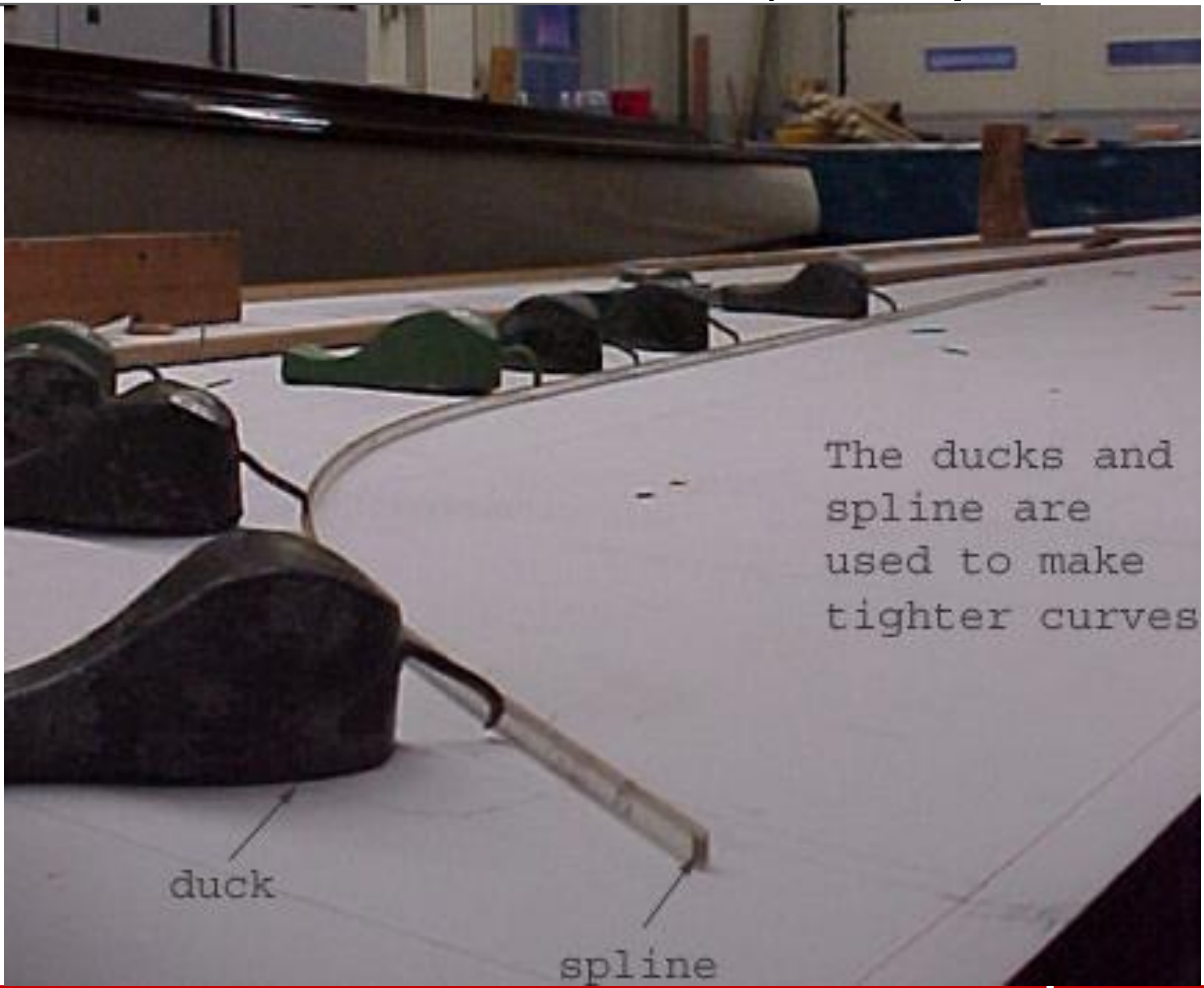
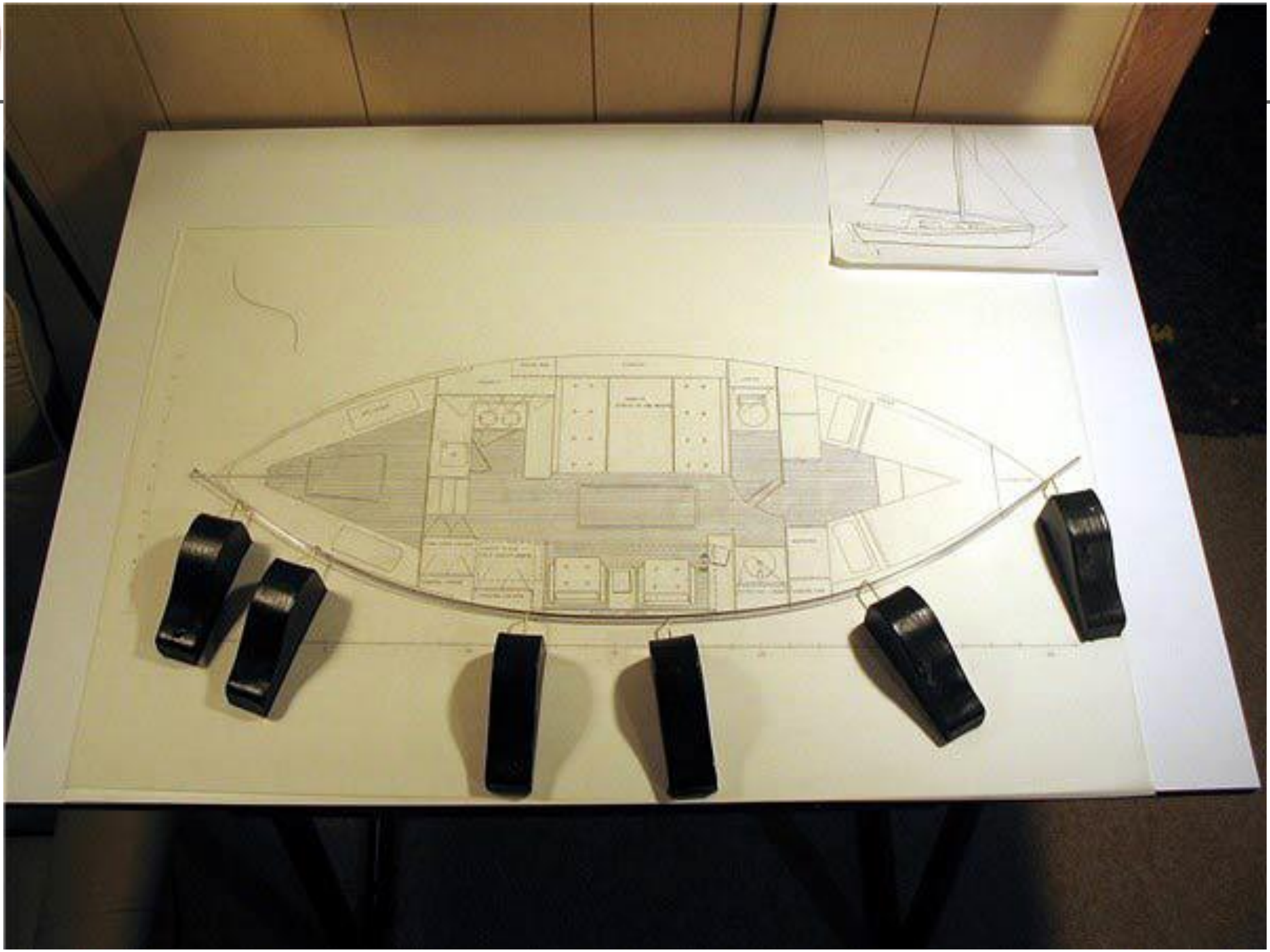
Geometric Modeling: curves

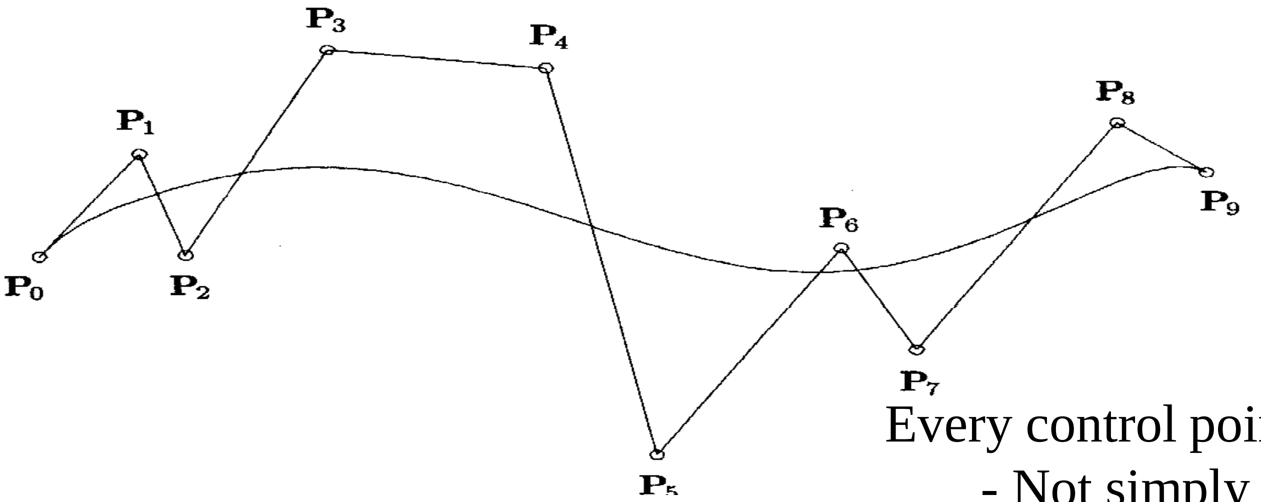
- Bézier Curves
- Spline Curves
- NURBS
- Subdivision curve

Courtesy of The Antique Boat Museum.

The term ‘spline’ comes from the shipbuilding industry: long, thin strips of wood or metal would be bent and held in place by heavy ‘ducks’, lead weights which acted as control points of the curve.



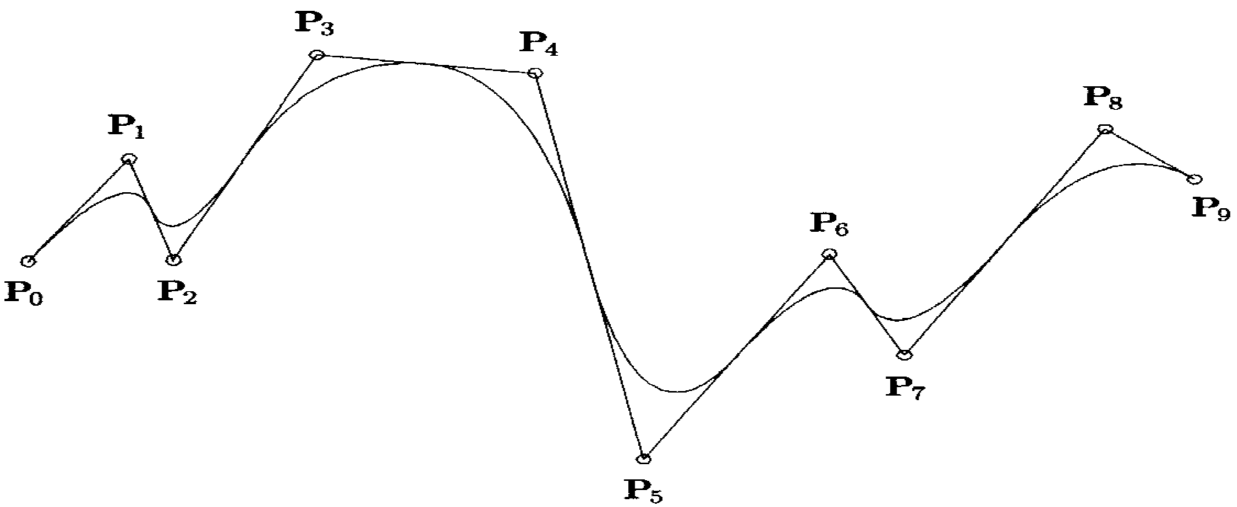




Every control point affects the entire curve

- Not simply a local effect
- More difficult to control for modeling

Bézier curve of degree 9



Spline curve of degree 2

Bézier Curve vs spline Curve

A Bézier curve of degree **n** in parametric form is defined by **ncp=n+1** control points $\mathbf{P_i}=(x_i, \mathbf{y_i})$ i=0,..,n in $\mathbf{R}^2$

$$C(t)=\sum_{i=0}^{n} P_i B_{i,n}(t)=\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \sum_{i=0}^{n} x_i B_{i,n}(t) \\ \sum_{i=0}^{n} y_i B_{i,n}(t) \end{pmatrix}$$

Polynomial of degree n

A spline curve of degree **n** in parametric form is defined by **ncp** control points $\mathbf{P_i}=(x_i, \mathbf{y_i})$ i=1,..,ncp in $\mathbf{R}^2$

$$C(t)=\sum_{i=1}^{ncp} P_i N_{i,n}(t)=\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \sum_{i=1}^{ncp} x_i N_{i,n}(t) \\ \sum_{i=1}^{ncp} y_i N_{i,n}(t) \end{pmatrix}$$

Piecewise Polynomials (spline) of degree n

CP, knot partition, Multeplicity Vector, Bases

Partition

Let [a,b] be a close and limited interval and $\Delta = \{x_i\}_{i=1,..,k}$ be a set of points (knots) such that

$$\mathbf{a = x_0 < x_1 < \ldots\ldots < x_k < x_{k+1} = b}$$

then Δ is a partition of [a,b] and it defines the subintervals

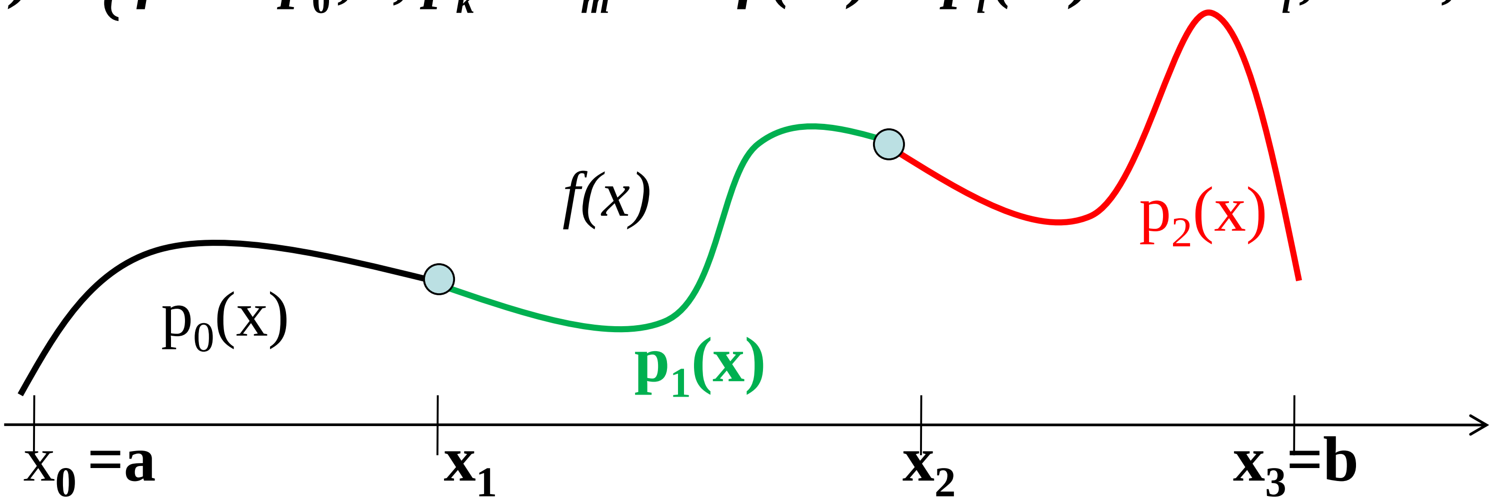
$$I_i=[x_i,x_{i+1}) \qquad I_k=[x_k,x_{k+1}]$$

The points x_i are called **KNOT (NODI)**

Piecewise polynomials
(polinomi a tratti)

Space of piecewise polynomials

$$PP_m(\Delta) = \{ f \,/\, \exists\, p_0,..,p_k \in P_m \text{ s.t. } f(x) \equiv p_i(x)\ \forall x \in I_i, i = 0,..,k \}$$



f(x) is defined in [a,b] and consists into 3 polynomial pieces of degree n (order m=n+1)

The pieces are joint with continuity C^0

Spline Functions s(x)
(Schoenberg 1946)

Given a partition Δ

an integer **m** – order - (degree n, m=n+1)

a vector **M** of knot multiplicities

$M=(m_1,m_2,..,m_k)$, $m_i<=m$

we can define a **space of spline functions**

$$S(P_m, M, \Delta) = \{ s(x) \,/\, \exists s_0(x),..,s_k(x) \in P_m \ s.t.$$

$$1. \quad s(x) \equiv s_j(x), \quad x \in I_j, \quad j = 0,..,k,$$

$$2. \quad continuity: D^{(l)} s_{j-1}(x_j) \equiv D^{(l)} s_j(x_j),$$

$$l = 0,..,m - m_j - 1 \ \}$$

Space dimension **m+K**,

$$K = \sum_{i=1}^{k} m_i$$

Special cases

$$M = (m, m, .., m) \quad \Rightarrow S(PP_m, M, \Delta)$$

Continuity C^0 at knots
S is the piecewise polynomial space PP_m

$$M = (1, 1, ..., 1) \quad \Rightarrow S(P_m, M, \Delta)$$

Max Continuity C^{m-1-1} at knots

Extended Partition

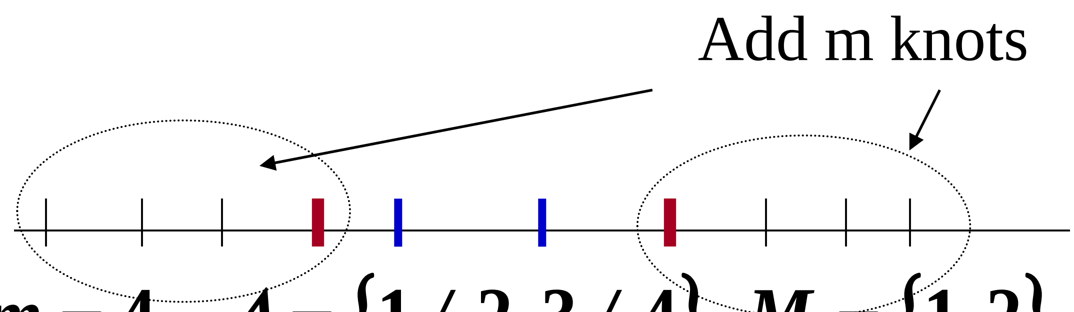
(Partizione estesa)

Starting from partition **$\Delta = \{x_i\}_{i=1,..,k}$**, the extended partition **$\Delta^* = \{t_i\}_{i=1,\dots,2m+K}$** is given by

$$1)\quad t_1 \le t_2 \le \dots \le t_{2m+K}$$

$$2)\quad t_m \equiv a \quad t_{m+K+1} \equiv b$$

$$3)\quad t_{m+1} < \dots < t_{m+K} \equiv (\underbrace{x_1 = \dots = x_1}_{m_1\ times} < \dots < \underbrace{x_k = \dots = x_k}_{m_k\ times})$$



Example

$$[a,b] = [0,1] \quad m = 4 \quad \Delta = \{1/2, 3/4\}, M = \{1,2\}$$

$$\Delta^* = \left\{-3/4, -1/2, -1/4, 0, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, 1, 5/4, 3/2, 7/4\right\}$$

Extended Partition

Open (o nonperiodico)

First and last knots have multiplicity n+1

Es: n=2 $\Delta^*=[0,0,0,1/2,1,1,1]$

Uniform

Equispaced internal knots

Non uniform

otherwise

Spline Functions

Each spline function **s(t)** of degree **n** (m=n+1) can be represented as a linear combination of Basis Functions of the spline space $S(P_m, M, \Delta)$

$$s(t) = \sum_{i=1}^{m+K} c_i N_{i,n}(t)$$

where $\left\{ N_{i,n}(t) \right\}_{i=1,..,m+K}$

is a set of normalized basis functions for $S(P_m, M, \Delta)$

B-Spline Bases

Recurrence FORMULA by Cox-de Boor

$$\left\{N_{i,n}(t)\right\}_{i=1}^{m+K}$$ Normalized B-spline functions

$$N_{i,n}(t)=\frac{t-t_i}{t_{i+n}-t_i}N_{i,n-1}(t)+\frac{t_{i+n+1}-t}{t_{i+n+1}-t_{i+1}}N_{i+1,n-1}(t)$$

$$N_{i,0}(t)=\begin{cases}1, & t_i\le t<t_{i+1}\\ 0, & altrimenti\end{cases}$$

Conventionally 0/0=0

Example:
B-spline Bases,n=1

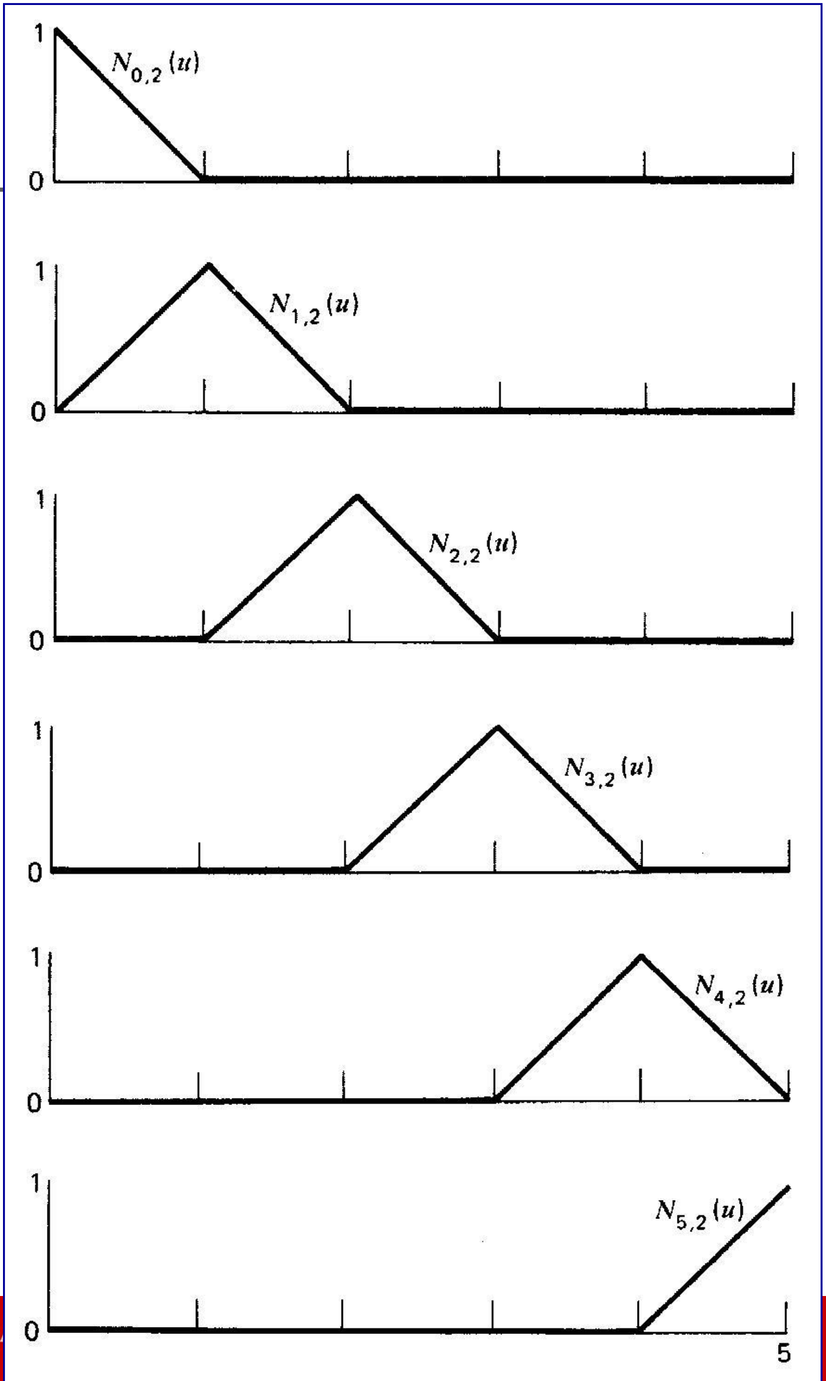
Number of knots:

N=Internal Knots K+

External Knots 2*m =8

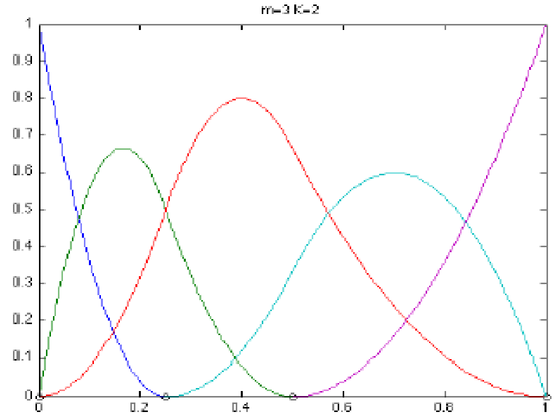
Nodal vector:

Δ *=(0,0,1,2,3,4,5,5)

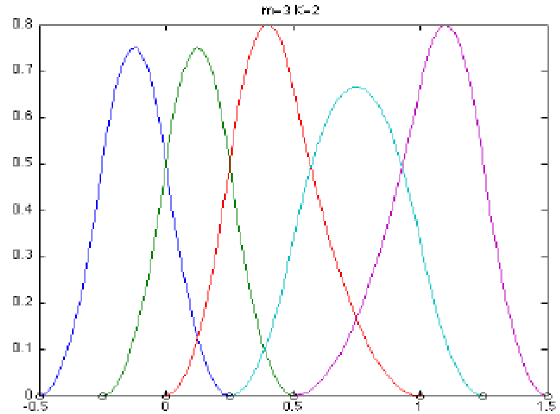


Example:
B-spline Bases, degree 2

n = 2 $\Delta^* = \{0\ 0\ 0\ 0.25\ 0.5\ 1\ 1\ 1\}$

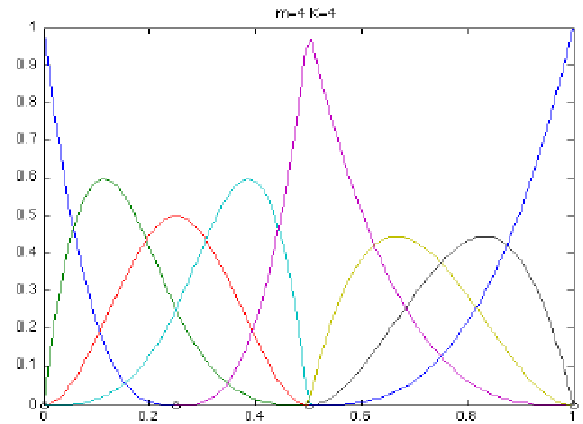


n = 2 $\Delta^* = \{-0.5\ \ -0.25\ 0\ 0.25\ 0.5\ 1\ 1.25\ 1.5\}$

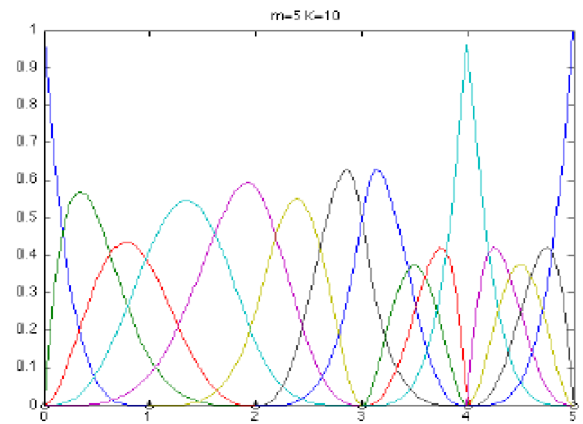


Example:
B-spline Bases, degree 3,4

n = 3 $\Delta^* = \{0\ 0\ 0\ 0\ 0.25\ 0.5\ 0.5\ 0.5\ 1\ 1\ 1\ 1\}$



n = 4 $\Delta^* = \{0\ 0\ 0\ 0\ 0\ 1\ 2\ 2\ 3\ 3\ 3\ 4\ 4\ 4\ 4\ 5\ 5\ 5\ 5\ 5\}$



Example: B-Spline Bases, n=2

$$\left\{N_{i,2}(t)\right\}_{i=1}^{3+K}$$

$$N_{i,2}(t) = \frac{t - t_i}{t_{i+2} - t_i} N_{i,1}(t) + \frac{t_{i+3} - t}{t_{i+3} - t_{i+1}} N_{i+1,1}(t)$$

$$N_{i,0}(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & altrimenti \end{cases}$$

Compute the triangular scheme

$$N_{i,2}(t)$$

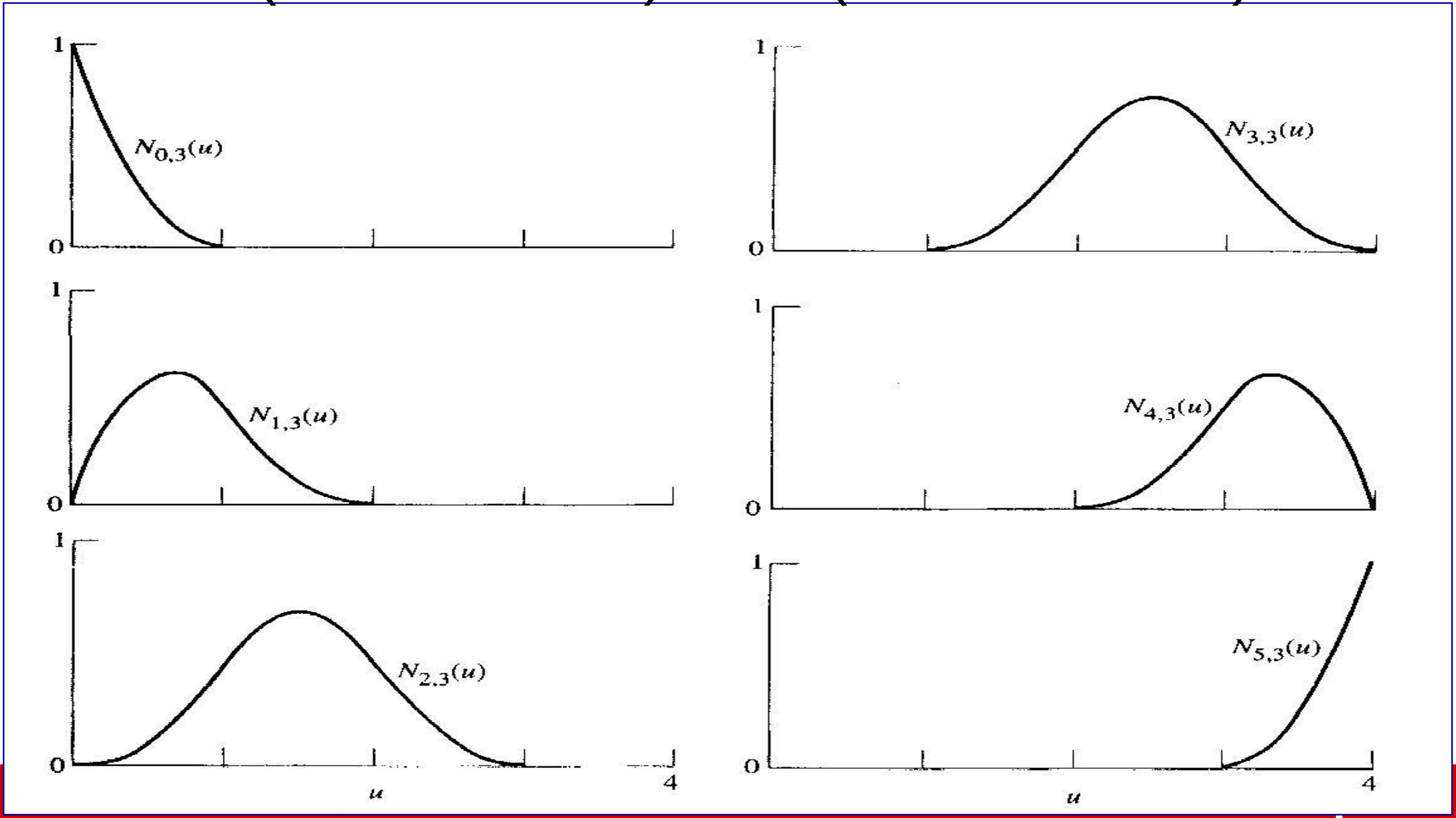
$$N_{i,1}(t) \quad N_{i+1,1}(t)$$

$$N_{i,0}(t) \quad N_{i+1,0}(t) \quad N_{i+2,0}(t)$$

Example: B-Spline Bases, n=2

Knot Vector: Δ^* =(0,0,0,**1,2,3**,4,4,4)

K (internal knots)+2*m (external knots)



B-Spline Bases: properties

1. Local Support

$$N_{i,n}(t) = 0 \quad \forall t \notin \left[t_i, t_{i+n+1}\right) \quad se\ t_i < t_{i+n+1}$$

2. Non-negativity

$$N_{i,n}(t) > 0 \quad \forall t \in (t_i, t_{i+n+1}) \quad t_i < t_{i+n+1}$$

3. Partition of Unity

$$\sum_{i=1}^{m+K} N_{i,n}(t) = 1 \quad \forall t \in [a,b]$$

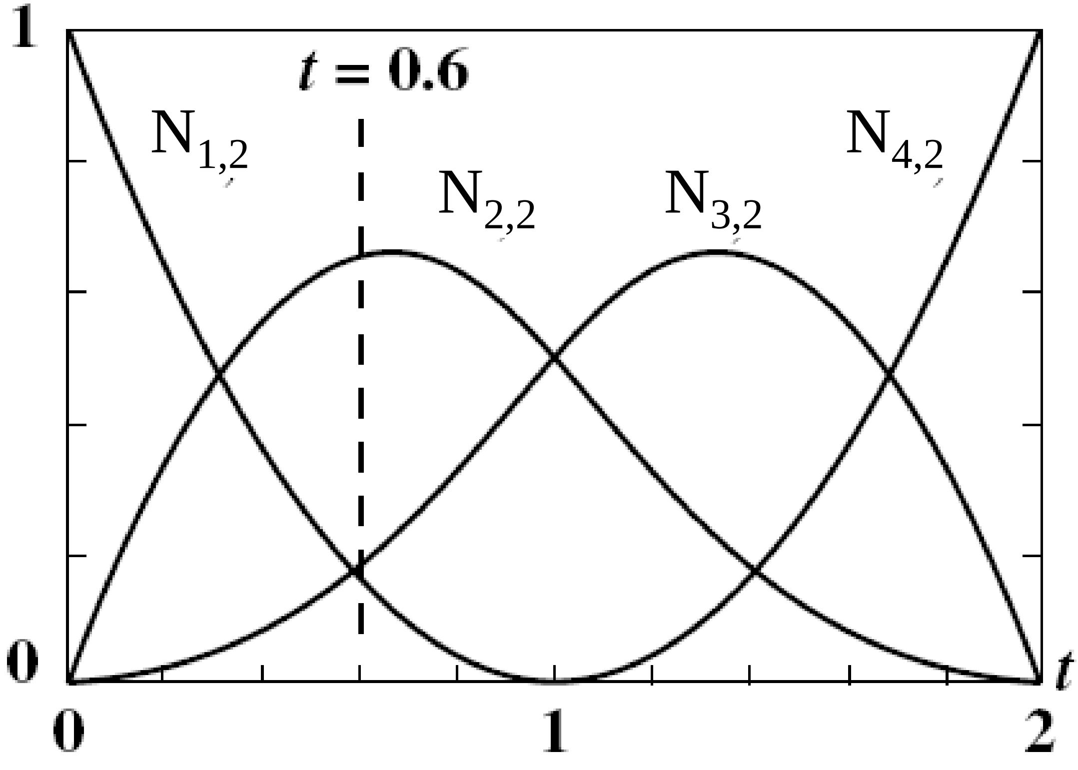
Partition of Unity

Example:

Degree n=2, Knot vector: Δ *=(0,0,**0**, **1,2**,2,2)

$$t = 0.6$$

$$N_{1,2} + N_{2,2} + N_{3,2} + N_{4,2} = 0.16 + 0.66 + 0.18 + 0.0 = 1.0$$



Spline Functions

Each spline function **s(t)** of degree **n** (m=n+1) can be represented as

$$(*) \quad s(t) = \sum_{i=1}^{m+K} c_i N_{i,n}(t)$$

For the local support property of $N_{i,n}$ (*) is reduced to:

$$s(t) = \sum_{i=\ell-n}^{\ell} c_i N_{i,n}(t), \qquad t \in [t_\ell, t_{\ell+1})$$

For each interval, s(t) is the sum of m B-spline Basis functions at most. Therefore s(t) has a local behaviour: by changing an arbitrary c_i the shape of s(t) changes only in m (=n+1) intervals.

Spline Curves

A parametric curve C(t) in $\mathbf{R}^2$ of degree **n** (m=n+1) in parametric form is represented as

$$C(t)=\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad t \in [a,b]$$

If the coordinate functions x(t) and y(t) are spline functions then the curve is a spline curve

$$C(t)=\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \sum_{i=1}^{ncp} x_i N_{i,n}(t) \\ \sum_{i=1}^{ncp} y_i N_{i,n}(t) \end{pmatrix}=\sum_{i=1}^{ncp} P_i N_{i,n}(t)$$

Control points $\mathbf{P_i}$ =(x_i, $\mathbf{y_i}$) i=1,..,ncp in $\mathbf{R}^2$ **ncp=K+m**

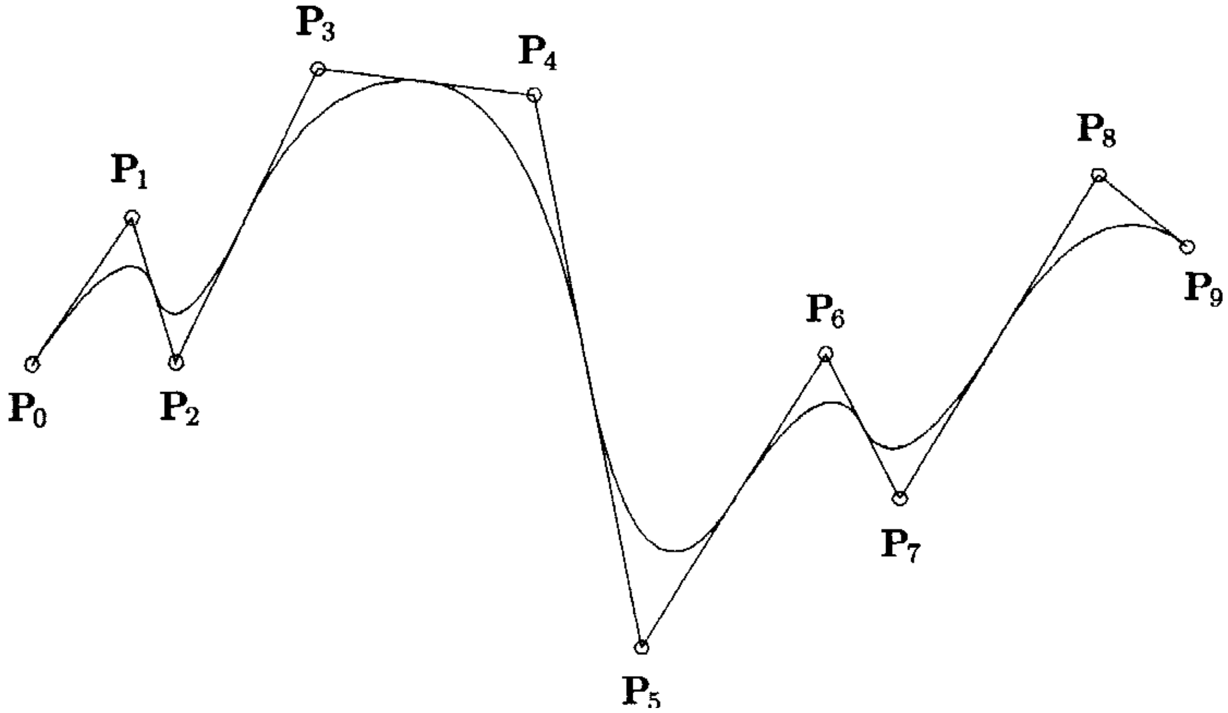
C(t) is C^{m-mj-1} continuous in knots with multiplicity m_j

Given the order **m**, and the number of control points **ncp**,

the total number of knots is **ncp+m**

the number of internal knots is **K=ncp-m**

Example : Spline Curve m=3



$\Delta^*=\{0,0,0,1/8,2/8,3/8,4/8,5/8,6/8,7/8,1,1,1\}$

How to control the shape of a spline curve

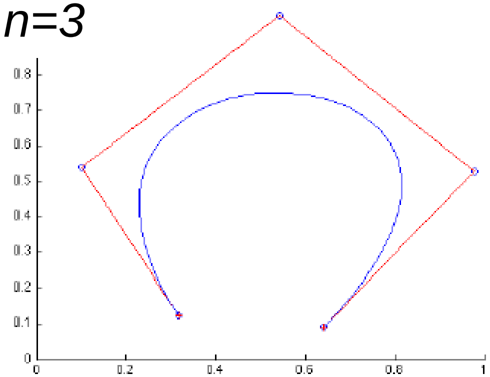
- Change of the degree
- Change of the number/position of the control points CP
- Use multiple coincident CP
- Use internal multiple knots:

  If a knot has multiplicity n, then the curve passes through the corresponding control point
- Modify the knot vector
  - Uniform
  - Nonuniform

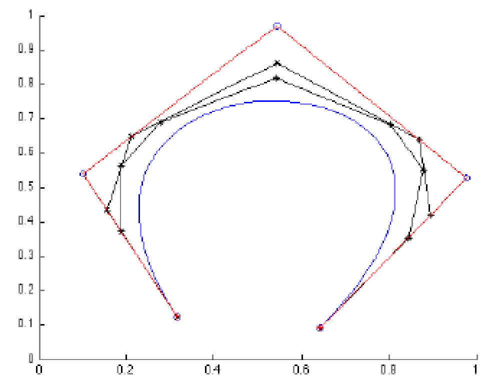
Degree elevation

$k = 1, \quad \Delta^* = \{0\ 0\ 0\ 0\ 0.5\ 1\ 1\ 1\ 1\}$

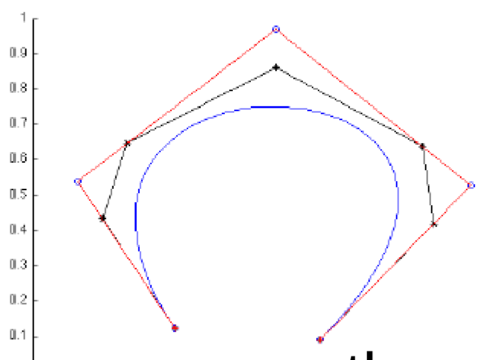
n=3



n=5 $\quad \Delta^* = \{0\ 0\ 0\ 0\ 0\ 0\ 0.5\ 0.5\ 0.5\ 1\ 1\ 1\ 1\ 1\ 1\}$

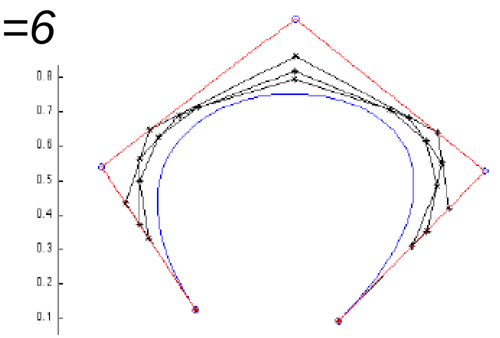


n=4 $\quad \Delta^* = \{0\ 0\ 0\ 0\ 0\ 0.5\ 0.5\ 1\ 1\ 1\ 1\ 1\}$



$\Delta^* = \{0\ 0\ 0\ 0\ 0\ 0\ 0\ 0.5\ 0.5\ 0.5\ 0.5\ 1\ 1\ 1\ 1\ 1\ 1\ 1\}$

n=6



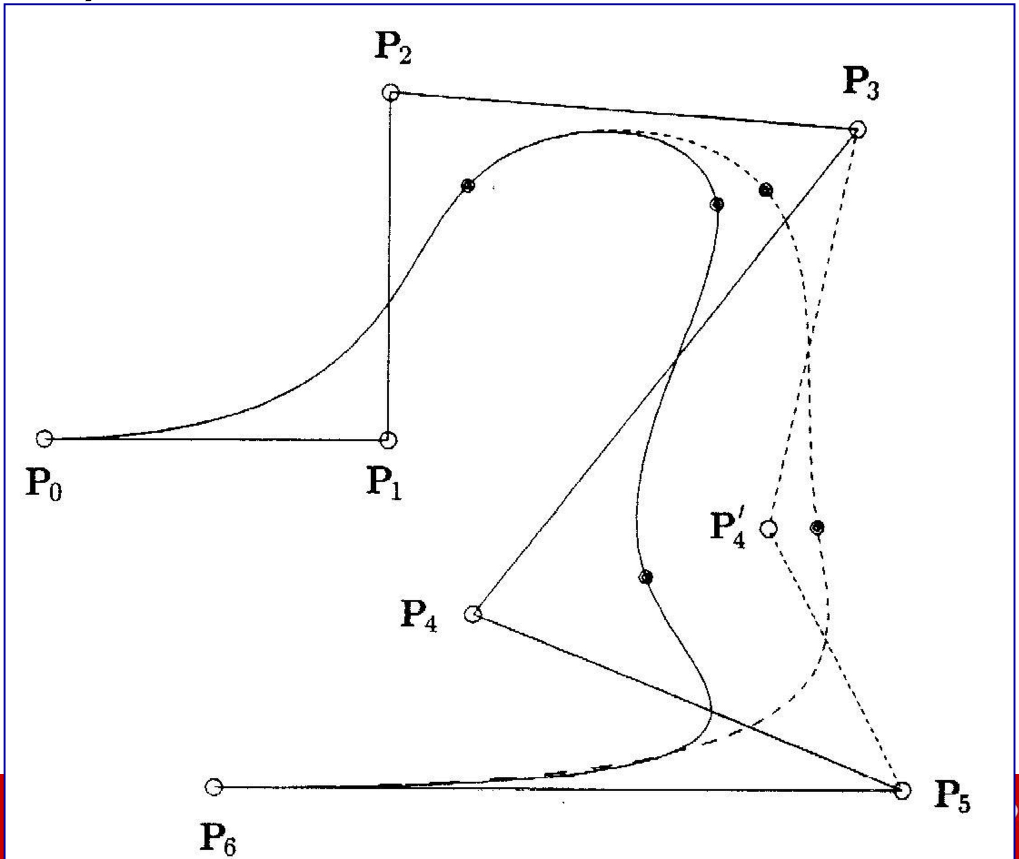
the curve shape is unchanged

Move control points

Example cubic spline (n=3)

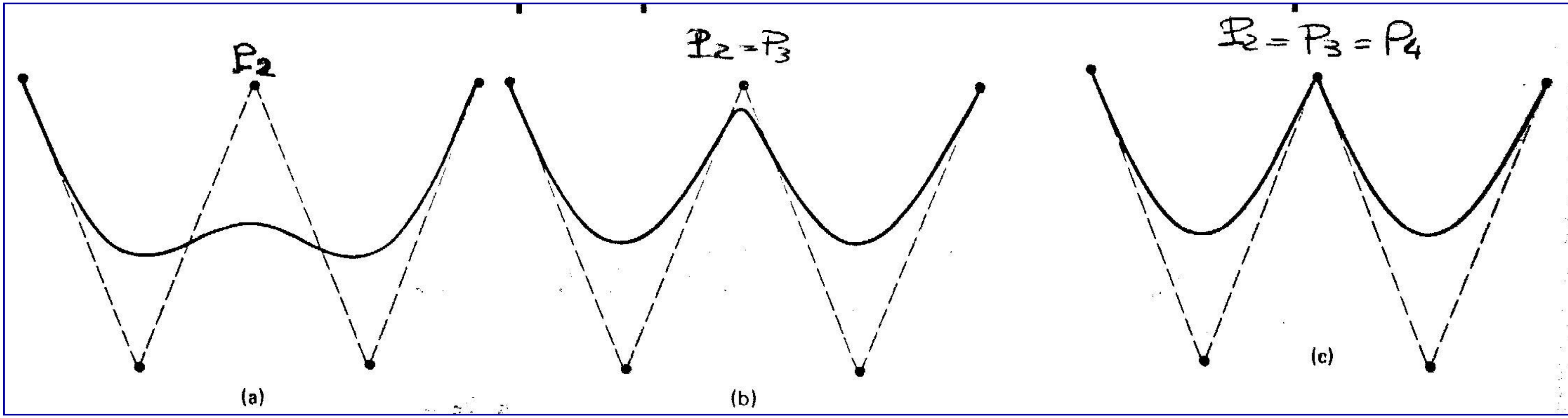
Nodal vector: $\Delta^*=\{0,0,0,0,\mathbf{1/4,1/2,3/4},1,1,1,1\}$

Move control point P_4 in P_4', the curve changes in the interval [1/4,1)



Multiple control points

Example cubic spline (m=4)



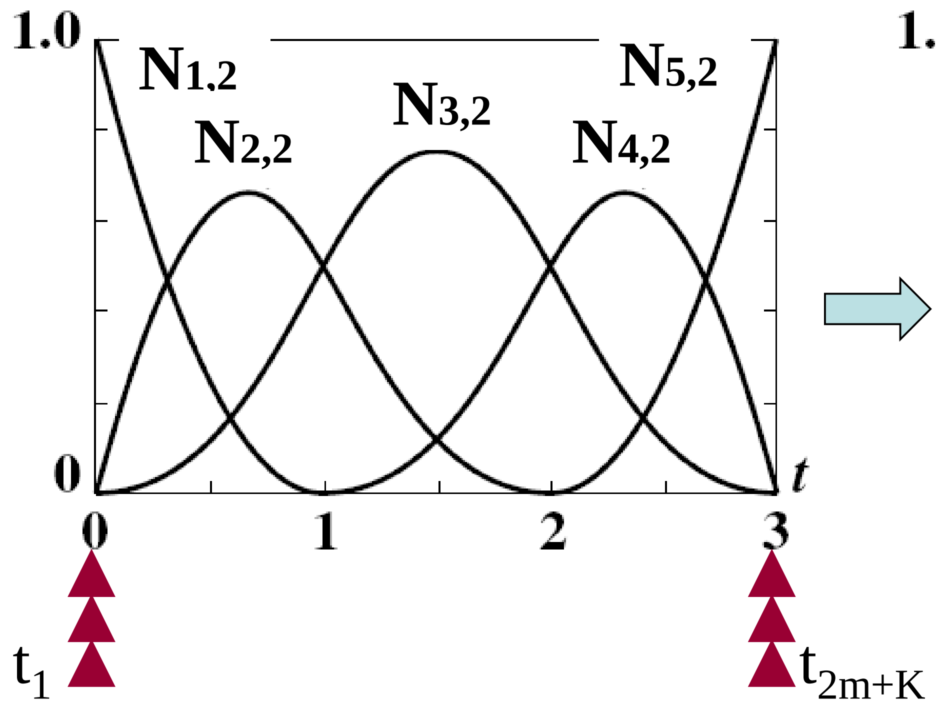
- m-1 coincident vertices are required for the curve to pass through the vertices
- The curve smoothly transitions through the coincident vertices with C^{m-mi-1} continuity

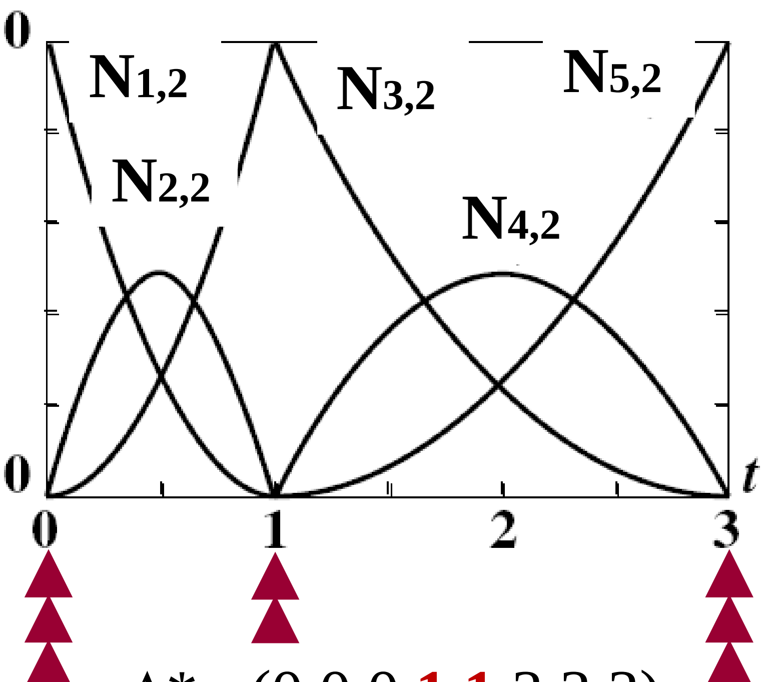
Multiple knots

$$C(t) = \sum_{i=2}^{4} P_i N_{i,2}(t)$$

Example: degree n=2, knot vector with internal multiple knots

In **t =1** we have $N_{2,2}=0$, $N_{3,2}=1$, $N_{4,2}=0$, $C(1)=P_3$, the curve interpolates P_3, the continuity is reduced C^{3-2-1}





$\Delta^* = (0,0,0,$**1,2,**$3,3,3)$

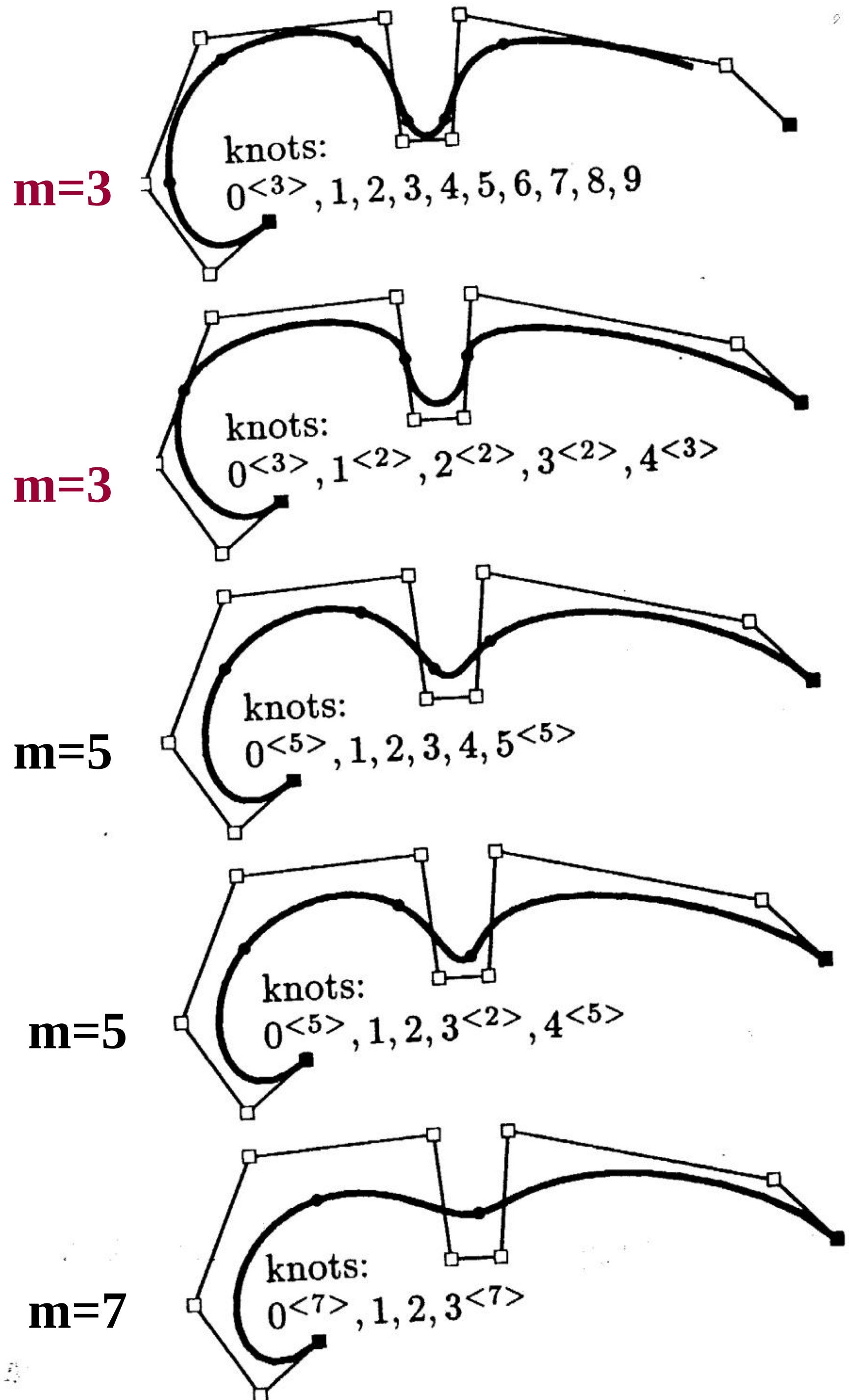
$\Delta^* = (0,0,0,$**1,1,**$3,3,3)$

$\ell = 4 \qquad t \in [t_4, t_5)$

Modify the knot vector

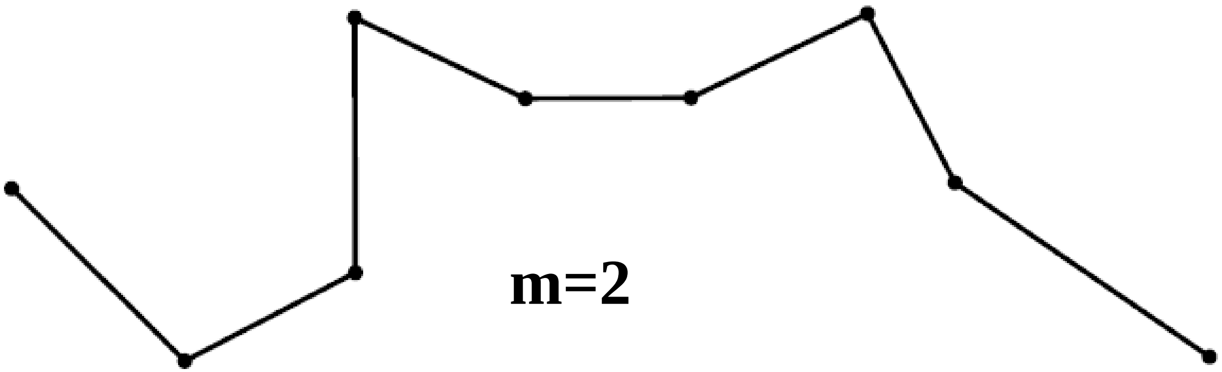
Examples:

Same CP but different order m and knot vector



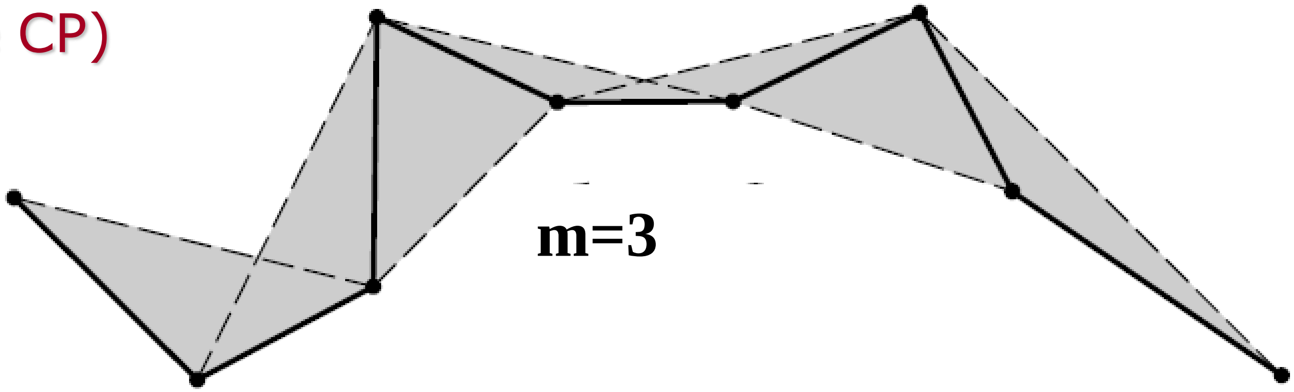
Convex hull (Guscio convesso)

- Degree n=1, order m=2, the curve is the control polygon



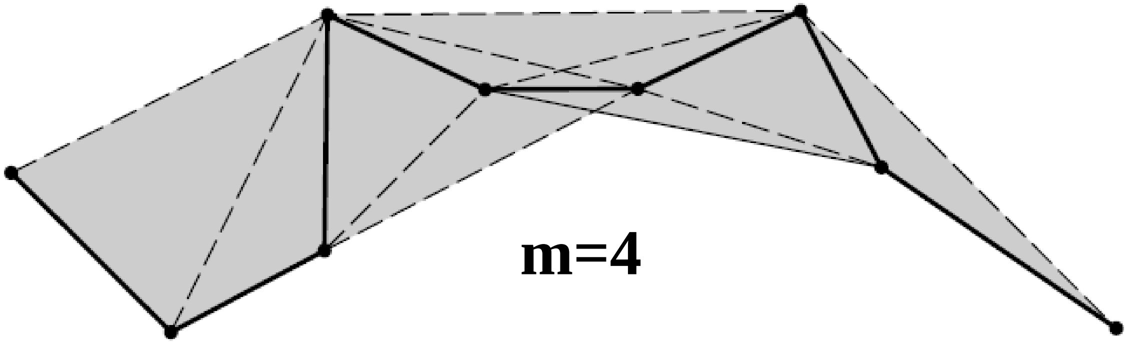
- Degree n=2, order m=3, the curve lies in the union of the convex hulls

(CH=m consecutive CP)

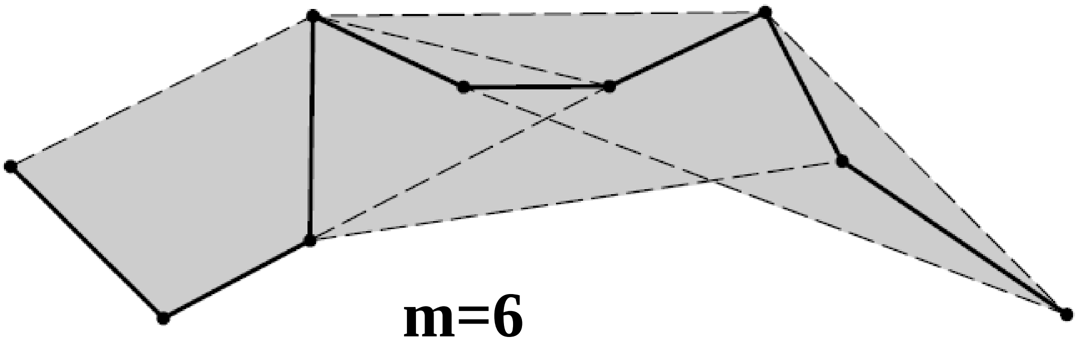


Convex hull

- Degree n=3, order m=4, the curve lies in inside the union of the convex hulls

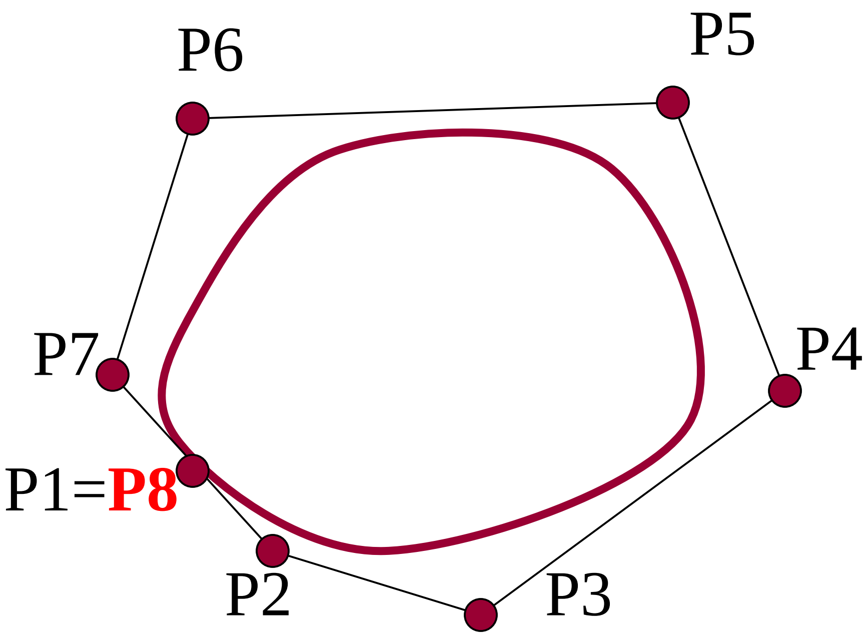


- Degree n=5, order m=6, higher degrees involve larger convex hull



The higher the order the less closely the B-spline curve follows the control polygon

Close Spline Curve



Close Curve:
First and last CP are the same

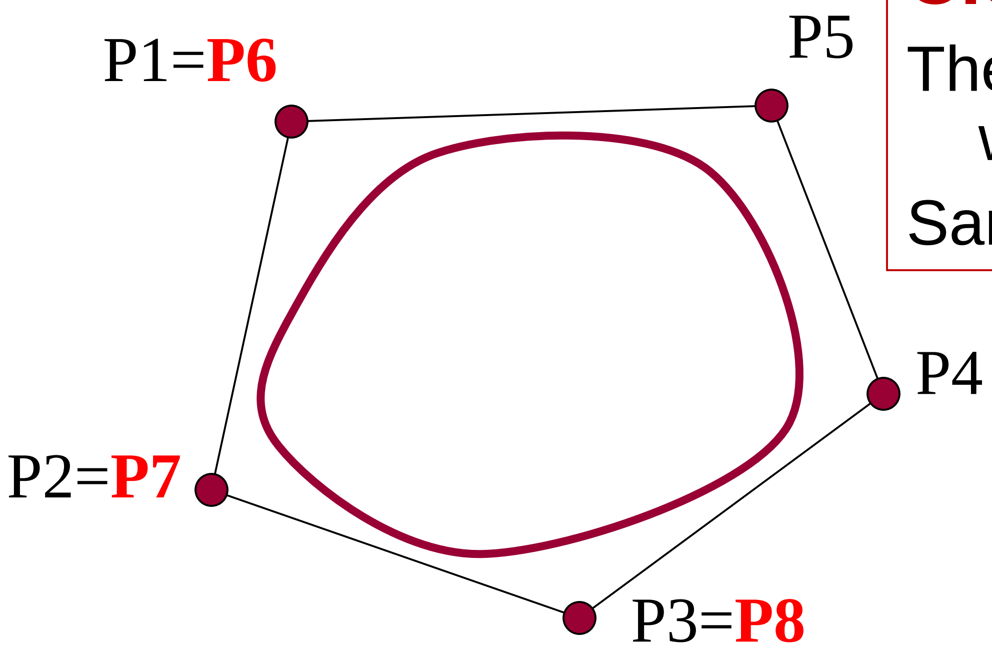
$$P_1 \equiv P_{m+K}$$

External knots coincide in Δ^*

Cubic Spline continuous C^{m-2} (C^2), **close**, defined on the knot vector:

[3 3 3 3 **4 5 6 7** 8 8 8 8]

Periodic Spline Curve



Close and Periodic Curve:

The **first n** control points coincide with the **last n**,

Same external knot intervals

C^2 also in the joint knot

Cubic n=3 spline continuous C^{m-2} (C^2), **periodic**, defined on the knot vector:

[0 1 2 3 **4 5 6 7** 8 9 10 11]

Knot insertion

Add a new knot in the Nodal Vector does not change neither the degree nor the curve shape.

Each new inserted knot leads to a new CP.

Useful for:

- Curve Evaluation/Editing
- Control over the curve continuity
- Convert spline to piecewise Bézier curves
- Draw the curve (the control polygon approaches to the curve)
- Compatibility between two curves (same knot vectors)

Knot insertion: Bohm's Method

Given the partition $\Delta^* = [t_1, .., t_{K+2m}]$

insert *knot* $\hat{t} \in [t_\ell, t_{\ell+1})$

$$\rightarrow \bar{\Delta}^* = [\hat{t}_1, .., \hat{t}_{K+2m+1}]$$

where $$\hat{t}_i = \begin{cases} t_i & i \le \ell \\ \hat{t} & i = \ell + 1 \\ t_{i-1} & i > \ell + 1 \end{cases}$$

Example

$$\Delta^* \rightarrow \bar{\Delta}^*$$

t = $7_{3/4}$ ℓ=8

[0,1,2,3,4,5,6,7,8,9,10,11] $\Rightarrow$ [0,1,2,3,4,5,6,7,$7_{3/4}$,8,9,10,11]

Knot insertion: Bohm's Method

The space dimension $S(P_m, M, \Delta)$ increases

Represent the spline $s(x) = \sum_{i=1}^{m+K} p_i N_{i,m}(x) \quad \in S(P_m, M, \Delta)$

in the new space $S(P_m, \overline{M}, \overline{\Delta})$ defined on the extended partition $\overline{\Delta}^*$

$$s(x) = \sum_{i=1}^{m+K+1} \hat{p}_i \, \overline{N}_{i,m}(x) \quad \in S(P_m, \overline{M}, \overline{\Delta})$$

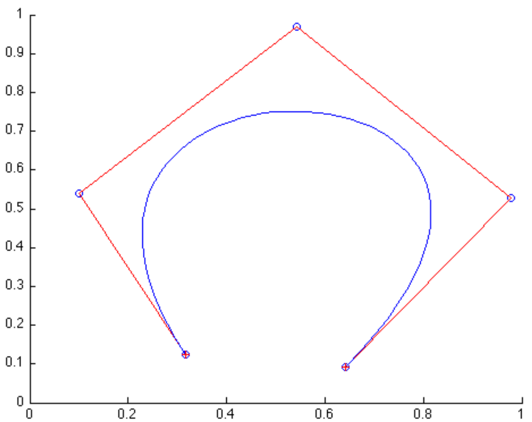
$$\hat{p}_i = \begin{cases} p_i & i \le \ell - n \\ \lambda_i p_i + (1-\lambda_i) p_{i-1} & \ell - n < i \le \ell \\ p_{i-1} & i \ge \ell + 1 \end{cases} \qquad \lambda_i = \frac{\hat{t} - t_i}{t_{i+n} - t_i}$$

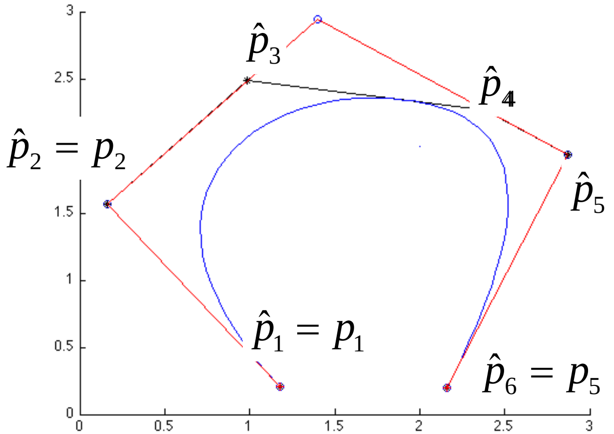
Example n=3

$$\Delta^* = \{0\ 0\ 0\ 0\ 2\ 3\ 3\ 3\ 3\} \qquad \textbf{\textit{insert}} \quad t = 2 \in [t_5, t_6)$$

$$\bar{\Delta}^* = \{0\ 0\ 0\ 0\ 2\ 2\ 3\ 3\ 3\ 3\}$$

$$\text{modify } p_i \quad \ell - n < i \le \ell$$





Bézier Curves are special cases of spline curves

Example: n=2, order m=3

- Nodal vector: [0,0,0,1,1,1]

- Basis Functions

$$N_{1,2}(t) = (1-t)^2$$
$$N_{2,2}(t) = 2t(1-t)$$
$$N_{3,2}(t) = t^2$$

- The spline curve is the Bézier curve of degree 2 with the same CP

From spline curve to piecewise Bézier Curve

- Convert a degree n spline curve to a piecewise Bézier curve of degree n

- Each Bézier curve piece is a polynomial of degree n

ALGORITHM:

Insert knots in Δ in order to have all the internal knots with multiplicity m_j=n (continuity $C^{m-mj-1} = C^0$) and external knots with multiplicity n+1.

Spline Function Evaluation

- Given $\underline{t}$, evaluate $s(\underline{t})$

$$s(\underline{t}) = \sum_{i=\ell-n}^{\ell} c_i N_{i,n}(\underline{t})$$

- Determine the interval $[t_\ell, t_{\ell+1})$ which contains $\underline{t}$; (dicotomic search)
- Then use:

Algorithm 1 via Basis Functions or

Algorithm 2 via coefficients **(de Boor)**

(repeat knot insertion at $\underline{t}$ multiple (n) times)

Algorithm 1: Basis Functions

- 1. Find *l* , index of the interval $[t_\ell, t_{\ell+1})$
- 2. Compute the nonvanishing Basis Functions $N_{i,n}$ in the interval;
- 3. Compute the linear combinations:

$$s(\underline{t}) = \sum_{i=\ell-n}^{\ell} c_i N_{i,n}(\underline{t}) \quad se \quad t \in [t_\ell, t_{\ell+1})$$

Algorithm 2: coefficient formula (de Boor)

$$
\begin{aligned}
s(\underline{t}) &= \sum_{i=l-m+1}^{l} c_i N_{i,m}(\underline{t}) = \\
&= \sum_{i=l-m+1}^{l} c_i \frac{\underline{t}-t_i}{t_{i+m-1}-t_i} N_{i,m-1} + \frac{t_{i+m}-\underline{t}}{t_{i+m}-t_{i+1}} N_{i+1,m-1} = \\
&= \sum_{i=l-m+2}^{l} c_i \frac{\underline{t}-t_i}{t_{i+m-1}-t_i} N_{i,m-1} + \sum_{i=l-m+2}^{l} c_{i-1} \frac{t_{i+m-1}-\underline{t}}{t_{i+m-1}-t_i} N_{i,m-1} = \\
&= \sum_{i=l-m+2}^{l} \frac{c_i(\underline{t}-t_i)+c_{i-1}(t_{i+m-1}-\underline{t})}{t_{i+m-1}-t_i} N_{i,m-1}(\underline{t}) = \\
&= \sum_{i=l-m+2}^{l} c_i^{(1)} N_{i,m-1}(\underline{t})
\end{aligned}
$$

Algorithm 2: coefficient formula (de Boor)

$$s(\underline{t}) = \sum_{i=l-m+1}^{l} c_i N_{i,m}(\underline{t}) = \sum_{i=l-m+2}^{l} c_i^{(1)} N_{i,m-1}(\underline{t}) = \sum_{i=l-m+3}^{l} c_i^{(2)} N_{i,m-2}(\underline{t}) =$$

$$= \ldots = \sum_{i=l}^{l} c_i^{(m-1)} N_{i,1}(\underline{t}) = c_l^{(m-1)} N_{l,1}(\underline{t}) = c_l^{(m-1)}$$

$$\boxed{c_i^{[j]} = \frac{t - t_i}{t_{i+m-j} - t_i} c_i^{[j-1]} + \frac{t_{i+m-j} - t}{t_{i+m-j} - t_i} c_{i-1}^{[j-1]} \quad , c_i^{[0]} = c_i \qquad \begin{array}{c} j = 1,..,m-1 \\ i = l-m+j-1,...,l \end{array}}$$

$$\boxed{\begin{array}{ccccc}
c_{l-m+1} & c_{l-m+2} & .. & .. & c_l \\
 & c_{l-m+2}^{[1]} & .. & .. & c_l^{[1]} \\
 & .. & .. & .. & .. \\
 & & & .. & .. \\
 & & & & c_l^{[m-1]} = s(t)
\end{array}}$$

Perspective projection

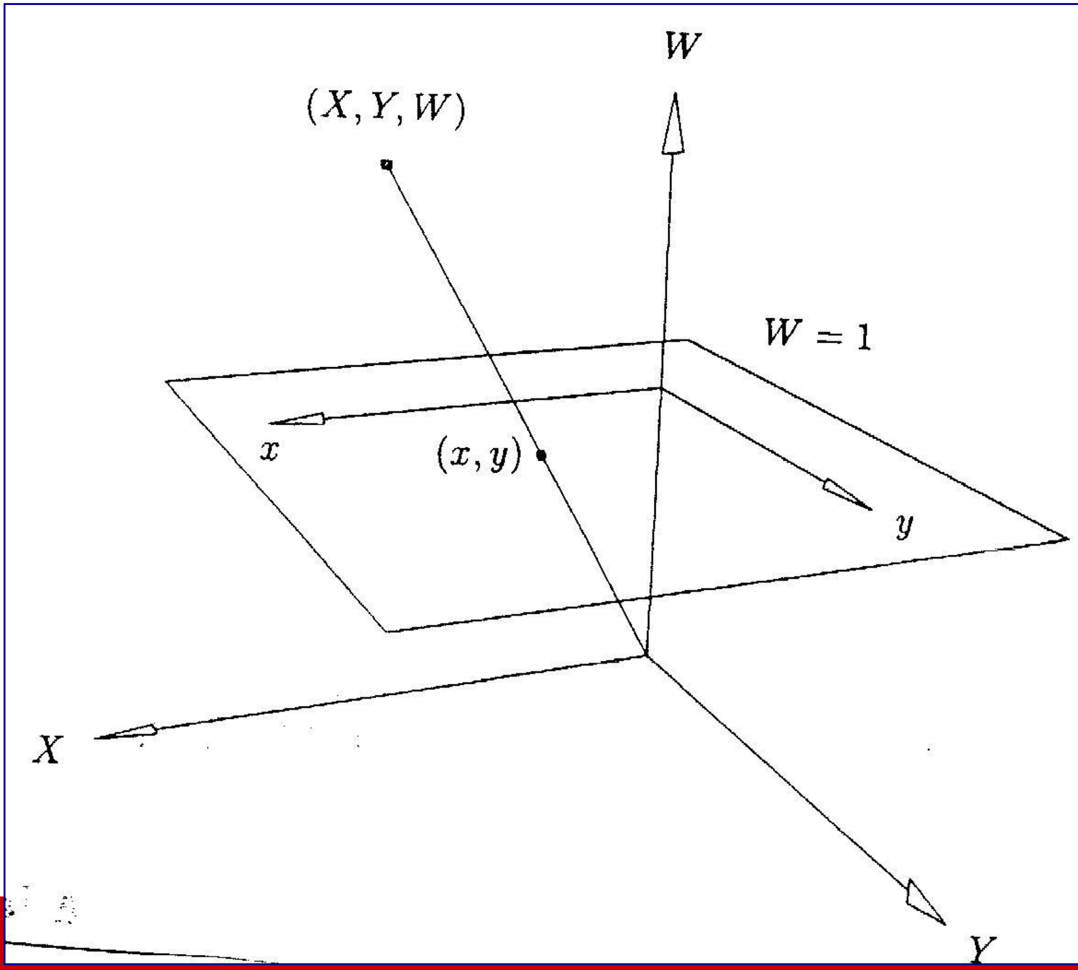
Perspective projection of a point in nD space to a plane which is parallel to n-1 axes.

Homogeneous point in 3D:

$$P^w = (X, Y, W)$$

Projection to 2D

$$P = (x, y) = \left(\frac{X}{W}, \frac{Y}{W}\right)$$



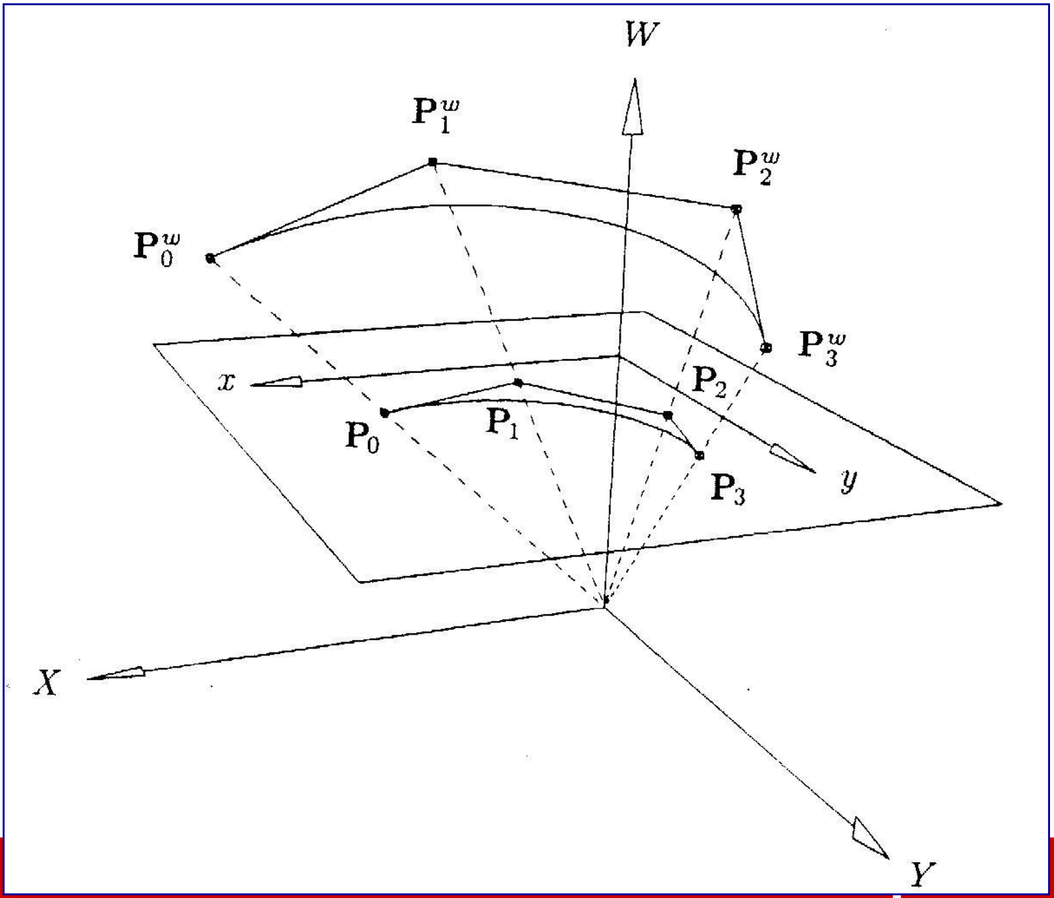
Rational Curve

Perspective projection of each point of a 3D curve

$$C^w(t) = \left(x(t), y(t), w(t)\right)$$

into the 2D plane:

$$C(t) = \begin{pmatrix} \dfrac{x(t)}{w(t)} & \dfrac{y(t)}{w(t)} \end{pmatrix}$$



Rational curve

NURBS curves
(NonUniform Rational Bspline)
(Ken Versprille)

Given a non-rational spline curve

$$s^w(t) = \sum \begin{pmatrix} w_i x_i \\ w_i y_i \\ w_i \end{pmatrix} N_{i,m}(t)$$

Apply perspective projection 3D -> 2D

get a rational spline (NURBS)

$$s(t) = \begin{pmatrix} \dfrac{\sum w_i x_i N_{i,m}(t)}{\sum w_i N_{i,m}(t)} \\ \\ \dfrac{\sum w_i y_i N_{i,m}(t)}{\sum w_i N_{i,m}(t)} \end{pmatrix}$$

NURBS

Given a partition Δ

an integer **m** – order - (degree n, m=n+1)

a vector **M** of knot multiplicities **M**=(m_1,m_2,..,m_k), m_i<=m

a vector **W** of weights **W**=(w_1,w_2,..,w_k), w_i>0

A NURBS curve of degree n is represented as

$$s(t) = \frac{\sum_{i=1}^{ncp} P_i w_i N_{i,n}(t)}{\sum_{i=1}^{ncp} w_i N_{i,n}(t)},$$

The weight $\mathbf{w_i > 0}$ is associated to the Control Point $\mathbf{P_i}$ and give more flexibility

Non-uniform = different spacing between the knots

rational = ratio of polynomials

Weights as shape parameters

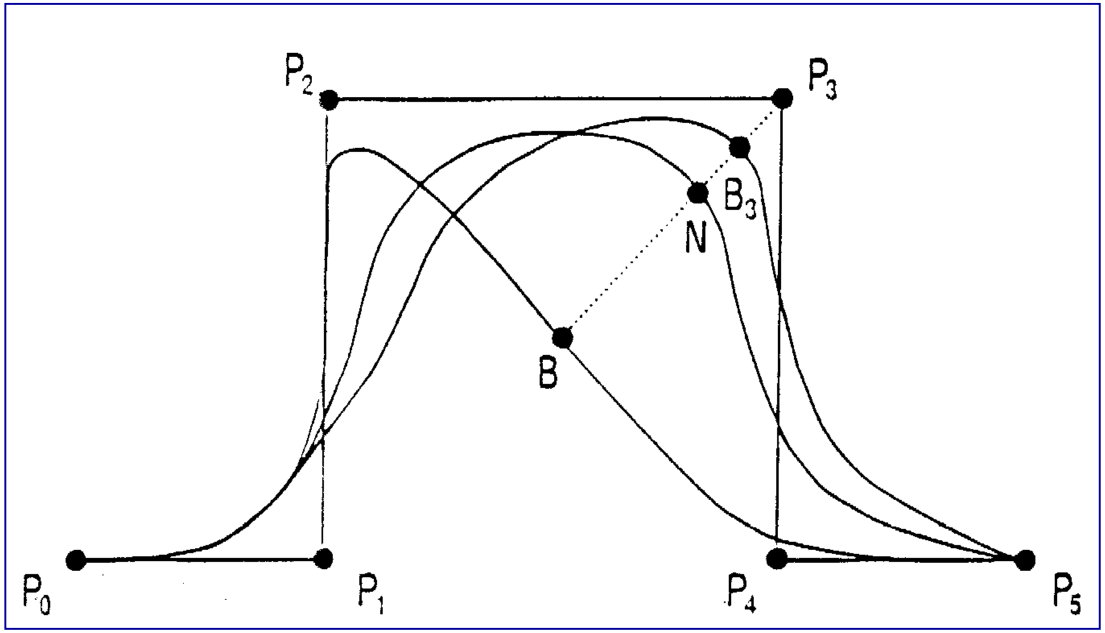
$$w_i > 0,$$

$$if \quad w_i = 0 \Rightarrow P_i \; \infty$$

$$B = s(t) \;\; for \; w_i \to 0$$

$$N = s(t) \;\; for \; w_i = 1$$

$$B_i = s(t) \;\; for \; w_i \neq [0,1]$$



w_i affects the curve locally in $[t_i, t_{i+m})$

When w_i=1 for all i , then the spline is nonrational

Why NURBS?

- Conic sections (Quadrics) are a special case of NURBS
  (es. Circle, ellipse, sphere, ..)

- Spline Curves and Bézier curves are NURBS
  → they have all their properties

- More degree of freedom for design

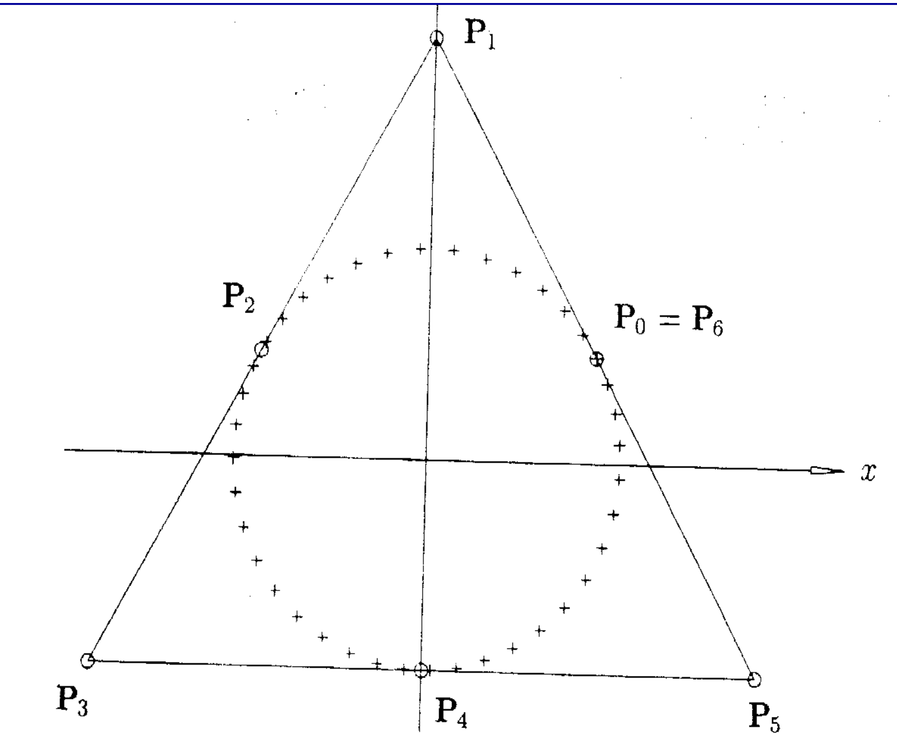
NURBS Circle

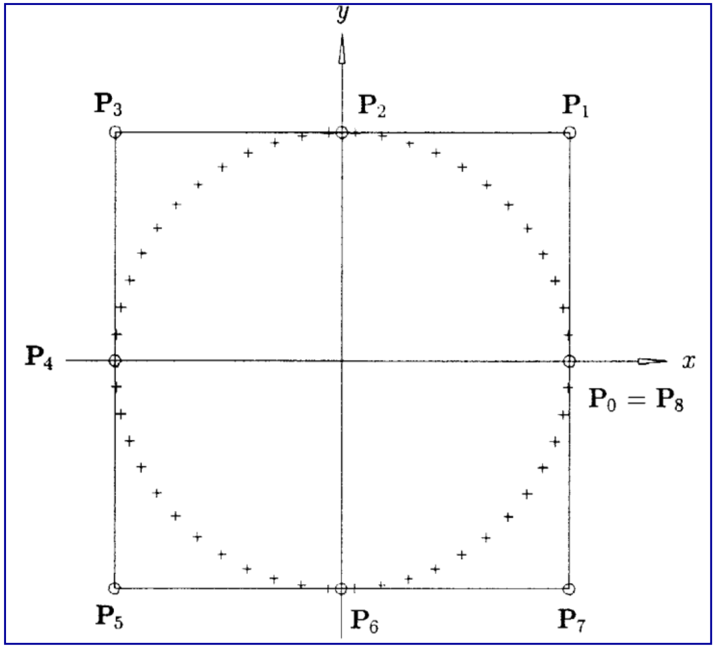
Conic Sections are represented exactly by NURBS of degree 2

Circle 7 points:

$\Delta^* = \{0,0,0,1/3,1/3,2/3,2/3,1,1,1\}$

$w_i = \{1,1/2,1,1/2,1,1/2,1\}$, $w_1 = \cos(120) = 1/2$





Circle 9 points:

$\Delta^* = \{0,0,0,1/4,1/4,1/2,1/2,3/4,3/4,1,1,1\}$

$w_i = \{1,\sqrt{2}/2,1,\sqrt{2}/2,1,\ \sqrt{2}/2,1,\ \sqrt{2}/2,1\}$

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