Conic sections

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Contents

Invariants

Theorem (Projective group)

The $\left\{ \begin{array}{c} \textit{projectivities} \\ \textit{affinities} \end{array} \right\}$ of the $\left\{ \begin{array}{c} \textit{projective} \\ \textit{affine} \end{array} \right\}$ plane form a group with respect to their composition, the $\left\{ \begin{array}{c} \textit{projective} \\ \textit{affine} \end{array} \right\}$ group. A quantity that is invariant with respect to a $\left\{ \begin{array}{c} \textit{projectivity} \\ \textit{affinity} \end{array} \right\}$ is called $\left\{ \begin{array}{c} \textit{projectively} \\ \textit{affinely} \end{array} \right\}$ invariant. The determination of all $\left\{ \begin{array}{c} \textit{projective} \\ \textit{affine} \end{array} \right\}$ invariants of the $\left\{ \begin{array}{c} \textit{projective} \\ \textit{affine} \end{array} \right\}$ group is referred to as the theory of invariants of the $\left\{ \begin{array}{c} \textit{projective} \\ \textit{affine} \end{array} \right\}$ group.

Invariants

Definition of "Geometry", Erlanger Programm, F. Klein, 1872

Geometry = Theory of invariants of a transformation group

Invariants

Definition of "Geometry", Erlanger Programm, F. Klein, 1872

Geometry = Theory of invariants of a transformation group

$$\Rightarrow \left\{ \begin{array}{c} \textit{projective} \\ \textit{affine} \end{array} \right\} \text{geometry} = \text{theory of invariants of the} \left\{ \begin{array}{c} \textit{projective} \\ \textit{affine} \end{array} \right\} \text{group}$$

Invariants

Examples

Affine invariants	Projective invariants
collinear points	collinear points
parallelity of lines	—
ratio of 3 collinear points	cross ratio of 4 collinear points
affine classification of conic sections	projective classification of conic sections

Invariants

Examples

Affine invariants	Projective invariants
collinear points	collinear points
parallelity of lines	—
ratio of 3 collinear points	cross ratio of 4 collinear points
affine classification of conic sections	projective classification of conic sections

Remark

The notion of parallelity of two lines does not exist in the projective plane; two lines always intersect in one point.

Projective classification of conic sections

Conic section in the projective plane

$$a_{00}x_0^2 + 2a_{01}x_0x_1 + a_{11}x_1^2 + 2a_{02}x_0x_2 + a_{22}x_2^2 + 2a_{12}x_1x_2 = 0$$

$\Longleftrightarrow$

$$\vec{x}^T A \vec{x} = 0\,,\; A = (a_{ij}) \in R^{3,3}, symmetric$$

Projective classification of conic sections

Conic section in the projective plane

$$a_{00}x_0^2 + 2a_{01}x_0x_1 + a_{11}x_1^2 + 2a_{02}x_0x_2 + a_{22}x_2^2 + 2a_{12}x_1x_2 = 0$$

$\Longleftrightarrow$

$$\vec{x}^T A \vec{x} = 0\,,\; A = (a_{ij}) \in R^{3,3}, symmetric$$

Applying a projectivity

$$\vec{x} = T\vec{y}\,,\; rk(T) = 3$$

$\Longrightarrow$

$$\vec{y}^T \underbrace{T^T A T}_{B} \vec{y} = 0\,,\; B := T^T A T$$

Projective classification of conic sections

Congruent matrix normal forms

$A, B \dots$ congruent matrices
$\Longrightarrow$ normal forms of real 3×3 congruent matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Projective classification of conic sections

Conic normal forms

a) $rk(A) = 3$:

a1) $x_0^2 + x_1^2 + x_2^2 = 0$ imaginary

a2) $x_0^2 + x_1^2 - x_2^2 = 0$ real

b) $rk(A) = 2$:

b1) $x_0^2 + x_1^2 = 0$ imaginary

b2) $x_0^2 - x_1^2 = 0$ real

a) $rk(A) = 1$:

$x_0^2 = 0$

Projective classification of conic sections

Conic normal forms

a) $rk(A) = 3$: non–degenerate conic section

a1) $x_0^2 + x_1^2 + x_2^2 = 0$ imaginary

a2) $x_0^2 + x_1^2 - x_2^2 = 0$ real

b) $rk(A) = 2$: pair of intersecting lines

b1) $x_0^2 + x_1^2 = 0$ imaginary

b2) $x_0^2 - x_1^2 = 0$ real

a) $rk(A) = 1$: double line

$x_0^2 = 0$

Projective classification of conic sections

Conic normal forms

a) $rk(A) = 3$: non–degenerate conic section

a1) $x_0^2 + x_1^2 + x_2^2 = 0$ imaginary
a2) $x_0^2 + x_1^2 - x_2^2 = 0$ real

b) $rk(A) = 2$: pair of intersecting lines

b1) $x_0^2 + x_1^2 = 0$ imaginary
b2) $x_0^2 - x_1^2 = 0$ real

a) $rk(A) = 1$: double line
$x_0^2 = 0$

Projective versus affine classification

projective classification of conics $\leftrightarrow$ affine classification of conics

Illustration

Conic sections as rational Bézier curves

Homogeneous form

$$\begin{pmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \end{pmatrix} = \sum_{i=0}^{2} w_i \begin{pmatrix} 1 \\ \vec{b}_i \end{pmatrix} B_i^2(t)\,, \ \textit{where}\ \vec{b}_i = \begin{pmatrix} b_i^1 \\ b_i^2 \end{pmatrix}\,, \ w_i \in I\!R$$

Conic sections as rational Bézier curves

Homogeneous form

$$\begin{pmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \end{pmatrix} = \sum_{i=0}^{2} w_i \begin{pmatrix} 1 \\ \vec{b}_i \end{pmatrix} B_i^2(t)\,, \; where \; \vec{b}_i = \begin{pmatrix} b_i^1 \\ b_i^2 \end{pmatrix}\,, \; w_i \in I\!R$$

Projection into the plane $x_0 = 1$

$$\vec{x}(t) = \frac{\sum_{i=0}^{2} w_i \vec{b}_i B_i^2(t)}{\sum_{i=0}^{2} w_i B_i^2(t)}$$

rational Bézier curve of degree 2

$B_i(\vec{b}_i) \ldots$ control points
$w_i \ldots$ weights

Conic sections as rational Bézier curves

Theorem

Polynomial parametric curves of degree 2

$$\vec{x}(t) = \sum_{i=0}^{2} \vec{b}_i B_i^2(t) \qquad (*)$$

are parabolas and every parabola can be represented in the form $(*)$.

Conic sections as rational Bézier curves

Theorem

Polynomial parametric curves of degree 2

$$\vec{x}(t) = \sum_{i=0}^{2} \vec{b}_i B_i^2(t) \qquad (*)$$

are parabolas and every parabola can be represented in the form $(*)$.

Theorem

Rational parametric curves of degree 2

$$\vec{x}(t) = \frac{\sum_{i=0}^{2} w_i \vec{b}_i B_i^2(t)}{\sum_{i=0}^{2} w_i B_i^2(t)} \qquad (**)$$

are conic sections and every conic section can be represented in the form $(**)$.

Conic sections as rational Bézier curves

Corollary (***)

A rational parametric Bézier curve of degree 2

$$\vec{x}(t) = \frac{\sum_{i=0}^{2} w_i \vec{b}_i B_i^2(t)}{\sum_{i=0}^{2} w_i B_i^2(t)}, \quad t \in [0,1]$$

with $w_i \in I\!R^+$, and $B_0(\vec{b}_0)$, $B_1(\vec{b}_1)$, $B_2(\vec{b}_2)$ not collinear, represents a connected segment of a non–degenerate conic section.

Conic sections as rational Bézier curves

Theorem

A rational parametric Bézier curve of degree 2 as in Corollary (***) has the following properties:

a) $\overrightarrow{x}(0) = \overrightarrow{b}_0,\ \overrightarrow{x}(1) = \overrightarrow{b}_2$

Conic sections as rational Bézier curves

Theorem

A rational parametric Bézier curve of degree 2 as in Corollary (***) has the following properties:

a) $\overrightarrow{x}(0) = \overrightarrow{b}_0,\ \overrightarrow{x}(1) = \overrightarrow{b}_2$

b) $\frac{d}{dt}\overrightarrow{x}(t)|_{t=0} = \frac{2w_1}{w_0}(\overrightarrow{b}_1 - \overrightarrow{b}_0),\ \frac{d}{dt}\overrightarrow{x}(t)|_{t=1} = \frac{2w_1}{w_2}(\overrightarrow{b}_2 - \overrightarrow{b}_1)$

Conic sections as rational Bézier curves

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A rational parametric Bézier curve of degree 2 as in Corollary (***) has the following properties:

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c) $\overrightarrow{x}(t)|_{t\in[0,1]} \subset H(B_0(\overrightarrow{b}_0), B_1(\overrightarrow{b}_1), B_2(\overrightarrow{b}_2))$ *convex hull property*

Conic sections as rational Bézier curves

Theorem

A rational parametric Bézier curve of degree 2 as in Corollary (***) has the following properties:

a) $\vec{x}(0) = \vec{b}_0,\ \vec{x}(1) = \vec{b}_2$

b) $\frac{d}{dt}\vec{x}(t)|_{t=0} = \frac{2w_1}{w_0}(\vec{b}_1 - \vec{b}_0),\ \frac{d}{dt}\vec{x}(t)|_{t=1} = \frac{2w_1}{w_2}(\vec{b}_2 - \vec{b}_1)$

c) $\vec{x}(t)|_{t\in[0,1]} \subset H(B_0(\vec{b}_0), B_1(\vec{b}_1), B_2(\vec{b}_2))$ *convex hull property*

d) *For* $w_0 = w_2 = 1$,

$$\overline{\vec{x}}(t) := \frac{\vec{b}_0 B_0^2(t) - w_1 \vec{b}_1 B_1^2(t) + \vec{b}_2 B_2^2(t)}{B_0^2(t) - w_1 B_1^2(t) + B_2^2(t)}$$

represents the complementary conic segment with respect to $\vec{x}(t)$.

Conic sections as rational Bézier curves

Role of the weights

Let

$$\vec{x}(t) = \frac{\sum_{i=0}^{2} w_i \vec{b}_i B_i^2(t)}{\sum_{i=0}^{2} w_i B_i^2(t)}, \quad t \in [0,1]$$

and

$$\vec{\tilde{x}}(\tau) = \frac{\sum_{i=0}^{2} \tilde{w}_i \vec{b}_i B_i^2(\tau)}{\sum_{i=0}^{2} \tilde{w}_i B_i^2(\tau)}, \quad \tau \in [0,1]$$

Are there weights $(w_i, \tilde{w}_i)$, $i = 0, 1, 2$ such that $\vec{x}(t)$, $\vec{\tilde{x}}(\tau)$ represent the same curve ?

Conic sections as rational Bézier curves

Role of the weights

Parameter transformation

$$t = \frac{\tau}{(1-k)\tau + k}\,, \quad k \in I\!R \setminus \{0\}$$

Conic sections as rational Bézier curves

Role of the weights

Parameter transformation

$$t = \frac{\tau}{(1-k)\tau + k}\,,\ k \in I\!R \setminus \{0\}$$

$\Rightarrow$

$$B_i^2\left(\frac{\tau}{(1-k)\tau+k}\right) = \binom{2}{i}\left(1 - \frac{\tau}{(1-k)\tau+k}\right)^{2-i}\left(\frac{\tau}{(1-k)\tau+k}\right)^{i}$$

Conic sections as rational Bézier curves

Role of the weights

Parameter transformation

$$t = \frac{\tau}{(1-k)\tau + k}, \quad k \in I\!R \setminus \{0\}$$

$\Rightarrow$

$$\begin{aligned} B_i^2\left(\frac{\tau}{(1-k)\tau+k}\right) &= \binom{2}{i}\left(1-\frac{\tau}{(1-k)\tau+k}\right)^{2-i}\left(\frac{\tau}{(1-k)\tau+k}\right)^{i} \\ &= \binom{2}{i}\left(\frac{k(1-\tau)}{(1-k)\tau+k}\right)^{2-i}\left(\frac{\tau}{(1-k)\tau+k}\right)^{i} \end{aligned}$$

Conic sections as rational Bézier curves

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B_i^2\left(\frac{\tau}{(1-k)\tau+k}\right) &= \binom{2}{i}\left(1 - \frac{\tau}{(1-k)\tau+k}\right)^{2-i}\left(\frac{\tau}{(1-k)\tau+k}\right)^{i} \\
&= \binom{2}{i}\left(\frac{k(1-\tau)}{(1-k)\tau+k}\right)^{2-i}\left(\frac{\tau}{(1-k)\tau+k}\right)^{i} \\
&= \left(\frac{1}{(1-k)\tau+k}\right)^{2}\binom{2}{i} k^{2-i}(1-\tau)^{2-i}\tau^{i}
\end{aligned}$$

Conic sections as rational Bézier curves

Role of the weights

Parameter transformation

$$t = \frac{\tau}{(1-k)\tau + k}\,, \quad k \in I\!R \setminus \{0\}$$

$\Rightarrow$

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B_i^2\left(\frac{\tau}{(1-k)\tau+k}\right) &= \binom{2}{i}\left(1 - \frac{\tau}{(1-k)\tau+k}\right)^{2-i}\left(\frac{\tau}{(1-k)\tau+k}\right)^{i} \\
&= \binom{2}{i}\left(\frac{k(1-\tau)}{(1-k)\tau+k}\right)^{2-i}\left(\frac{\tau}{(1-k)\tau+k}\right)^{i} \\
&= \left(\frac{1}{(1-k)\tau+k}\right)^{2}\binom{2}{i} k^{2-i}(1-\tau)^{2-i}\tau^{i} \\
&= \frac{k^{2-i}}{((1-k)\tau+k)^2} B_i^2(\tau)
\end{aligned}$$

Conic sections as rational Bézier curves

Role of the weights

We thus obtain

$$\vec{x}(t) = \frac{\sum_{i=0}^{2} w_i k^{2-i} \vec{b}_i B_i^2(\tau)}{\sum_{i=0}^{2} w_i k^{2-i} B_i^2(\tau)}$$

Conic sections as rational Bézier curves

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$\Longrightarrow$

$$\tilde{w}_i = w_i k^{2-i}\,, \; i = 0, 1, 2$$

Conic sections as rational Bézier curves

Role of the weights

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$\Longrightarrow$

$$\tilde{w}_i = w_i k^{2-i}, \quad i = 0, 1, 2$$

in particular: $k = \sqrt{\frac{w_2}{w_0}} \Longrightarrow \tilde{w}_0 = w_2, \tilde{w}_2 = w_2$

Conic sections as rational Bézier curves

Role of the weights

We thus obtain

$$\overrightarrow{x}(t) = \frac{\sum_{i=0}^{2} w_i k^{2-i} \overrightarrow{b}_i B_i^2(\tau)}{\sum_{i=0}^{2} w_i k^{2-i} B_i^2(\tau)}$$

$\Longrightarrow$

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in particular: $k = \sqrt{\frac{w_2}{w_0}} \Longrightarrow \tilde{w}_0 = w_2, \tilde{w}_2 = w_2$

Theorem

Every rational parametric Bézier curve of degree 2 can be written in standard form, i.e., $w_0 = w_2 = 1$.

Conic sections as rational Bézier curves

Role of the weights

Let the conic section $\overrightarrow{x}(t)$ be in standard form ($w_0 = w_2 = 1$).

1) weight w_1 increases $\Longrightarrow \overrightarrow{x}(t)$ gets near B_1
   weight w_1 decreases $\Longrightarrow \overrightarrow{x}(t)$ gets away from B_1

Conic sections as rational Bézier curves

Role of the weights

Let the conic section $\overrightarrow{x}(t)$ be in standard form ($w_0 = w_2 = 1$).

1) weight w_1 increases $\Longrightarrow \overrightarrow{x}(t)$ gets near B_1
   weight w_1 decreases $\Longrightarrow \overrightarrow{x}(t)$ gets away from B_1
2) $w_1 \leftrightarrow$ affine type of the conic

$$\Longleftrightarrow$$

$w_1 \leftrightarrow$ behavior of the conic with respect to the line at infinity

Conic sections as rational Bézier curves

Role of the weights

2)

$$\vec{x}(t) = \frac{\sum_{i=0}^{2} w_i \vec{b}_i B_i^2(t)}{\sum_{i=0}^{2} w_i B_i^2(t)}, \quad t \in [0,1]$$

Conic sections as rational Bézier curves

Role of the weights

2)

$$\vec{x}(t) = \frac{\sum_{i=0}^{2} w_i \vec{b}_i B_i^2(t)}{\sum_{i=0}^{2} w_i B_i^2(t)}, \quad t \in [0,1]$$

$$\Longrightarrow x_0(t) = \sum_{i=0}^{2} w_i B_i^2(t) > 0, \quad t \in [0,1]$$

Conic sections as rational Bézier curves

Role of the weights

2)

$$\vec{x}(t) = \frac{\sum_{i=0}^{2} w_i \vec{b}_i B_i^2(t)}{\sum_{i=0}^{2} w_i B_i^2(t)}\,,\ t \in [0,1]$$

$$\Longrightarrow x_0(t) = \sum_{i=0}^{2} w_i B_i^2(t) > 0\,,\ t \in [0,1]$$

$\Longrightarrow$ eventual poles are in the complement $\vec{\bar{x}}(t)$:

$$\bar{x}_0(t) = B_0^2(t) - w_1 B_1^2(t) + B_2^2(t) = 0$$

Conic sections as rational Bézier curves

Role of the weights

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$$\vec{x}(t) = \frac{\sum_{i=0}^{2} w_i \vec{b}_i B_i^2(t)}{\sum_{i=0}^{2} w_i B_i^2(t)}, \quad t \in [0,1]$$

$$\Longrightarrow x_0(t) = \sum_{i=0}^{2} w_i B_i^2(t) > 0, \quad t \in [0,1]$$

$\Longrightarrow$ eventual poles are in the complement $\vec{x}(t)$:

$$\bar{x}_0(t) = B_0^2(t) - w_1 B_1^2(t) + B_2^2(t) = 0$$

$$\Longleftrightarrow t_{1/2} = \frac{1 + w_1 \pm \sqrt{w_1^2 - 1}}{2 + 2w_1}$$

Conic sections as rational Bézier curves

Theorem

The conic section in rational parametric Bézier representation $\overrightarrow{x}(t)$ *and in standard form* ($w_0 = w_2 = 1$) *is part of a* $\left\{ \begin{array}{c} \textit{ellipse} \\ \textit{parabola} \\ \textit{hyperbola} \end{array} \right\}$ *if*

$$\left\{ \begin{array}{l} w_1 \\ w_1 \\ w_1 \end{array} \right\}.$$

Conic sections as rational Bézier curves

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$$\left\{ \begin{array}{c} w_1 < 1 \\ w_1 = 1 \\ w_1 > 1 \end{array} \right\}.$$

Conic sections as rational Bézier curves

Theorem

The conic section in rational parametric Bézier representation $\overrightarrow{x}(t)$ *and in standard form* ($w_0 = w_2 = 1$) *with* $\| \overrightarrow{B_0 B_1} \| = \| \overrightarrow{B_1 B_2} \|$ *describes a circular arc if and only if* $w_1 = \cos\phi$, *where* $\phi = \angle B_2 B_0 B_1 = \angle B_0 B_2 B_1$.