l} (\vec{d}_{l-1} + t_i(\vec{d}_{l-n} - \vec{d}_{l-1})) N_i^n(u_l) \\
&= \vec{d}_{l-1} \underbrace{\sum_{i=l-n}^{l} N_i^n(u_l)} + (\vec{d}_{l-n} - \vec{d}_{l-1}) \underbrace{\sum_{i=l-n}^{l} t_i N_i^n(u_l)}
\end{aligned}$$

B–Spline curves

Demonstration, continued

C7) follows with C4)

C8) $D_{l-n}, \dots, D_{l-1}$ collinear

$$\Rightarrow \vec{d}_i = \vec{d}_{l-1} + t_i(\vec{d}_{l-n} - \vec{d}_{l-1})\,;\ i = l-n, \dots, l-1 \quad (*)$$

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B–Spline curves

Demonstration, continued

C7) follows with C4)

C8) $D_{l-n}, \ldots, D_{l-1}$ collinear

$$\Rightarrow \vec{d}_i = \vec{d}_{l-1} + t_i(\vec{d}_{l-n} - \vec{d}_{l-1})\,;\ i = l-n, \ldots, l-1 \quad (*)$$

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B–Spline curves

Proof.

C8)

$$\Rightarrow \vec{s}(u_l) = \vec{d}_{l-1} + \lambda(\vec{d}_{l-n} - \vec{d}_{l-1})$$

□

B–Spline curves

Proof.

C8)

$$\Rightarrow \vec{s}(u_l) = \vec{d}_{l-1} + \lambda(\vec{d}_{l-n} - \vec{d}_{l-1})$$

$$\Rightarrow \vec{s}(u_l) \in (D_{l-n} D_{l-1})$$

□

B–Spline curves

Proof.

C8)

$$\Rightarrow \overrightarrow{s}(u_l) \quad = \quad \overrightarrow{d}_{l-1} + \lambda(\overrightarrow{d}_{l-n} - \overrightarrow{d}_{l-1})$$

$\Rightarrow \overrightarrow{s}(u_l) \in (D_{l-n}D_{l-1})$
Furthermore we know:

□

B–Spline curves

Proof.

C8)

$$\Rightarrow \overrightarrow{s}(u_l) \quad = \quad \overrightarrow{d}_{l-1} + \lambda(\overrightarrow{d}_{l-n} - \overrightarrow{d}_{l-1})$$

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Furthermore we know:

- $\overrightarrow{s}(u) \in C^1$ for $u = u_l$

B–Spline curves

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C8)

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$\Rightarrow \vec{s}(u_l) \in (D_{l-n}D_{l-1})$
Furthermore we know:

- $\vec{s}(u) \in C^1$ for $u = u_l$
- C4) $\Rightarrow$

B–Spline curves

Proof.

C8)

$$\Rightarrow \overrightarrow{s}(u_l) = \overrightarrow{d}_{l-1} + \lambda(\overrightarrow{d}_{l-n} - \overrightarrow{d}_{l-1})$$

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Furthermore we know:

- $\overrightarrow{s}(u) \in C^1$ for $u = u_l$
- C4) $\Rightarrow$
  - $\overrightarrow{s}(u)|_{u\in[u_l,u_{l+1})} \subset H(D_{l-n},\ldots,D_l) = H(D_{l-n},D_{l-1},D_l)$

□

B–Spline curves

Proof.

C8)

$$\Rightarrow \vec{s}(u_l) = \vec{d}_{l-1} + \lambda(\vec{d}_{l-n} - \vec{d}_{l-1})$$

$\Rightarrow \vec{s}(u_l) \in (D_{l-n}D_{l-1})$

Furthermore we know:

- $\vec{s}(u) \in C^1$ for $u = u_l$
- C4) $\Rightarrow$
  - $\vec{s}(u)|_{u\in[u_l,u_{l+1})} \subset H(D_{l-n}, \ldots, D_l) = H(D_{l-n}, D_{l-1}, D_l)$
  - $\vec{s}(u)|_{u\in[u_{l-1},u_l)} \subset H(D_{l-n-1}, \ldots, D_{l-1}) = H(D_{l-n-1}, D_{l-n}, D_{l-1})$

B–Spline curves

Proof.

C8)

$$\Rightarrow \vec{s}(u_l) \; = \; \vec{d}_{l-1} + \lambda(\vec{d}_{l-n} - \vec{d}_{l-1})$$

$\Rightarrow \vec{s}(u_l) \in (D_{l-n}D_{l-1})$
Furthermore we know:

- $\vec{s}(u) \in C^1$ for $u = u_l$
- C4) $\Rightarrow$
  - $\vec{s}(u)|_{u\in[u_l,u_{l+1})} \subset H(D_{l-n},\ldots,D_l) = H(D_{l-n},D_{l-1},D_l)$
  - $\vec{s}(u)|_{u\in[u_{l-1},u_l)} \subset H(D_{l-n-1},\ldots,D_{l-1}) = H(D_{l-n-1},D_{l-n},D_{l-1})$

C9) follows with:

$$\frac{d}{du}N_i^n(u) = n(\frac{N_i^{n-1}(u)}{u_{i+n}-u_i} - \frac{N_{i+1}^{n-1}(u)}{u_{i+n+1}-u_{i+1}})$$

Demonstration by recurrence with respect to n □

Curve point computation according to de Boor

given:

$$\vec{s}(u) = \sum_i \vec{d}_i N_i^n(u)\,;\ \pi = \{u_i\}_{i=-\infty}^{\infty}$$

Curve point computation according to de Boor

given:

$$\vec{s}(u) = \sum_i \vec{d}_i N_i^n(u)\,;\; \pi = \{u_i\}_{i=-\infty}^{\infty}$$

we look for:

$$\vec{s}(x)\,;\quad x \in [u_l, u_{l+1})\ \mathit{fixed}$$

Curve point computation according to de Boor, continued

$$\vec{s}(x) = \sum_{i=l-n}^{l} \vec{d}_i N_i^n(x)$$

Curve point computation according to de Boor, continued

$$\vec{s}(x) = \sum_{i=l-n}^{l} \vec{d}_i N_i^n(x)$$

$$= \sum_{i=l-n}^{l} \vec{d}_i \left(\frac{x - u_i}{u_{i+n} - u_i} N_i^{n-1}(x) + \frac{u_{i+n+1} - x}{u_{i+n+1} - u_{i+1}} N_{i+1}^{n-1}(x) \right)$$

Curve point computation according to de Boor, continued

$$
\begin{aligned}
\vec{s}(x) &= \sum_{i=l-n}^{l} \vec{d}_i N_i^n(x) \\
&= \sum_{i=l-n}^{l} \vec{d}_i \left(\frac{x - u_i}{u_{i+n} - u_i} N_i^{n-1}(x) + \frac{u_{i+n+1} - x}{u_{i+n+1} - u_{i+1}} N_{i+1}^{n-1}(x) \right) \\
&= \sum_{i=l-n+1}^{l} \vec{d}_i \frac{x - u_i}{u_{i+n} - u_i} N_i^{n-1}(x)
\end{aligned}
$$

Curve point computation according to de Boor, continued

$$
\begin{aligned}
\vec{s}(x) &= \sum_{i=l-n}^{l} \vec{d}_i N_i^n(x) \\
&= \sum_{i=l-n}^{l} \vec{d}_i \left(\frac{x - u_i}{u_{i+n} - u_i} N_i^{n-1}(x) + \frac{u_{i+n+1} - x}{u_{i+n+1} - u_{i+1}} N_{i+1}^{n-1}(x) \right) \\
&= \sum_{i=l-n+1}^{l} \vec{d}_i \frac{x - u_i}{u_{i+n} - u_i} N_i^{n-1}(x) \\
&\quad + \sum_{i=l-n+1}^{l+1} \vec{d}_{i-1} \frac{u_{i+n} - x}{u_{i+n} - u_i} N_i^{n-1}(x)
\end{aligned}
$$

Curve point computation according to de Boor, continued2

$$= \sum_{i=l-n+1}^{l} \Big(\underbrace{\frac{x-u_i}{u_{i+n}-u_i}}\, \vec{d}_i + \underbrace{\frac{u_{i+n}-x}{u_{i+n}-u_i}}\, \vec{d}_{i-1} \Big) N_i^{n-1}(x)$$

Curve point computation according to de Boor, continued2

$$= \sum_{i=l-n+1}^{l} \Big(\underbrace{\underbrace{\frac{x-u_i}{u_{i+n}-u_i}}_{=:\alpha_i^1} \vec{d}_i + \underbrace{\frac{u_{i+n}-x}{u_{i+n}-u_i}} \vec{d}_{i-1}}\Big) N_i^{n-1}(x)$$

Curve point computation according to de Boor, continued2

$$= \sum_{i=l-n+1}^{l} \underbrace{(\underbrace{\frac{x-u_i}{u_{i+n}-u_i}}_{=:\alpha_i^1} \vec{d}_i + \underbrace{\frac{u_{i+n}-x}{u_{i+n}-u_i}}_{=1-\alpha_i^1} \vec{d}_{i-1})} N_i^{n-1}(x)$$

Curve point computation according to de Boor, continued2

$$= \sum_{i=l-n+1}^{l} \underbrace{\Big(\underbrace{\frac{x-u_i}{u_{i+n}-u_i}}_{=:\alpha_i^1} \vec{d}_i + \underbrace{\frac{u_{i+n}-x}{u_{i+n}-u_i}}_{=1-\alpha_i^1} \vec{d}_{i-1} \Big)}_{=:\vec{d}_i^{\,1}} N_i^{n-1}(x)$$

Curve point computation according to de Boor, continued2

$$= \sum_{i=l-n+1}^{l} \underbrace{\Big(\underbrace{\frac{x-u_i}{u_{i+n}-u_i}}_{=:\alpha_i^1} \vec{d}_i + \underbrace{\frac{u_{i+n}-x}{u_{i+n}-u_i}}_{=1-\alpha_i^1} \vec{d}_{i-1} \Big)}_{=:\vec{d}_i^1} N_i^{n-1}(x)$$

$$= \sum_{i=l-n+1}^{l} \vec{d}_i^1 N_i^{n-1}(x)$$

Curve point computation according to de Boor, continued3

Repeated application of the recurrence formula:

$$\vec{s}(x) = \sum_{i=l-n}^{l} \vec{d}_i^{\,0} N_i^n(x)$$

Curve point computation according to de Boor, continued3

Repeated application of the recurrence formula:

$$\begin{aligned}\vec{s}(x) &= \sum_{i=l-n}^{l} \vec{d}_i^{\,0} N_i^n(x) \\ &= \sum_{i=l-n+1}^{l} \vec{d}_i^{\,1} N_i^{n-1}(x)\end{aligned}$$

Curve point computation according to de Boor, continued3

Repeated application of the recurrence formula:

$$
\begin{aligned}
\vec{s}(x) &= \sum_{i=l-n}^{l} \vec{d}_i^{\,0} N_i^n(x) \\
&= \sum_{i=l-n+1}^{l} \vec{d}_i^{\,1} N_i^{n-1}(x) \\
&= \dots \\
&= \sum_{i=l-n+n}^{l} \vec{d}_i^{\,n} N_i^{n-n}(x)
\end{aligned}
$$

Curve point computation according to de Boor, continued3

Repeated application of the recurrence formula:

$$
\begin{aligned}
\vec{s}(x) &= \sum_{i=l-n}^{l} \vec{d}_i^{\,0} N_i^n(x) \\
&= \sum_{i=l-n+1}^{l} \vec{d}_i^{\,1} N_i^{n-1}(x) \\
&= \dots \\
&= \sum_{i=l-n+n}^{l} \vec{d}_i^{\,n} N_i^{n-n}(x) \\
&= \vec{d}_l^{\,n}
\end{aligned}
$$

Curve point computation according to de Boor, continued4

de Boor algorithm:

$$\alpha_i^k := \frac{x - u_i}{u_{i+n-k+1} - u_i}$$

Curve point computation according to de Boor, continued4

de Boor algorithm:

$$\alpha_i^k := \frac{x - u_i}{u_{i+n-k+1} - u_i}$$

$$\vec{d}_i^{\,k} := \alpha_i^k \vec{d}_i^{\,k-1} + (1 - \alpha_i^k)\vec{d}_{i-1}^{\,k-1}$$

$$i = l - n + k, \ldots, l, \; k = 1, \ldots, n.$$

Curve point computation according to de Boor, continued4

de Boor algorithm:

$$\alpha_i^k := \frac{x - u_i}{u_{i+n-k+1} - u_i}$$

$$\vec{d}_i^{\,k} := \alpha_i^k \vec{d}_i^{\,k-1} + (1 - \alpha_i^k)\vec{d}_{i-1}^{\,k-1}$$

$$i = l - n + k, \ldots, l,\ k = 1, \ldots, n.$$

Scheme and Illustration

Curve point computation according to de Boor, continued5

Remarks:

1. Property C9) of B–Spline curves $\Rightarrow \overrightarrow{d}_l^{n-1} - \overrightarrow{d}_{l-1}^{n-1}$ yields the tangent direction in $\overrightarrow{d}_l^{n}$ (curve point!)

Curve point computation according to de Boor, continued5

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1. Property C9) of B–Spline curves $\Rightarrow \overrightarrow{d}_l^{n-1} - \overrightarrow{d}_{l-1}^{n-1}$ yields the tangent direction in $\overrightarrow{d}_l^n$ (curve point!)
2. Derivative computation:
   By applying de Boor's algorithm to the B–Spline representation of the derivative.

Increasing flexibility by inserting knots

given:

$$\vec{s}(u) = \sum_i \vec{d}_i N_i^n(u) \; ; \; \pi = \{u_i\}_{i=-\infty}^{\infty}$$

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we look for:

$$\vec{s}(u) = \sum_i \vec{d}_i^{(1)} \hat{N}_i^n(u) \ \ sur \ \ \pi^1$$

Increasing flexibility by inserting knots, continued

Determination of the new de Boor points $D_i^1(\overrightarrow{d}_i^1)$:

Increasing flexibility by inserting knots, continued

Determination of the new de Boor points $D_i^1(\vec{d}_i^1)$:
Property N3) of normalized B–Splines $\Rightarrow$

$$N_i^n(u) = \left\{ \right.$$

Increasing flexibility by inserting knots, continued

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Increasing flexibility by inserting knots, continued

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Increasing flexibility by inserting knots, continued

Theorem

for $l - n \leq i \leq l$ we have:

$$N_i^n(u) = \beta_i^1 \hat{N}_i^n(u) + (1 - \beta_{i+1}^1)\hat{N}_{i+1}^n(u)$$

where $\beta_j^1 := \dfrac{u_{l+1}^1 - u_j^1}{u_{j+n+1}^1 - u_j^1}$.

Increasing flexibility by inserting knots, continued

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by recurrence with respect to n by using the recurrence formula for normalized B–Splines □

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for $l - n \leq i \leq l$ we have:

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Proof.

by recurrence with respect to n by using the recurrence formula for normalized B–Splines □

We have:

$$\beta_{l-n}^1 = 1\,,\ \beta_{l+1}^1 = 0$$

Increasing flexibility by inserting knots, continued

$$\vec{s}(u) = \sum_i \vec{d}_i N_i^n(u)$$

Increasing flexibility by inserting knots, continued

$$\vec{s}(u) = \sum_i \vec{d}_i N_i^n(u)$$

$$= \sum_{i \le l-n-1} \vec{d}_i \hat{N}_i^n(u) + \sum_{i=l-n}^{l} \vec{d}_i (\beta_i^1 \hat{N}_i^n(u) + (1-\beta_{i+1}^1)\hat{N}_{i+1}^n(u))$$

$$+ \sum_{i \ge l+1} \vec{d}_i \hat{N}_{i+1}^n(u)$$

Increasing flexibility by inserting knots, continued

$$\begin{aligned}
\vec{s}(u) &= \sum_i \vec{d}_i N_i^n(u) \\
&= \sum_{i \le l-n-1} \vec{d}_i \hat{N}_i^n(u) + \sum_{i=l-n}^{l} \vec{d}_i \left(\beta_i^1 \hat{N}_i^n(u) + (1-\beta_{i+1}^1)\hat{N}_{i+1}^n(u)\right) \\
&\quad + \sum_{i \ge l+1} \vec{d}_i \hat{N}_{i+1}^n(u) \\
&= \sum_{i \le l-n} \vec{d}_i \hat{N}_i^n(u) + \sum_{i=l-n+1}^{l} \vec{d}_i \beta_i^1 \hat{N}_i^n(u) \\
&\quad + \sum_{i=l-n+1}^{l} \vec{d}_{i-1}(1-\beta_i^1)\hat{N}_i^n(u)) + \sum_{i \ge l+1} \vec{d}_{i-1} \hat{N}_i^n(u)
\end{aligned}$$

Increasing flexibility by inserting knots, continued

$$= \sum_{i \le l-n} \vec{d}_i \hat{N}_i^n(u) + \sum_{i=l-n+1}^{l} (\beta_i^1 \vec{d}_i + (1-\beta_i^1)\vec{d}_{i-1}) \hat{N}_i^n(u)$$
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$$
\begin{aligned}
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&=: \sum_i \vec{d}_i^{\,1} \hat{N}_i^n(u)
\end{aligned}
$$

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$$= \sum_{i \le l-n} \vec{d}_i \hat{N}_i^n(u) + \sum_{i=l-n+1}^{l} (\beta_i^1 \vec{d}_i + (1-\beta_i^1)\vec{d}_{i-1}) \hat{N}_i^n(u)$$
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where

$$\vec{d}_i^{\,1} = \beta_i^1 \vec{d}_i + (1-\beta_i^1)\vec{d}_{i-1}$$

Increasing flexibility by inserting knots, continued

with

$$\beta_i^1 := \begin{cases} 1 & , i \leq l-n \\ \dfrac{u_{l+1}^1 - u_i^1}{u_{i+n+1}^1 - u_i^1} & , i = l-n+1,\ldots,l \\ 0 & , i \geq l+1 \end{cases}$$

Increasing flexibility by inserting knots, continued

with

$$\beta_i^1 := \begin{cases} 1 & , i \leq l-n \\ \dfrac{u_{l+1}^1 - u_i^1}{u_{i+n+1}^1 - u_i^1} = \dfrac{x - u_i}{u_{i+n} - u_i} & , i = l-n+1, \ldots, l \\ 0 & , i \geq l+1 \end{cases}$$

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Scheme and Example

Increasing flexibility by inserting knots, continued

Remarks:

1) Successive refinement of the partition π, i.e., successive knot insertion $\Rightarrow$ the successive control polygons converge towards the curve.

Increasing flexibility by inserting knots, continued

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2) In practice often:

$$\vec{s}(u) = \sum_{i=0}^{m} \vec{d}_i N_i^n(u)\,; \quad u \in [u_0, u_{m+n+1})\,; \quad \pi = \{u_i\}_{i=0}^{m+n+1}$$

Increasing flexibility by inserting knots, continued

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1) Successive refinement of the partition π, i.e., successive knot insertion $\Rightarrow$ the successive control polygons converge towards the curve.
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where

- either $u_1 = \ldots = u_n$ and $u_{m+1} = \ldots = u_{m+n}$
- or $u_0 = u_1 = \ldots = u_n$ and $u_{m+1} = \ldots = u_{m+n} = u_{m+n+1}$

Increasing flexibility by inserting knots, continued

Remarks, continued:

Consequences of this choice of knots:

Increasing flexibility by inserting knots, continued

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i) Property C6) of the theorem on the properties of B–Spline curves $\Rightarrow \overrightarrow{s}(u_1) =$, $\overrightarrow{s}(u_{m+1}) =$

Increasing flexibility by inserting knots, continued

Remarks, continued:

Consequences of this choice of knots:

i) Property C6) of the theorem on the properties of B–Spline curves $\Rightarrow \vec{s}(u_1) = \vec{d}_0$, $\vec{s}(u_{m+1}) = \vec{d}_m$

Increasing flexibility by inserting knots, continued

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i) Property C6) of the theorem on the properties of B–Spline curves $\Rightarrow \vec{s}(u_1) = \vec{d}_0,\ \vec{s}(u_{m+1}) = \vec{d}_m$

ii)

$$N_0^n(u)|_{u\in[u_n,u_{n+1})} =$$

$$N_m^n(u)|_{u\in[u_m,u_{m+1})} =$$

Increasing flexibility by inserting knots, continued

Remarks, continued:

Consequences of this choice of knots:

i) Property C6) of the theorem on the properties of B–Spline curves $\Rightarrow \vec{s}(u_1) = \vec{d}_0, \ \vec{s}(u_{m+1}) = \vec{d}_m$

ii)

$$N_0^n(u)|_{u \in [u_n, u_{n+1})} = B_0^n\left(\frac{u - u_n}{u_{n+1} - u_n}\right)$$

$$N_m^n(u)|_{u \in [u_m, u_{m+1})} = B_n^n\left(\frac{u - u_m}{u_{m+1} - u_m}\right)$$

Increasing flexibility by inserting knots, continued

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iii) $(D_0 D_1) \ldots$
$(D_{m-1} D_m) \ldots$

Increasing flexibility by inserting knots, continued

Remarks, continued:

Consequences of this choice of knots:

i) Property C6) of the theorem on the properties of B–Spline curves $\Rightarrow \vec{s}(u_1) = \vec{d}_0$, $\vec{s}(u_{m+1}) = \vec{d}_m$

ii)

$$\begin{aligned} N_0^n(u)|_{u\in[u_n,u_{n+1})} &= B_0^n\left(\frac{u-u_n}{u_{n+1}-u_n}\right) \\ N_m^n(u)|_{u\in[u_m,u_{m+1})} &= B_n^n\left(\frac{u-u_m}{u_{m+1}-u_m}\right) \end{aligned}$$

iii) $(D_0 D_1)\ldots$ tangent of $\vec{s}(u)$ in D_0
$(D_{m-1} D_m)\ldots$

Increasing flexibility by inserting knots, continued

Remarks, continued:

Consequences of this choice of knots:

i) Property C6) of the theorem on the properties of B–Spline curves $\Rightarrow \vec{s}(u_1) = \vec{d}_0,\ \vec{s}(u_{m+1}) = \vec{d}_m$

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$$
\begin{aligned}
N_0^n(u)|_{u\in[u_n,u_{n+1})} &= B_0^n\left(\frac{u-u_n}{u_{n+1}-u_n}\right) \\
N_m^n(u)|_{u\in[u_m,u_{m+1})} &= B_n^n\left(\frac{u-u_m}{u_{m+1}-u_m}\right)
\end{aligned}
$$

iii) $(D_0 D_1) \ldots$ tangent of $\vec{s}(u)$ in D_0
$(D_{m-1} D_m) \ldots$ tangent of $\vec{s}(u)$ in D_m

Increasing flexibility by inserting knots, continued

Remarks, continued:

3)

$$\vec{s}(u) = \sum_i \vec{d}_i N_i^n(u)\,; \quad \pi = \{u_i\}_{i=-\infty}^{\infty}$$

Increasing flexibility by inserting knots, continued

Remarks, continued:

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$$\vec{s}(u) = \sum_i \vec{d}_i N_i^n(u) \, ; \quad \pi = \{u_i\}_{i=-\infty}^{\infty}$$

Increasing the multiplicities of the knots u_i to n has the following consequences:

Increasing flexibility by inserting knots, continued

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Increasing the multiplicities of the knots u_i to n has the following consequences:

- de Boor points = Bézier points

Increasing flexibility by inserting knots, continued

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Increasing the multiplicities of the knots u_i to n has the following consequences:

- de Boor points = Bézier points
- B–Spline curve = Bézier curve (composition of Bézier segments)

Increasing flexibility by inserting knots, continued

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Increasing the multiplicities of the knots u_i to n has the following consequences:

- de Boor points = Bézier points
- B–Spline curve = Bézier curve (composition of Bézier segments)
- de Boor algorithm = de Casteljau algorithm

Increasing flexibility by inserting knots, continued

Remarks, continued:

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Increasing the multiplicities of the knots u_i to n has the following consequences:

- de Boor points = Bézier points
- B–Spline curve = Bézier curve (composition of Bézier segments)
- de Boor algorithm = de Casteljau algorithm
- variation diminishing property for B–Spline curves (property C5) of the theorem on the properties of B–Spline curves)