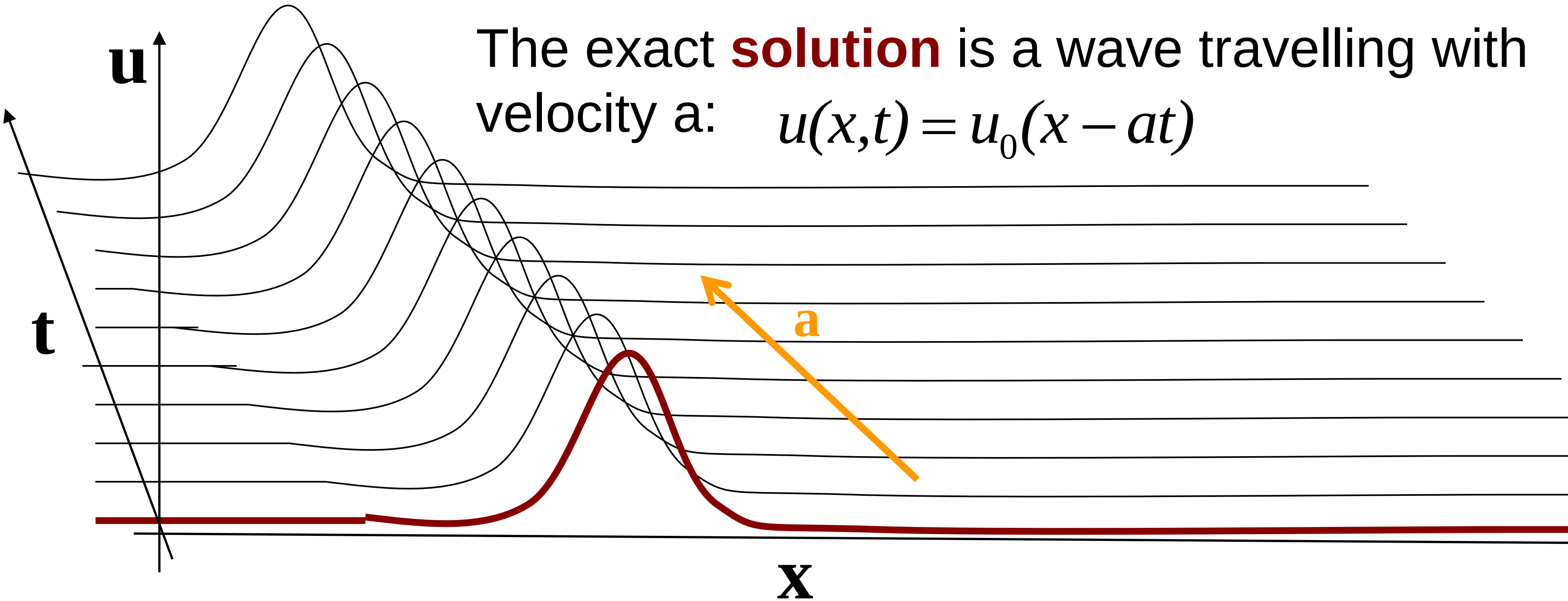
Numerical Methods for Partial Differential Equations (PDE) (4)

Finite Difference Methods

Advection Equations (Transport)

$$\begin{cases} u_t + a u_x = 0 & a \text{ constant } \neq 0,\ 0 \le x \le 1 \\ u(x,0) = u_0(x) & IC \end{cases}$$



The exact **solution** is a wave travelling with velocity a: $u(x,t) = u_0(x - at)$

1D Linear hyperbolic PDE (First order)

Advection Equations (Transport)

$$\begin{cases} u_t + a u_x = 0 & a \text{ constant } \neq 0,\ 0 \le x \le 1 \\ u(x,0) = u_0(x) & IC \end{cases}$$

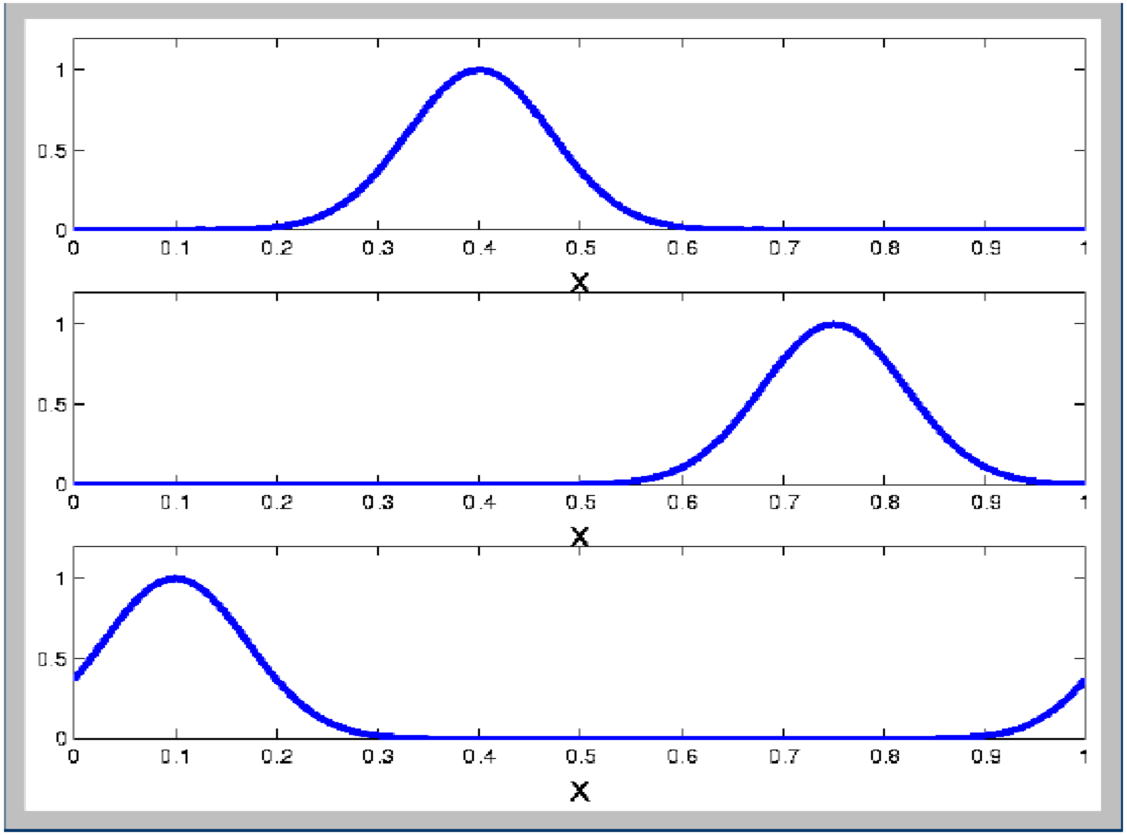
Boundary Conditions (BC) on [0,1]

- **Inflow & outflow**
- If a>0 ➡ the solution profile does not change shape but moves in the positive x direction with constant speed
  x=0 inflow boundary **x=1 outflow boundary**
  ↘
  where the BC has to be imposed
  u(0,t)=g(t)
- If a<0 ⬅ then **x=1 inflow boundary** **x=0 outflow boundary**

Boundary Conditions

- **Periodic** $u(0,t) = u(1,t) \quad t \geq 0, \quad a > 0$

Whatever flows out at the outflow boundary flows back in at the inflow boundary



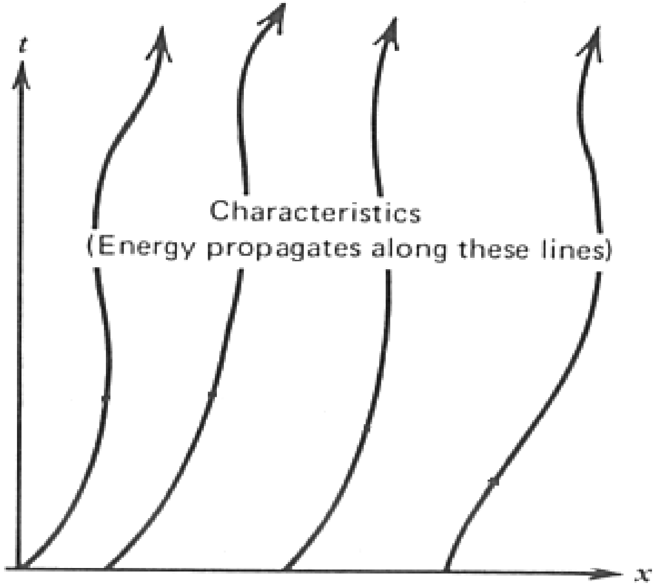
t=0

t=T

t=2T

Characteristic curves

The curves (x(t),t)
in the plane (x,t)
are called
characteristic curves



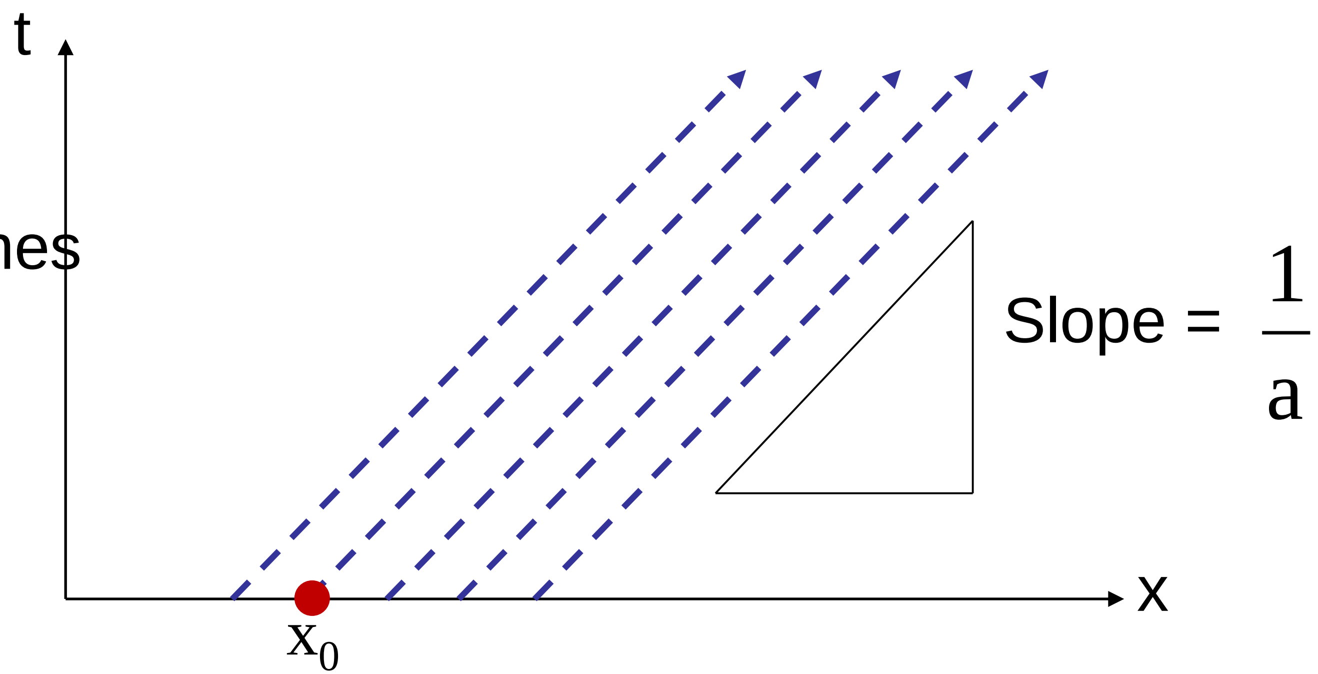
The initial solution at *x* affects the solution only along a line in the *xt*-plane.

A value of a function u_0 (i.e., of a signal) at a given point x, propagates in the (x,t)-plane along a line, named characteristic line.
For a=constant , is a straight line with constant slope.

Characteristic curves

Case a = const

Parallel straight lines



The characteristic curves x(t) in the plane (x,t):

$$x(t) = x_0 + at \quad \forall x_0$$

are the solutions of the following ODEs:

$$\begin{cases} \dfrac{dx}{dt} = a, \\ x(0) = x_0 \end{cases}$$

Characteristic curves

The solution u is constant along each characteristic, in fact

$$let \quad u(t, x(t)):$$

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\underbrace{\frac{dx}{dt}}_{a} = 0$$

Any discontinuities in the initial data u_0 propagate along the characteristic curves and are maintained by the solution.

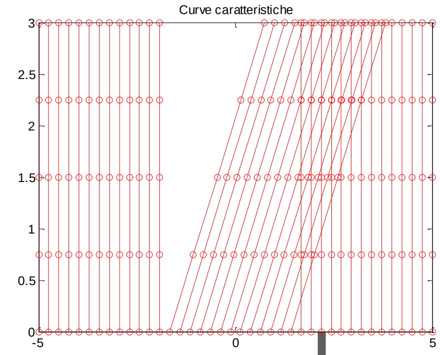
Characteristic curves

$$\begin{cases} u_t + a(u)u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

The exact solution is a wave travelling with velocity a(u):

$$u(x,t) = u_0(x - a(u(x,t))t)$$

Yet the characteristic curves are straight lines as u is constant along the directions characteristics, even if they are no longer parallel. They can intersect.



Finite Difference Method

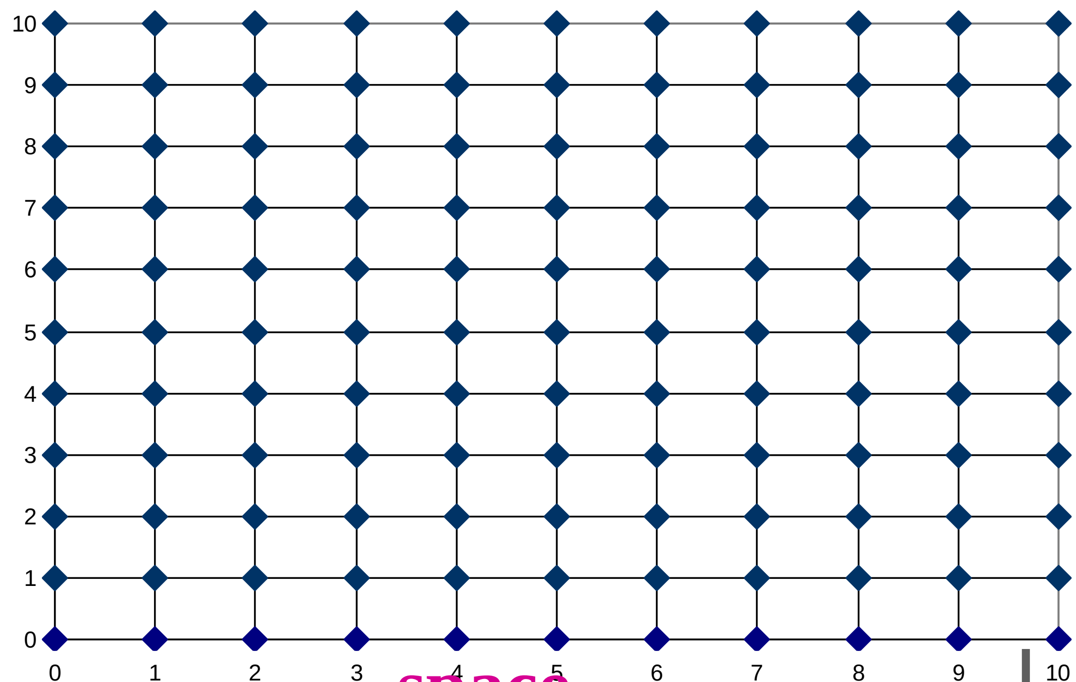
$$u_t + \color{red}{a} u_x = 0$$

a pos. constant

$$IC: \quad u(x,0) = u_0(x) \qquad \{0 < x < 1\},\ t > 0$$

$$BC: \quad u(0,t) = f_1(t) \quad u(1,t) = f_2(t)$$

t



$h = \Delta x$

$k = \Delta t$

Time
(j index)

N

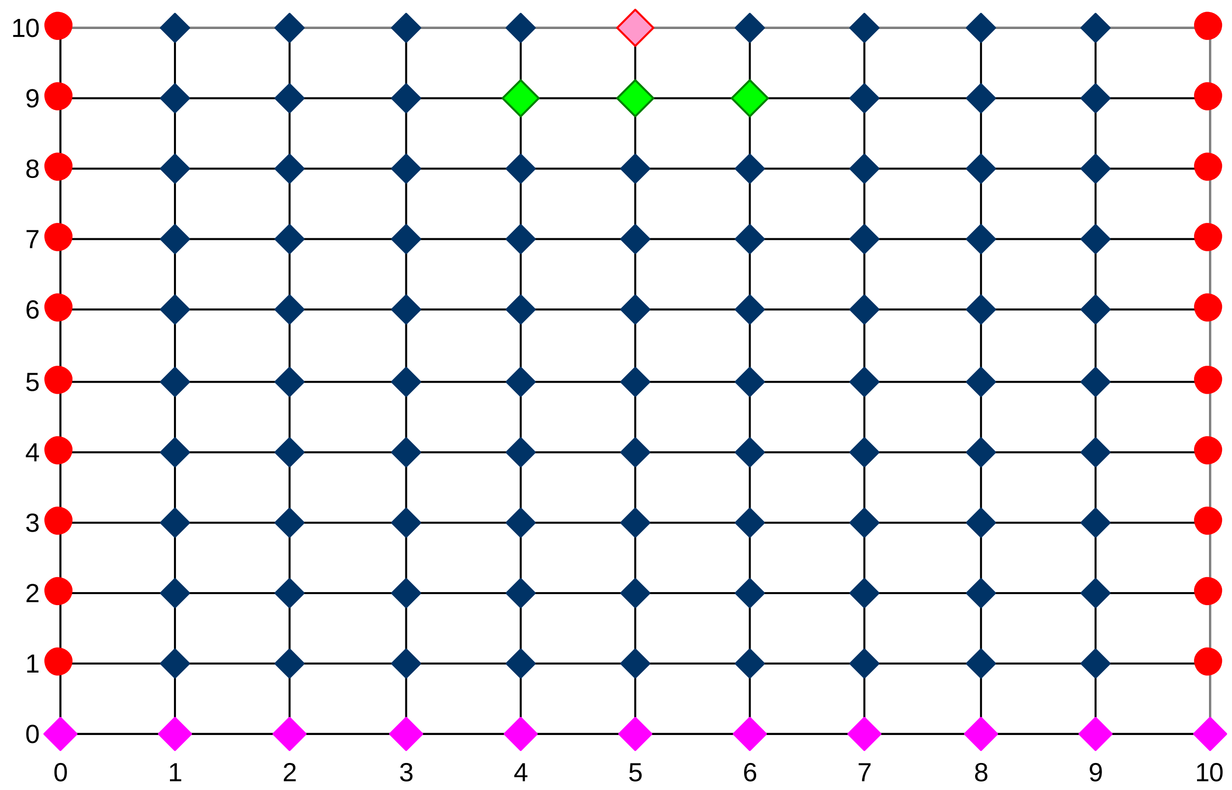
space
(i index)

x

Initial and boundary conditions

Dirichlet BC
$u(0, t) = f1(t)$

t

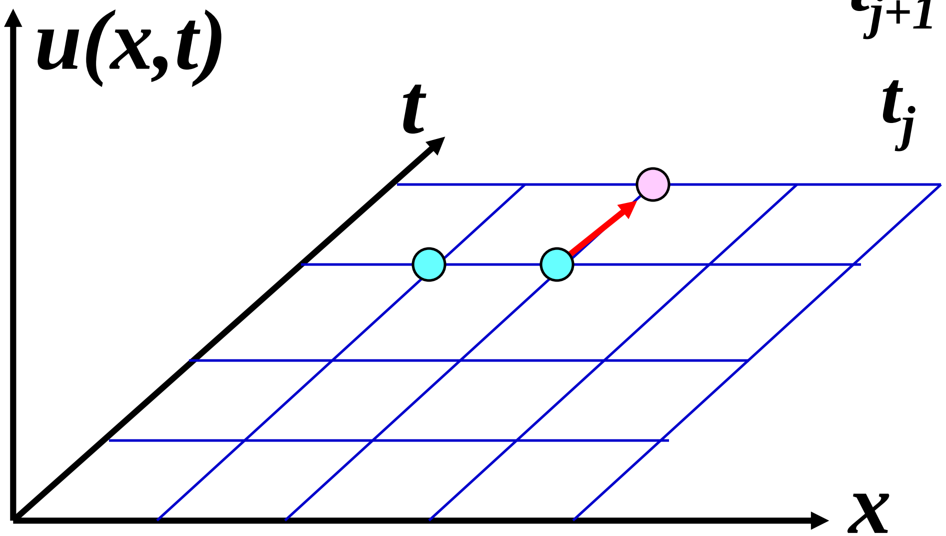


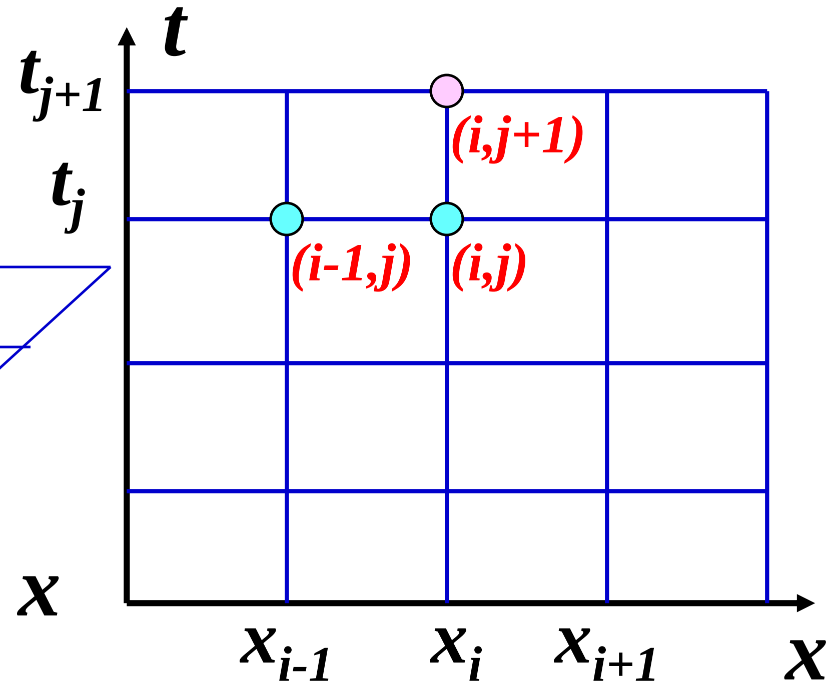
Dirichlet BC
$u(1, t) = f2(t)$

Initial Conditions: $u(x,0) = u_0(x)$

Euler Method (Explicit): I MODE

$$\frac{u_{i,j+1}-u_{i,j}}{k}+a\frac{u_{i,j}-u_{i-1,j}}{h}=0$$





FT Forward in time

$$u_t=\frac{1}{k}(u_{i,j+1}-u_{i,j})+O(k)$$

BS Backward in space
at time j

$$u_x=\frac{1}{h}(u_{i,j}-u_{i-1,j})+O(h)$$

Algorithm I MODE

Let u_{ij} be the approximation of $u(x_i,t_j)$,

$$u_{0,j} = f(t_j) \qquad\qquad j = 1,2,..$$

$$u_{i,0} = u_0(x_i) \qquad\qquad i = 0,1,.., N$$

$$\alpha = ak/h$$

$$for \quad j = 1,2,..$$

$$for \quad i = 0,1,2,.., N$$

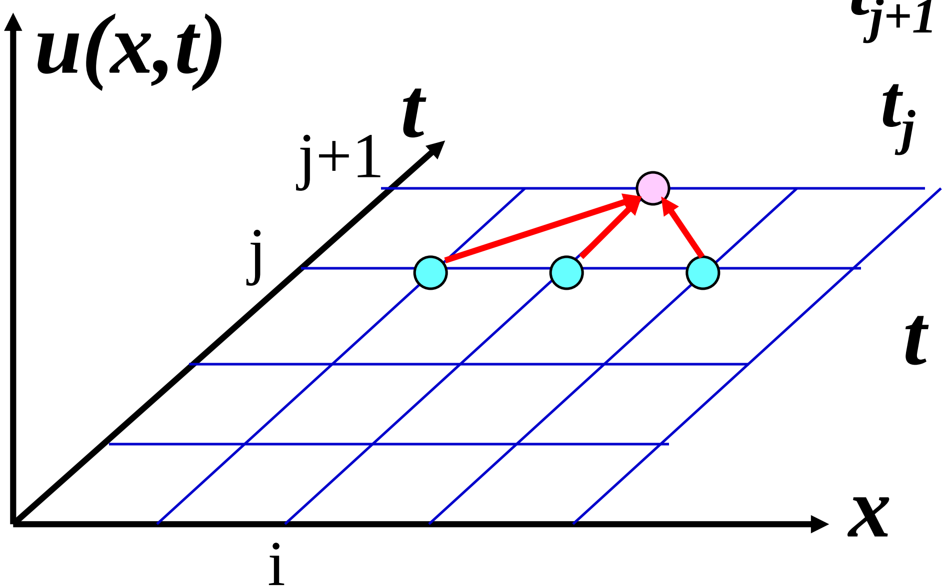
$$u_{i,j+1} = (1-\alpha)u_{i,j} + \alpha u_{i-1,j} \qquad j = 0,1,.. \quad i = 1,.., N$$

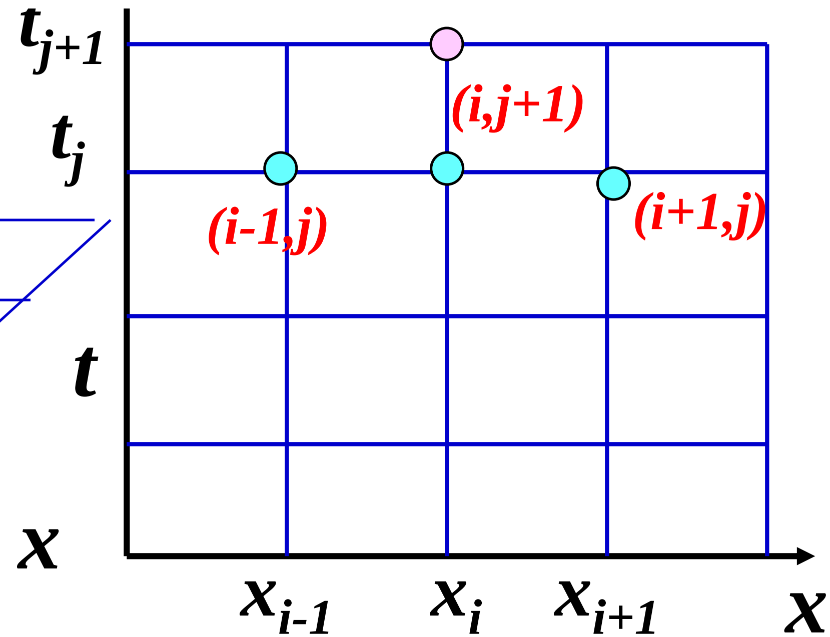
$$end$$

$$end$$

Euler Method (Explicit): II MODE

$$\frac{u_{i,j+1} - u_{i,j}}{k} + a\frac{u_{i+1,j} - u_{i-1,j}}{2h} = 0 \quad \rightarrow u_{i,j+1} = u_{i,j} - \frac{ak}{2h}(u_{i+1,j} - u_{i-1,j})$$





FT Forward Difference

$$u_t = \frac{1}{k}(u_{i,j+1} - u_{i,j}) + O(k)$$

CS Centered Difference in Space (at time j)

$$u_x = \frac{1}{2h}(u_{i+1,j} - u_{i-1,j}) + O\left(h^2\right)$$

NO Stability!!

Method of lines (MOL)

Consider a semidiscretization with FD in space of the PDE that provides a large system of ODEs with each component of the system that corresponds to the solution in a certain grid point as a function of time. Then we solve the system of ODEs using one of the methods already seen for ODE.

$$u_t + a u_x = 0 \qquad BC: u(0,t) = u(1,t)$$

$$U(t) = [U_1(t), U_2(t), \ldots, U_{n+1}(t)]$$
$$U_0(t) = U_{n+1}(t)$$

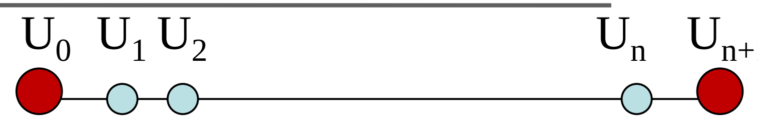
Apply MODE II Explicit Euler FT, CS

$$U_i'(t) = -a\frac{1}{2h}(U_{i+1}(t) - U_{i-1}(t)) \quad 1 \le i \le n+1$$

$$\boxed{U'(t) = AU(t)}$$

System of ODEs

STABILITY for Explicit Euler



$$U_1'(t) = -\frac{a}{2h}\left(U_2(t) - U_{n+1}(t)\right)$$

$$U_{n+1}'(t) = -\frac{a}{2h}\left(U_1(t) - U_n(t)\right)$$

$$A = -\frac{a}{2h}\begin{bmatrix} 0 & 1 & & -1 \\ -1 & 0 & 1 & \\ & -1 & 0 & 1 \\ 1 & & -1 & 0 \end{bmatrix}$$

$$A \in R^{(n+1)x(n+1)} \qquad U \in R^{(n+1)}$$

$$U'(t) = AU(t)$$

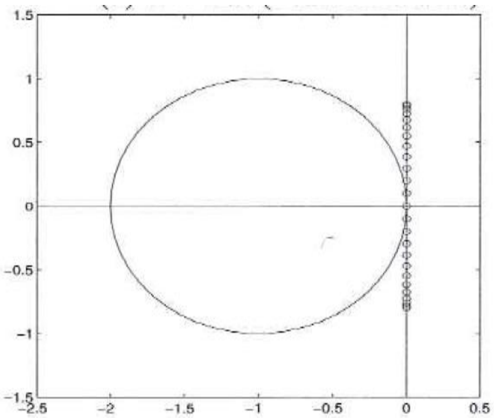
Discretize in time:

$$U^{j+1} = U^j + kAU^j$$

$$\Downarrow$$

$$U^{j+1} = (I + kA)U^j$$

STABILITY for Explicit Euler



$$eig(A)=\lambda_p=-\frac{ia}{h}\sin(2\pi ph) \quad p=1,..,n+1$$

$$-\frac{ia}{h}\le\lambda_p\le\frac{ia}{h}$$

stability for $\left|1+k\lambda_p\right|\le 1$

Ra= Region of absolute stability

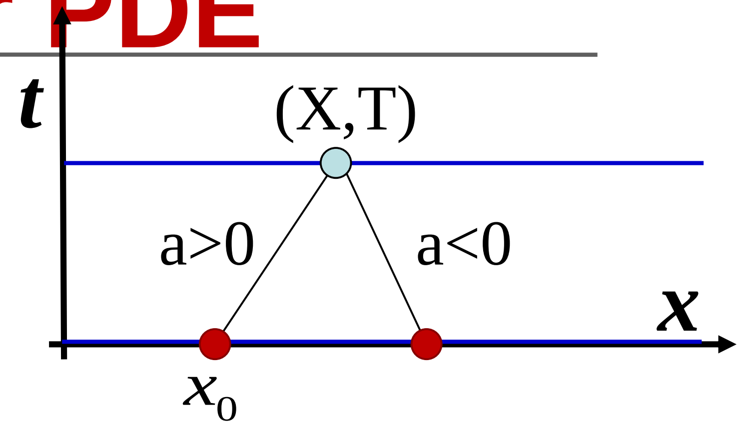
Since the eigenvalues are pure imaginary values, $k\lambda_p$ will not belong to Ra.

So the method is UNCONDITIONALLY UNSTABLE for any fixed ratio k / h!

Stability Analysis

Necessary Stability Condition (CFL)

Domain of dependence of the scalar PDE



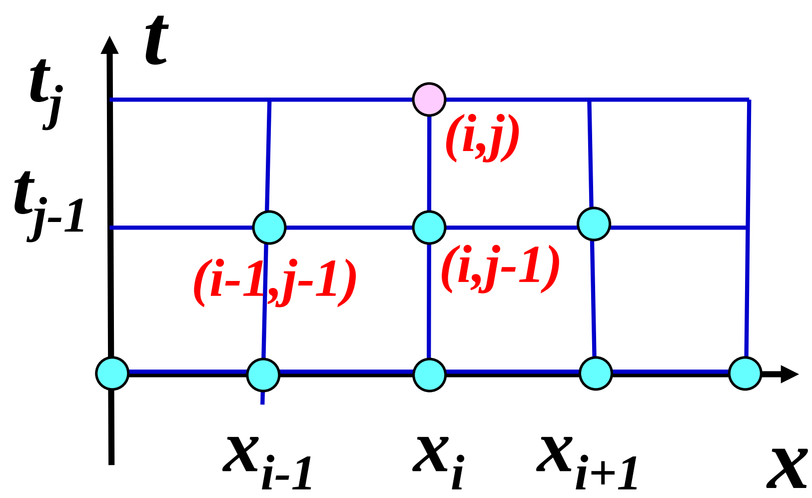
$$u_t + a(u)u_x = 0$$

- The solution u(X,T) at a certain point (X,T) only depends on the initial value u_0 at a point x_0 =X-aT s.t. (X,T) is on the characteristic curve for x_0.
- The domain of dependence of the point (X,T) is the set D(X,T)= $\{x_0\}$

  If we change the initial value at x_0 the solution u(X,T) changes, while changing the data in every other point does not affect the solution in (X,T).

Numerical Domain of dependence (of the FD method)

From a grid point (x_i,t_j) the numerical domain of dependence is given by the grid points x at the initial time t = 0 with the property that the data have effect on the solution $u_{i,j}$



u_{ij} depends on the initial values at the points $X_{i-j},..,X_{i+j}$
The numerical solution in (X,T) will converge to the exact solution

$$u(X,T) = u_0(X - aT)$$

Only if $X - Th/k \le X - aT \le X + Th/k \quad \rightarrow |ak/h| \le 1$

That is the numerical domain includes the exact domain of (X,T)

STABILITY: CFL condition

- Courant, Friedrichs and Lewy (1928) have shown that, a necessary condition for a numerical explicit scheme for the transport equation to be stable, is that the discretization step in space and time are related by the condition:

$$|a|\frac{k}{h} \leq 1$$

- For a hyperbolic system the CFL condition is

$$u_t + Au_x = 0 \quad u \in R^s, A \in R^{sxs}$$

$$eig(A) = \lambda_1, .., \lambda_s$$

$$max_{1 \leq p \leq s} \left| \lambda_p \frac{k}{h} \right| \leq 1$$

CFL Condition

$$u_t + u_x = 0 \qquad (a = 1)$$

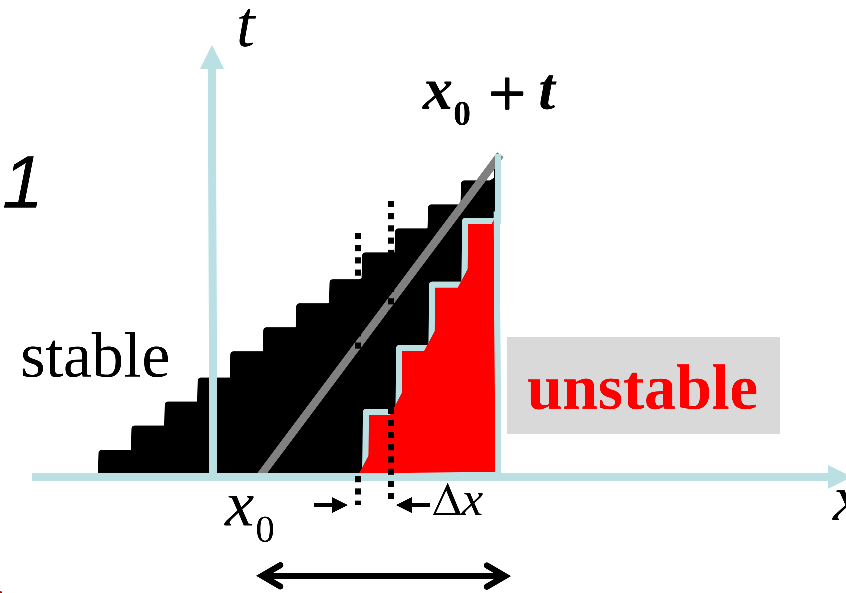
Solution $\quad u(x,t) = u(x-t,0)$

Characteristic slope *dx/dt=1*

CFL $\quad \dfrac{\Delta t}{\Delta x} \leq 1$

Explicit Euler (I Mode)

A numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE.



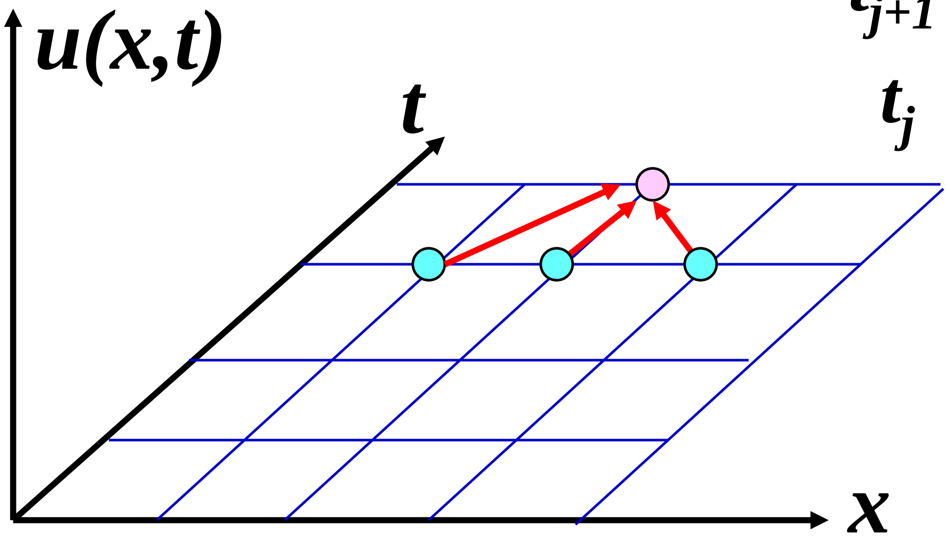
Domain of dependence of the solution

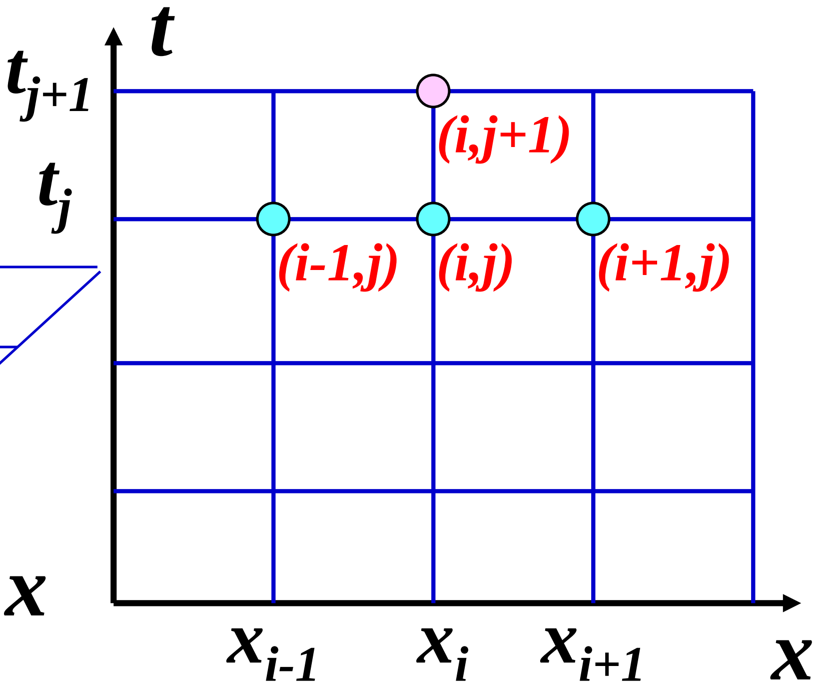
Numerical Domain of dependence

$\Delta t < \Delta x$

$\Delta t > \Delta x$

Upwind Method





- Consider one-sided approximation of the space derivative according to the flow direction
- The flow velocity a can be function of (x,t)

Upwind Method

$$u_t \approx \frac{u_{i,j+1} - u_{i,j}}{k}$$

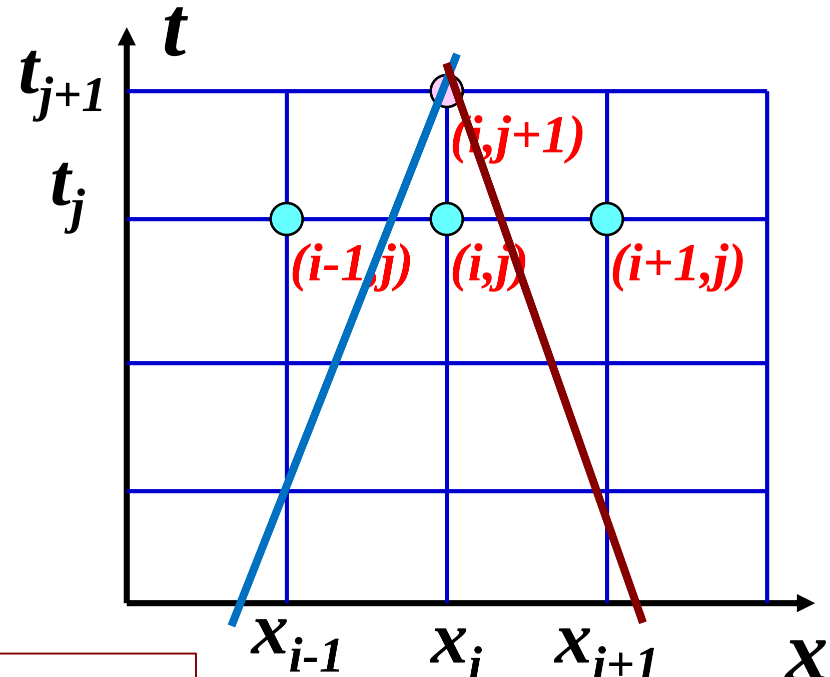
FD in time

Time j

$$u_x \approx \begin{cases} a\dfrac{u_{i+1,j} - u_{i,j}}{h} & \text{if } a < 0 \\ a\dfrac{u_{i,j} - u_{i-1,j}}{h} & \text{if } a > 0 \end{cases}$$

FD in space (if $a < 0$)

BD in space (if $a > 0$)



$$u_{i,j+1} = \begin{cases} u_{i,j} - a\dfrac{k}{h}(u_{i+1,j} - u_{i,j}) & \text{if } a < 0 \\ u_{i,j} - a\dfrac{k}{h}(u_{i,j} - u_{i-1,j}) & \text{if } a > 0 \end{cases}$$

$$u_{i,j+1} = u_{i,j} - k\left(max(a_i,0)D_-^x u_{i,j} + min(a_i,0)D_+^x u_{i,j}\right)$$

Upwind Method

The upwind method can be written as

$$\underbrace{u_{i,j+1} = u_{i,j} - ak\frac{(u_{i+1,j} - u_{i-1,j})}{2h}}_{\text{Original PDE}} \pm a\frac{kh}{2}\frac{(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})}{h^2}$$

Diffusive Term

Which is the FD discretization of the PDE

$$u_t + au_x = \varepsilon u_{xx} \qquad \varepsilon = \frac{ah}{2}$$

That is the advection-diffusion equation (diffusive term with diffusive coefficient ε that vanishes as h →0)

- Conditional Stable under CFL condition
- **First Order of Accuracy in space and time O(h+k)**

Convergence Theorem

A finite difference approximation scheme converges (towards the solution of the PDE) if and only if:

- The scheme is **consistent**

for k-> 0 the LTE $\rightarrow$ *0*, i.e., the FD scheme tends to the continuous differential PDE

- The scheme is **Lax Richtmyer stable**.

Stability and LTE

- the *forward Euler/centered (FTCS,* method II) is unconditionally unstable
- the *upwind* method, *Lax-Friedrichs* and *Lax-Wendroff* schemes are conditionally stable provided that the CFL condition is satisfied;
- the *backward Euler/centered* method is unconditionally stable

$$FT,CS \quad \alpha = ak/h$$

$$u_{i,j+1} + \frac{\alpha}{2}(u_{i+1,j+1} - u_{i-1,j+1}) = u_{i,j}$$

- truncation error for Lax-Friedrichs $O(h^2+k)$, Lax-Wendroff $O(h^2+k^2)$ and upwind $O(h+k)$ methods

Lax-Friedrichs Method

Peter Lax, born 1926

A simple way to stabilize the FTCS method has been proposed by Peter Lax:

$u_{i,j}$ replaced by the average $\frac{1}{2}(u_{i+1,j}+u_{i-1,j})$

$$\boxed{u_{i,j+1}=\underbrace{\frac{1}{2}(u_{i+1,j}+u_{i-1,j})}_{u_{i,j}}-\frac{ak}{2h}(u_{i+1,j}-u_{i-1,j})}$$

We write the terms a bit different:

$$\frac{1}{2}(u_{i+1,j}+u_{i-1,j})=u_{i,j}+\frac{1}{2}(u_{i-1,j}-2u_{i,j}+u_{i+1,j})$$

$$u_{i,j+1}=u_{i,j}-\frac{ak}{2h}(u_{i+1,j}-u_{i-1,j})+\frac{1}{2}(u_{i-1,j}-2u_{i,j}+u_{i+1,j})$$

$$\underbrace{\frac{u_{i,j+1}-u_{i,j}}{k}+a\left(\frac{u_{i+1,j}-u_{i-1,j}}{2h}\right)}_{\text{Original PDE}}=\frac{h^2}{2k}\left(\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}\right)$$

Diffusive Term

Lax-Friedrichs Method

- But it solves the wrong PDE!

$$u_t + au_x = \varepsilon u_{xx} \qquad \varepsilon = \frac{h^2}{2k}$$

artificial viscosity

How bad is that?

- Answer: Not that bad.
  The dissipative term mainly damps small spatial structures on grid resolution, which we are not interested in => **Numerical dissipation**
- The unstable FTCS-method blows this small scale structures up and spoils the solution.
- Lax-Richtmeyer stable numerical scheme (if CFL fulfilled)

MOL: $\boxed{U'(t) = A_\varepsilon U(t)}$ BC periodic $U_0 = U_{n+1}$

$$A_\varepsilon = -\frac{a}{2h}\begin{bmatrix} 0 & 1 & & -1 \\ -1 & 0 & 1 & \\ & .. & .. & 1 \\ 1 & & -1 & 0 \end{bmatrix} + \frac{\varepsilon}{h^2}\begin{bmatrix} -2 & 1 & & 1 \\ 1 & -2 & 1 & \\ & 1 & .. & 1 \\ 1 & & 1 & -2 \end{bmatrix}$$

FT: Explicit Euler

$$U_{j+1} = (I + kA_\varepsilon)U_j \qquad A_\varepsilon \in R^{(n+1)x(n+1)} \quad U \in R^{(n+1)}$$

$$\lambda_p = -\frac{ia}{h}\sin(2\pi ph) - \frac{2\varepsilon}{h^2}(1 - \cos(2\pi ph)) \quad p = 1, 2, .., n+1$$

λ_p *lie on the ellipse centered at* $c = -2k\varepsilon / h^2$

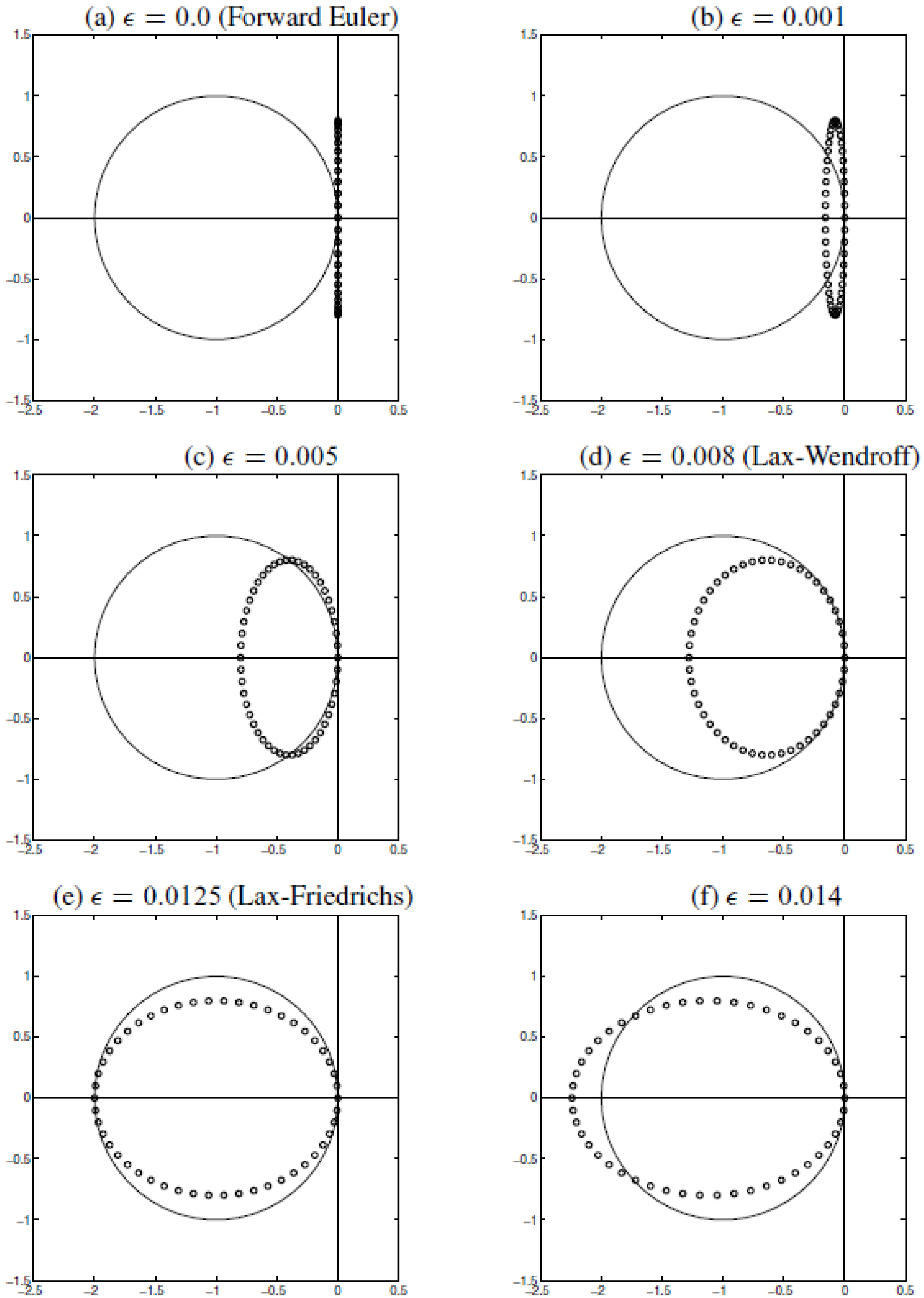
since $\varepsilon = h^2 / 2k \quad c = -1$ *this ellipse lies entirely inside the unit circle centered at -1*

Stability provided $|ak / h| \le 1$ *(if* $|ak / h| > 1$ *the ellipse is out)* h=1/50

k=0.8h

a=1

ak/h=0.8



$$\varepsilon = \frac{a^2 k}{2}$$

$$\varepsilon = \frac{h^2}{2k}$$

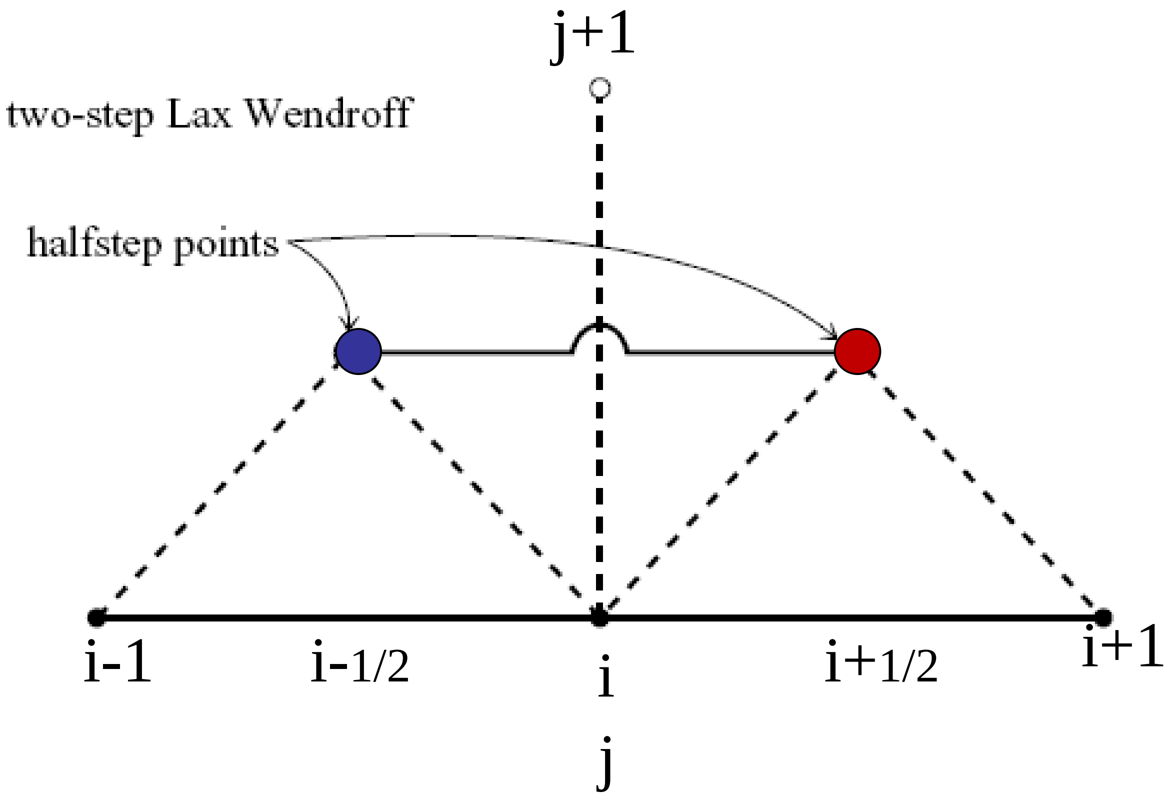
Lax-Wendroff Method

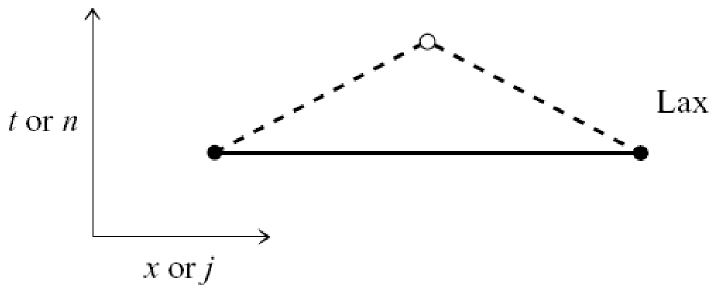
2 step method based on LaxF Method

- Apply first one step “Lax step” but advance only half a time step.
- Compute fluxes at this points $\mathbf{t}^{j+1/2}$
- Now advance to step $\mathbf{t}^{j+1}$ by using points at $\mathbf{t}^{j}$ and $\mathbf{t}^{j+1/2}$
- Intermediate results at $\mathbf{t}^{j+1/2}$ not needed anymore.

Scheme is **second order** in space and time.

Lax-Wendroff Method





Lax-Wendroff Method

$$u_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}(u_{i+1,j} + u_{i,j}) - \frac{ak}{2h}(u_{i+1,j} - u_{i,j})$$

$$u_{i-\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}(u_{i,j} + u_{i-1j}) - \frac{ak}{2h}(u_{i,j} - u_{i-1,j}) \qquad (1)$$

Compute the flux in $\mathbf{t}^{\mathbf{j+1/2}}$ then:

$$u_{i,j+1} = u_{i,j} - \frac{ak}{h}(u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i-\frac{1}{2},j+\frac{1}{2}}) \qquad (2)$$

Lax-Wendroff Method

By replacing (1) in (2) we have

$$\underbrace{u_{i,j+1} = u_{i,j} - ak\frac{(u_{i+1,j} - u_{i-1,j})}{2h}}_{\text{Original PDE}} + a^2\frac{k^2}{2}\frac{(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})}{h^2}$$

Diffusive Term

That can be seen as the central differences with an additional numerical diffusion with diffusion coefficient very small ($a^2k/2$)

$$u_t + au_x = \varepsilon u_{xx} \qquad \varepsilon = \frac{a^2k}{2}$$

- Stable if CFL-condition fulfilled.
- Still diffusive, but here this is only 4th order in k, compared to 2th order for Lax method.

=> Much smaller effect.

MOL: $$\boxed{U'(t) = A_\varepsilon U(t)}$$ BC periodic $U_0=U_{n+1}$

$$A_\varepsilon = -\frac{a}{2h}\begin{bmatrix} 0 & 1 & & -1 \\ -1 & 0 & 1 & \\ & .. & .. & 1 \\ 1 & & -1 & 0 \end{bmatrix} + \frac{\varepsilon}{h^2}\begin{bmatrix} -2 & 1 & & 1 \\ 1 & -2 & 1 & \\ & 1 & .. & 1 \\ 1 & & 1 & -2 \end{bmatrix}$$

FT:Explicit Euler

$$U_{j+1} = (I + kA_\varepsilon)U_j \qquad A_\varepsilon \in R^{(n+1)x(n+1)} \quad U \in R^{(n+1)}$$

$$\lambda_p = -\frac{ia}{h}\sin(2\pi ph) - \frac{2\varepsilon}{h^2}(1-\cos(2\pi ph)) \quad p = 1,2,..,n+1$$

$k\lambda_p$ *lie on an ellipse centered at* $c = -(ak/h)^2$

If $|ak/h| \le 1$ *then* $k\lambda_p$ *lies inside Ra of Explicit Euler*

Numerical Stabilization

The three methods upwind (up), Lax-Friedrichs (LF) e Lax-Wandroff (LW) can be written in advection-diffusion form with different diffusion coefficients:

$$\boxed{u_t + a u_{xx} = \varepsilon u_{xx}} \qquad \varepsilon_{LW} = \frac{a^2 k}{2} \qquad \varepsilon_{up} = \frac{ah}{2} \qquad \varepsilon_{LF} = \frac{h^2}{2k}$$

$$If \quad 0 < \frac{ak}{h} < 1, then \quad \varepsilon_{LW} < \varepsilon_{up} < \varepsilon_{LF}$$

and the method is stable for every ε value

$$\varepsilon_{LW} \leq \varepsilon \leq \varepsilon_{LF}$$

h=1/50

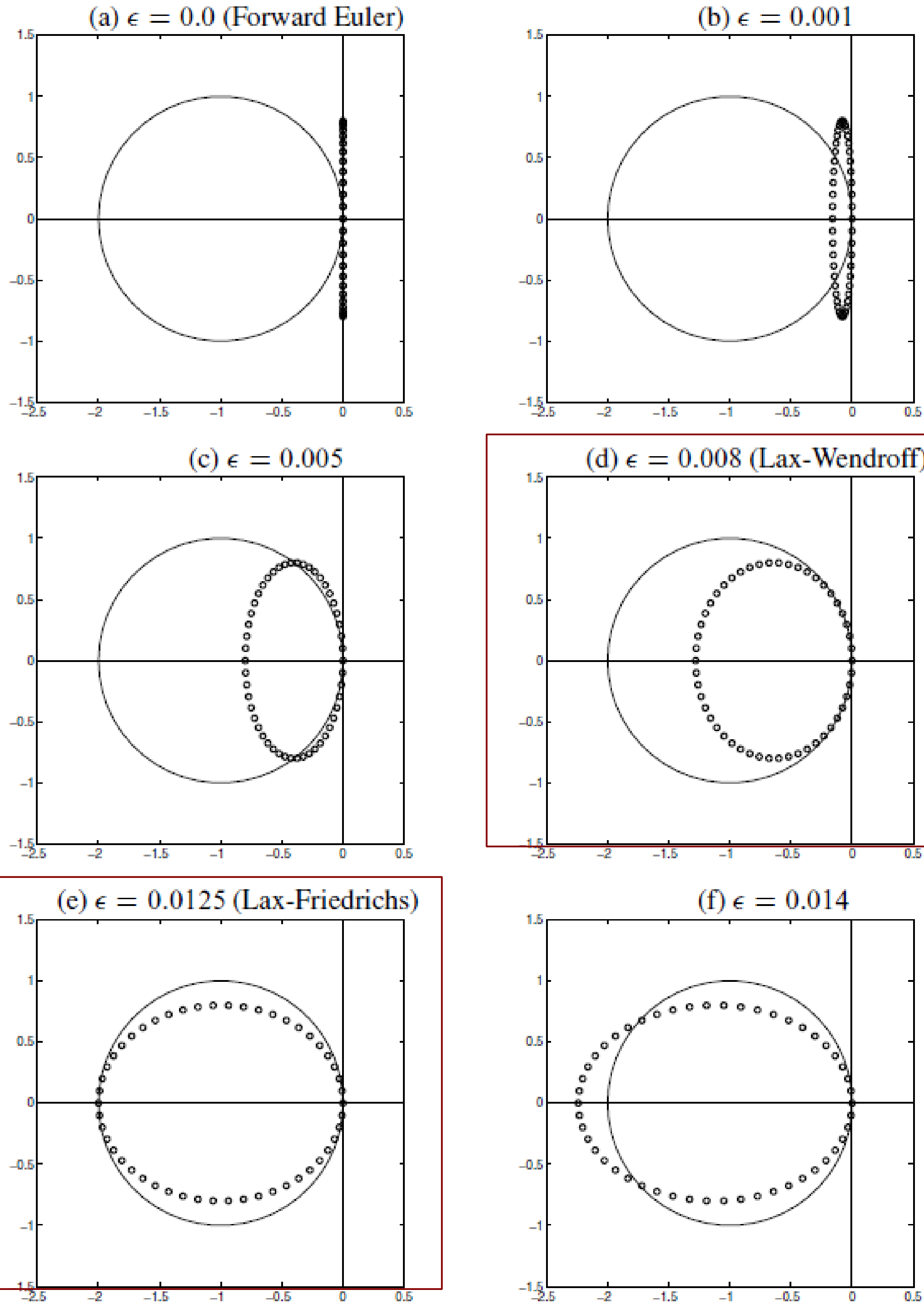
k=0.8h

a=1

ak/h=0.8

$$\varepsilon = \frac{a^2 k}{2}$$

$$\varepsilon = \frac{h^2}{2k}$$



Numerical Stabilization

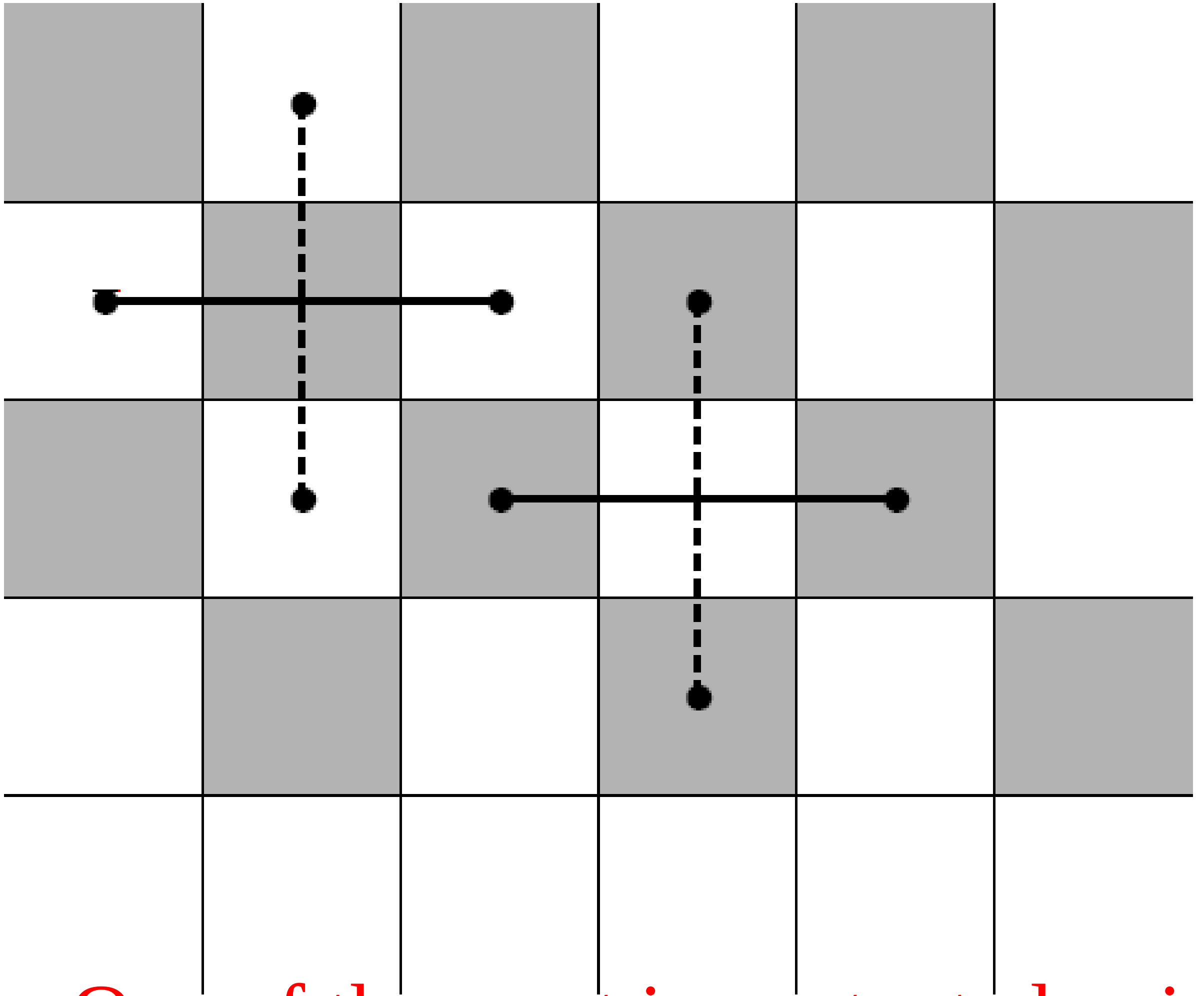
- Add a term of artificial diffusion in the direction of the field V:

$$\frac{du}{dt} + \vec{V} \bullet \nabla u = \mu \Delta u$$

The coefficient of viscosity is chosen proportional to the spatial step μh

- The artificial viscosity tends to zero as h→ 0, while preserving the consistency of the method

Leap-Frog Method



Children playing leapfrog Harlem, ca. 1930.

Scheme uses second order **central differences** in space and time.

One of the most important classical methods.

Leap-Frog Method

CT in time $u_t = \frac{1}{2k}(u_{i,j+1} - u_{i,j-1}) \quad + O\left(k^2\right)$

CS in space $u_x = \frac{1}{2h}(u_{i+1,j} - u_{i-1,j}) \quad + O\left(h^2\right)$

$$\boxed{u_{i,j+1} = u_{i,j-1} - \frac{ak}{h}(u_{i+1,j} - u_{i-1,j})}$$

- Explicit
- Consistent (accuracy of second order in space and time)
- Requires storage of previous time step.
  (3 levels method)

Leap-Frog Method

$$\boxed{u_{i,j+1} = u_{i,j-1} - \frac{ak}{h}(u_{i+1,j} - u_{i-1,j})}$$

- Corresponds to the **midpoint method** for ODE

$$U_{j+1} = U_{j-1} + 2kAU_j$$

with Ra defined in the interval imaginary axis

$$i\alpha, \quad -1 < \alpha < 1$$

- Stability under CFL-condition
- **No amplitude diffusion, but possible dispersion**

Von Neumann Analysis

- Decomposes the solution of the problem in the Fourier series, assuming that it is periodic of period 2π
- Consider the expansion of the periodic initial data

$$u_0(x) = \sum_{k=-\infty}^{\infty} \alpha_k e^{ikx}$$

kth Fourier Coefficient

- The numerical approximation of a FD explicit scheme for the transport problem satisfies

$$u_j^n = u^n(x_j) = \sum_{k=-\infty}^{\infty} \gamma_k^n \alpha_k e^{ikjh}, \quad j = 0, \pm 1, \pm 2, .. \; n = 1, 2, ..$$

γ_k amplification coefficient of the k-th harmonic

Von Neumann Analysis

The **exact solution** of a transport problem in general can be written in the form

$$u(x,t^n) = u_0(x - cn\Delta t), \quad t^n = n\Delta t,$$

$$u(x_j,t^n) = \sum_{k=-\infty}^{\infty} g_k^n \alpha_k e^{ikjh},$$

$g_k^n = e^{-cik\Delta t}$ complex coefficient of unit magnitude

While $|g_k| = 1\,, \quad |\gamma_k| \le 1$ is a necessary and sufficient condition for a given numerical scheme to satisfy the stability

$$\forall \mathrm{k}, \quad |\gamma_k| \le 1 \quad if \;\; \Delta t \le \frac{h}{|a|}$$

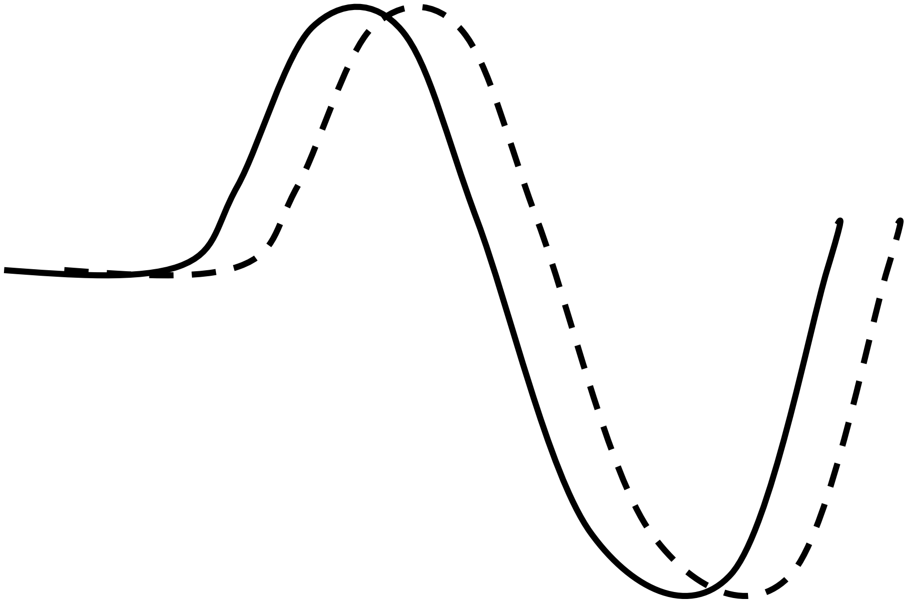
Von Neumann Analysis

Numerical Error: compare γ_k vs. g_k in terms of modules and **phase angles** $\rightarrow$ $\phi_k = k\Delta x$

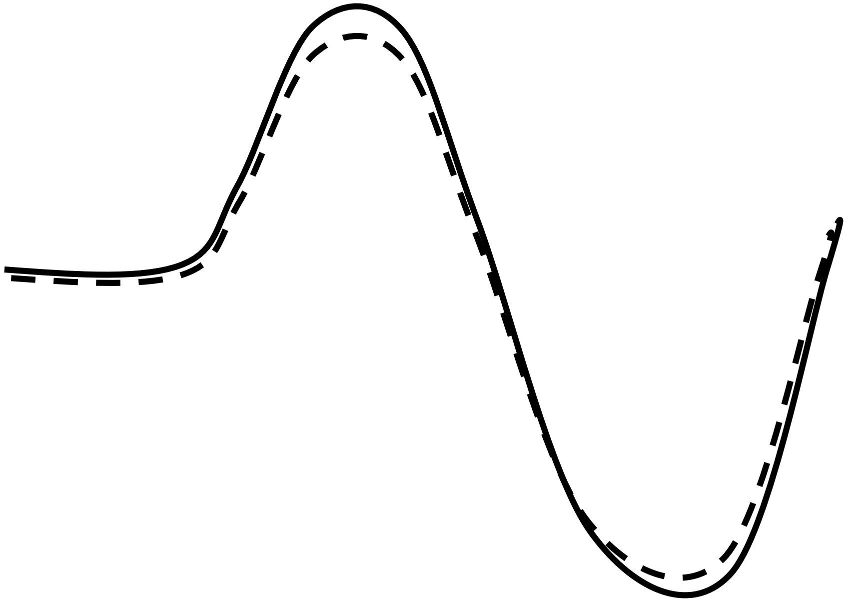
$$\mathrm{E}_{\mathrm{a,k}} \equiv \frac{|\gamma_k|}{|g_k|}, \qquad \mathrm{E}_{\mathrm{d,k}} \equiv \frac{\prec(\gamma_k)}{\prec(g_k)} = \frac{\omega}{kc}, \qquad \gamma_k = |\gamma_k| \mathrm{e}^{-i\frac{\omega}{k}k\Delta t}$$

$\frac{\omega}{k}$ Speed of propagation of the numerical solution (for the kth armonic)

- $\mathrm{E}_{\mathrm{a,k}}$ **Error of dissipation (or amplification)** effects of discretization on the amplitude of the k-th harmonic.
- $\mathrm{E}_{\mathrm{d,k}}$ **Error of dispersion** measures the effects on the phase of the k-th harmonic, i.e. on the speed of propagation.



Dispersion effects: that is either a delay or an advance in the wave propagation.

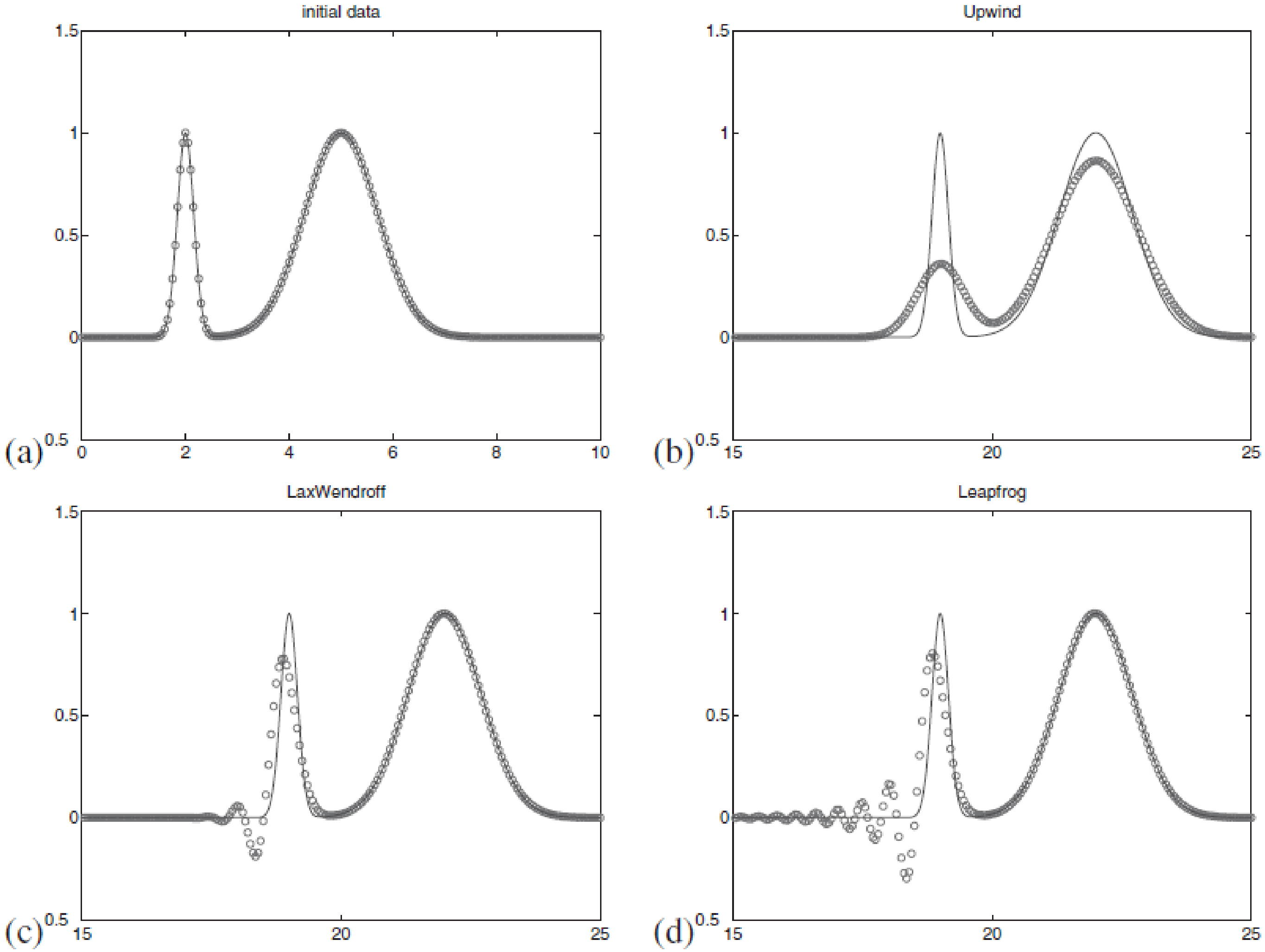


Effects of dissipation: dumping of the wave amplitude

Example

$$\begin{cases} u_t + u_x = 0 \\ 0 \le x \le 25,\ 0 \le t \le 17 \\ IC: u(x,0) = \exp(-20(x-2)^2) + \exp(-(x-5)^2) \end{cases}$$

$$h = 0.05,\ k = 0.8h$$



Esempio 2D $u_t = |\nabla u|$

Curva che si propaga lungo la normale con velocità costante V_N=1

$$u_t + \left\langle \nabla u, \vec{V} \right\rangle = u_t + \left\langle \nabla u, V_N \vec{N} \right\rangle = u_t + V_N \left\langle \nabla u, \frac{\nabla u}{|\nabla u|} \right\rangle = u_t + V_N |\nabla u| = 0$$

$$Sia \quad \Delta x = \Delta y = 1 \quad \Rightarrow$$

$$CFL = V_N \frac{\Delta t}{h}, \quad h = \sqrt{\frac{1}{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}}} : quindi \quad \Delta t \leq \frac{1}{\sqrt{2}}$$

Schema numerico (upwind)

$$u_{i,j}^{k+1} = u_{i,j}^k + \Delta t\Big((max(-D_-^x u_{i,j}^k, D_+^x u_{i,j}^k, 0))^2 + (max(-D_-^y u_{i,j}^k, D_+^y u_{i,j}^k, 0))^2 \Big)^{1/2}$$

Numerical Methods for Hyperbolic Linear Systems

$$U_t + AU_x = 0 \qquad \textbf{(*)}$$

A is a constant matrix

The system is called hyperbolic if A is diagonalizable with real eigenvalues, so that we can decompose

$$A = H\Lambda H^{-1},$$

$$\Lambda \equiv diag(\lambda_1, \ldots, \lambda_p) \quad \lambda_i \in R$$

$$H \equiv (h^1, h^2, \ldots, h^p)$$

H: matrix of right eigenvectors of A

$$Ah^k = \lambda_k h^k$$

We rewrite **(*)** in the form:

$$H^{-1}U_t + H^{-1}H\Lambda H^{-1}U_x = 0$$

$$sia \quad w = H^{-1}U$$

characteristic variables

$$\Rightarrow \quad \frac{\partial w}{\partial t} + \Lambda \frac{\partial w}{\partial x} = 0$$

Numerical Methods for Hyperbolic Linear Systems

This decouples into p independent scalar transport equations:

$$\frac{\partial w_k}{\partial t} + \lambda_k \frac{\partial w_k}{\partial x} = 0 \qquad k = 1, \ldots, p$$

(solve by the methods discussed earlier)

Every solution w_k is constant along the kth characteristic

$$solution \quad w_k(x,t) = w_k(0, x - \lambda_k t)$$

The solution to the original system (*) is finally recovered via

$$u = Hw$$

$$u(x,t) = \sum_{k=1}^{p} w_k(0, x - \lambda_k t) h^k$$

The solution depends only on the initial data at the p points $x-\lambda_p t$

1D wave equation (second order hyperbolic PDE)

Example: model of a vibrating elastic rope of length (b-a), fixed at the ends, c coefficient dependent on the specific mass of the rope and on its tension, the rope is subjected to a vertical force of density f. The solution u represents the vertical displacement

$$\begin{cases} u_{tt} - c^2 u_{xx} = f \\ IC: u(x,0) = u_0(x) \\ \qquad u_t(x,0) = v_0(x) \\ BC: \quad u(a,t) = 0, u(b,t) = 0 \end{cases} \qquad x \in (a,b)$$

solution $u(x,t) = u_0(x+ct) + u_1(x-ct)$

The kinetic energy of the system is preserved

Wave equation: convert in a first order hyperbolic system

change of variables: $w_1 = u_x, \quad w_2 = u_t$

The second order PDE is transformed into a system of 2 Hyperbolic first order independent PDE:

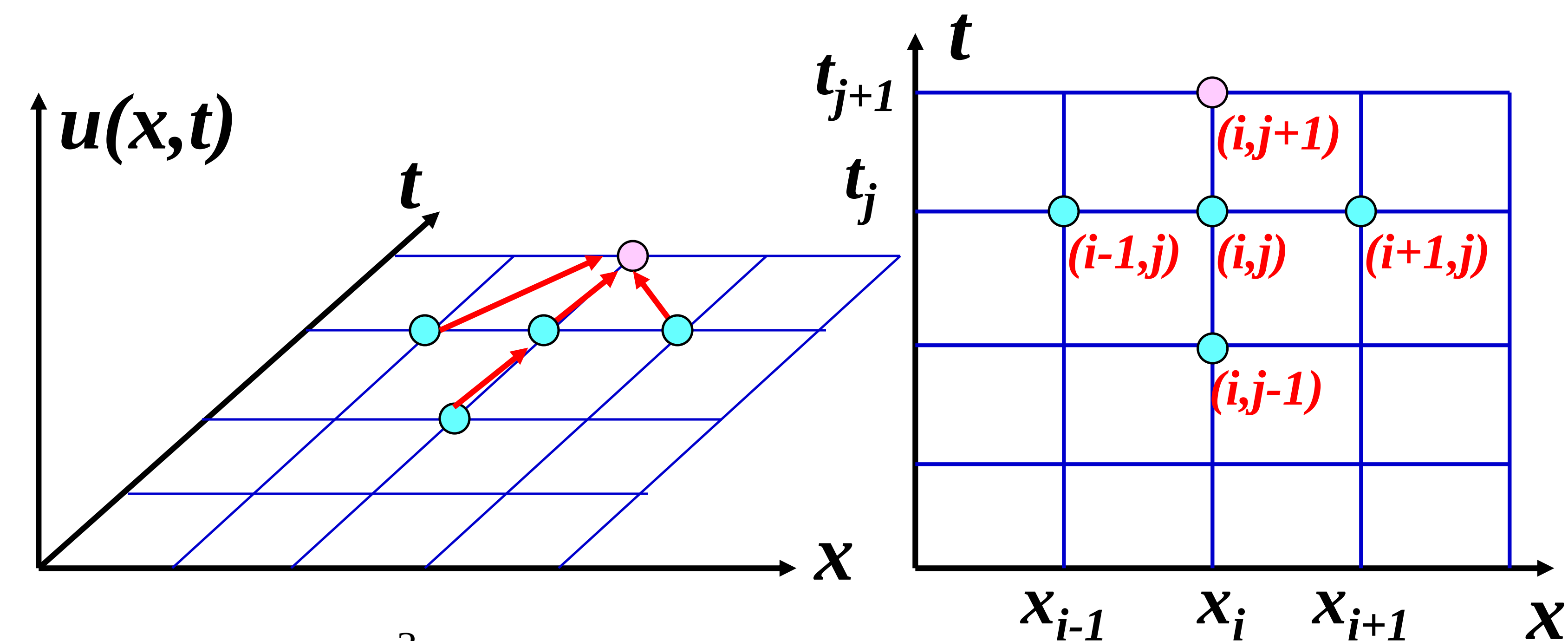
$$\frac{\partial \mathrm{w}}{\partial \mathrm{t}} + A\frac{\partial w}{\partial x} = 0 \quad x \in (a,b)$$

$$w = [w_1 \ w_2]^T \qquad A = \begin{bmatrix} 0 & -1 \\ -c^2 & 0 \end{bmatrix}$$

$$CI: \quad w_1(x,0) = u_0'(x), \quad w_2(x,0) = v_0(x)$$

Solve each scalar PDE by a method for advection eq.(eg Upwind)

Wave Equation: Explicit Method



CT

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_j) = \frac{1}{k^2}(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}) + O(k^2)$$

CS
At time j

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) = \frac{c^2}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + O(h^2)$$

Explicit Method

$$\begin{cases} h = \Delta x = 1/n, & \mathrm{x_i} = ih \\ k = \Delta t = T/m, & \mathrm{t_j} = jk \end{cases}$$

$$\frac{1}{k^2}(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}) = c^2 \frac{1}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

Define: $r = c\dfrac{k}{h}$

$$u_{tt} - c^2 u_{xx} = 0$$

$$\Downarrow$$

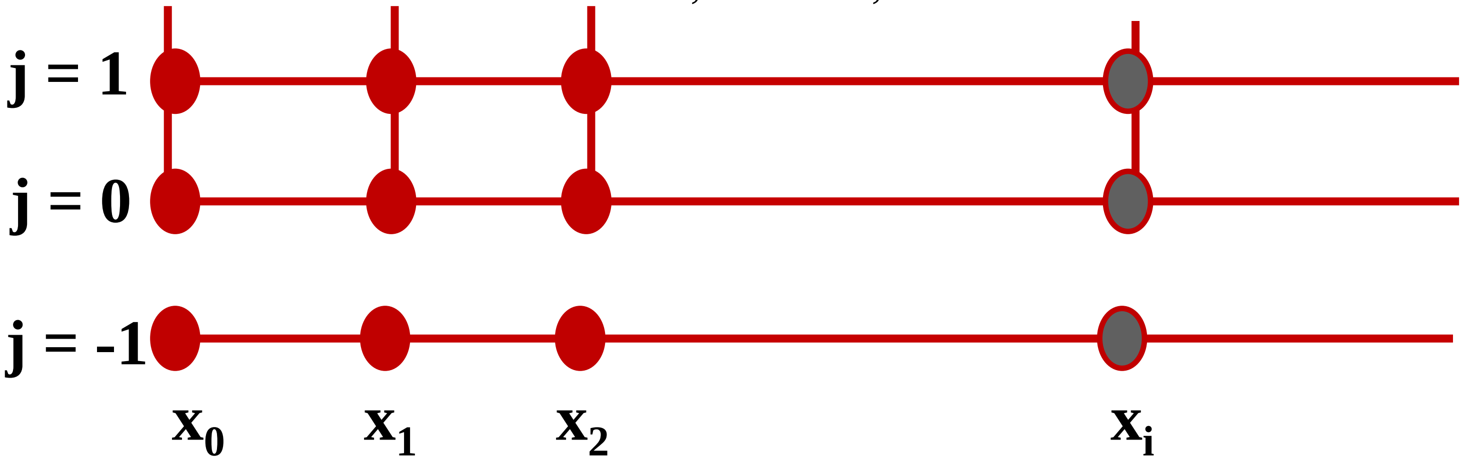
$$u_{i,j+1} = r^2 u_{i-1,j} + 2(1 - r^2)u_{i,j} + r^2 u_{i+1,j} - u_{i,j-1}$$

Explicit Method

Centered Difference for the Initial Condition

$$u_t(x,0) = v_0(x) \quad \Rightarrow \quad \frac{u_{i,1} - u_{i,-1}}{2k} = v_0(x_i)$$

$$\text{ghost point:} \quad u_{i,-1} = u_{i,1} - 2kv_0(x_i)$$



Replace $u_{i,-1}$ with the given relation in the numerical scheme for the node j=0

Explicit Method

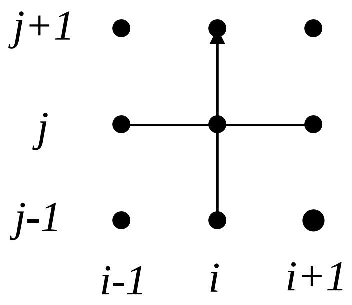
$$u_{i,-1} = u_{i,1} - 2kv_0(x_i)$$

$$\begin{cases} u_{i,0} = u_0(x_i) & \text{CI} \\ u_{i,1} = \dfrac{r^2}{2}u_{i-1,0} + (1-r^2)u_{i,0} + \dfrac{r^2}{2}u_{i+1,0} + kv_0(x_i) & j = 0 \\ \boxed{u_{i,j+1} = r^2 u_{i-1,j} + 2(1-r^2)u_{i,j} + r^2 u_{i+1,j} - u_{i,j-1} \qquad j = 1,2,.. \quad i = 1,..,n-1} \\ u_{0,j} = 0, u_{n,j} = 0 & \text{CB} \end{cases}$$

Conditional stability

Wave Equation: Numerical Solution

$$\boxed{u_{i,j+1} = r^2 u_{i-1,j} + 2(1-r^2)u_{i,j} + r^2 u_{i+1,j} - u_{i,j-1}}$$



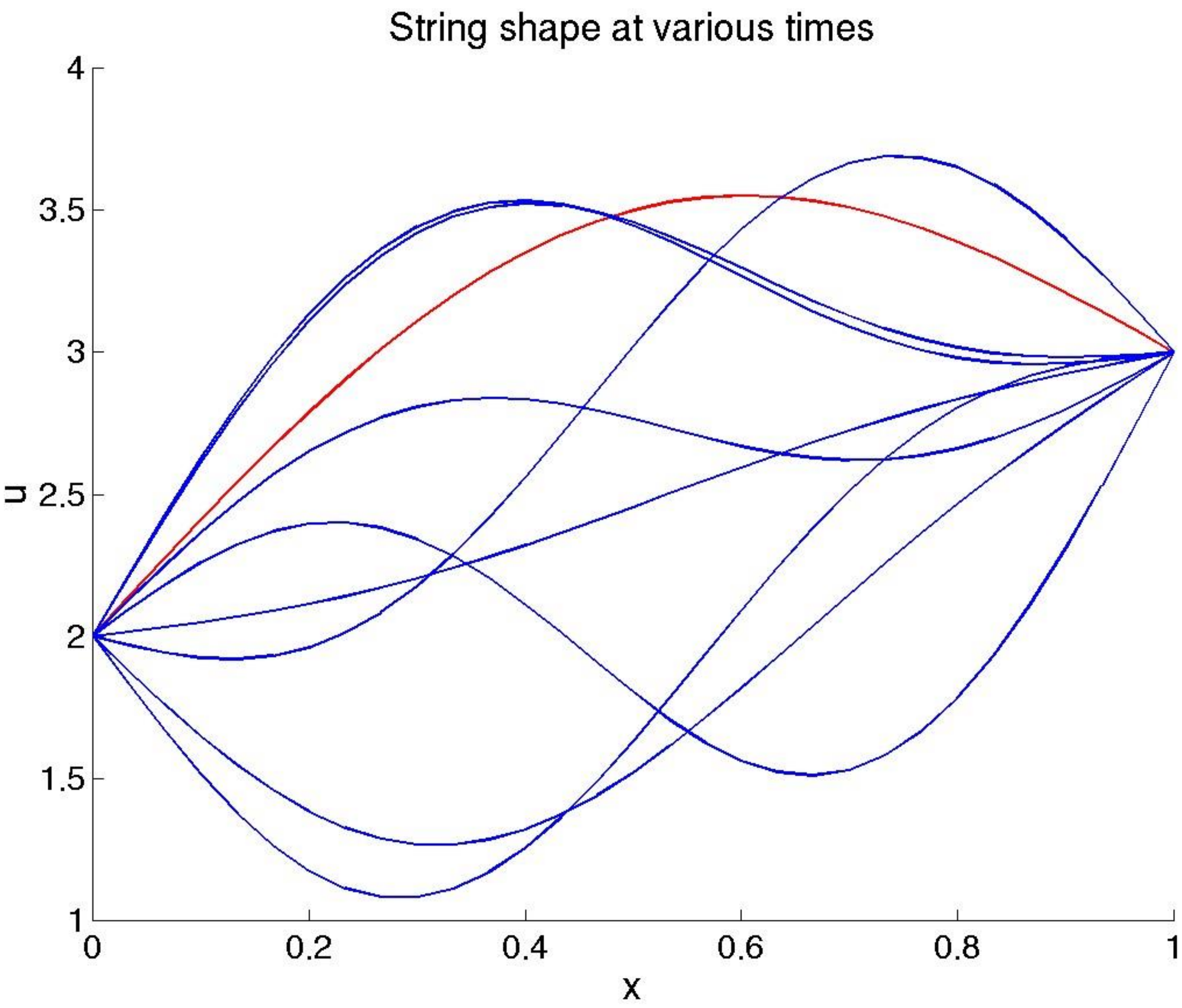

```
u0 = ...
u1 = ...
r  = c*(dt/dx)
for t = 2*dt:dt:endt
   u2(2:n) = r^2*u1(1:n-1)+
             2*(1-r^2)*u1(2:n)+
             r^2*u1(3:n+1)-u0(2:n)
   u0 = u1;
   u1 = u2;
end
```

Example: Wave Equation

$$u_{tt} = c^2\, u_{xx} \quad \text{for } 0 \le x \le 1,\ t \ge 0$$

- IC: $u(0,x) = \sin(\pi x)+x+2,$
  $u_t(0,x) = 4\sin(2\pi x)$
- BC: $u(t,0) = 2,\ u(t,1)=3$
- $c = 1$ propagation speed
- unknown: $u(t,x)$
- discretize unknown function: $u_j^k \approx u(k\Delta t, j\Delta x)$

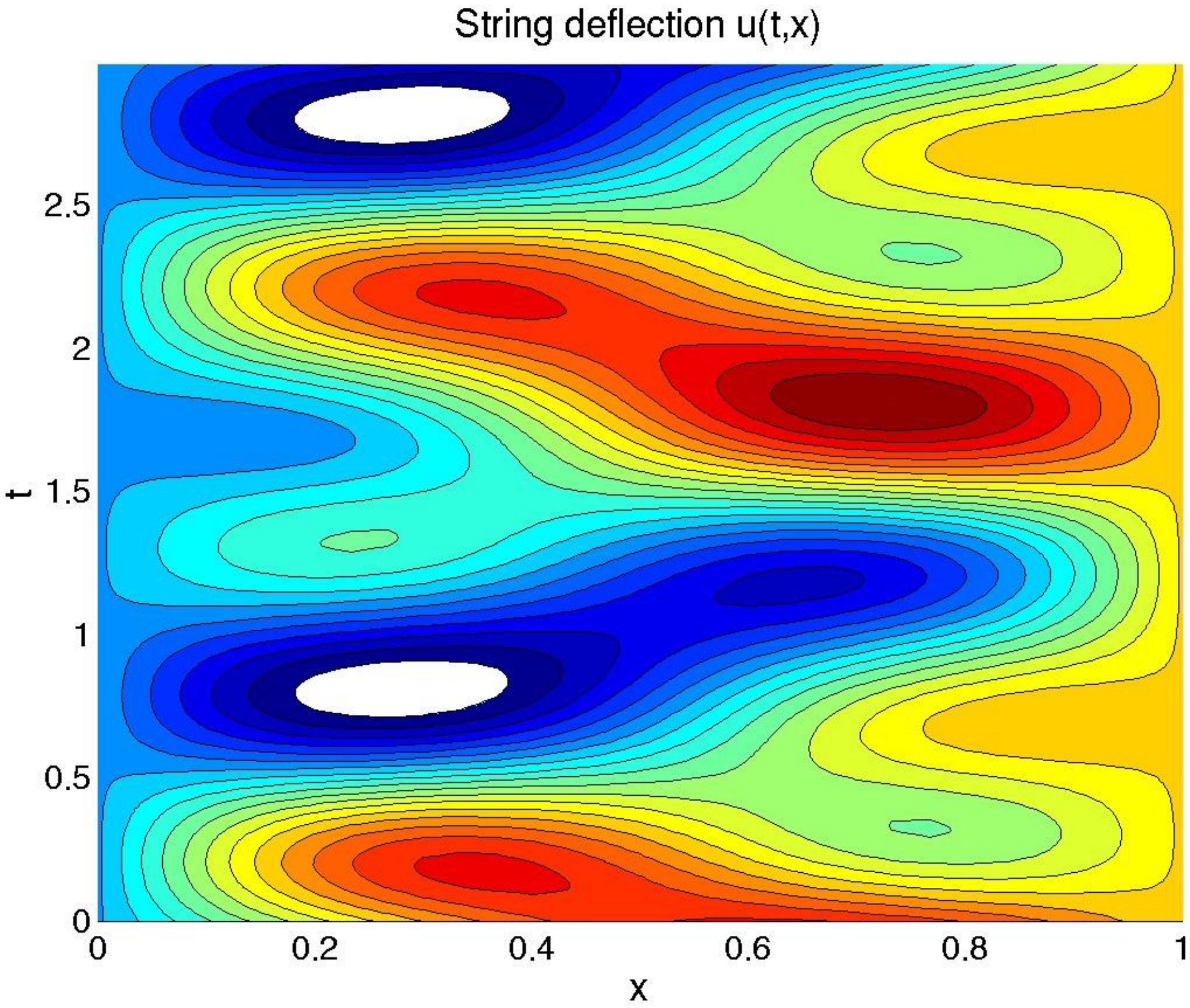
Wave Equation Results



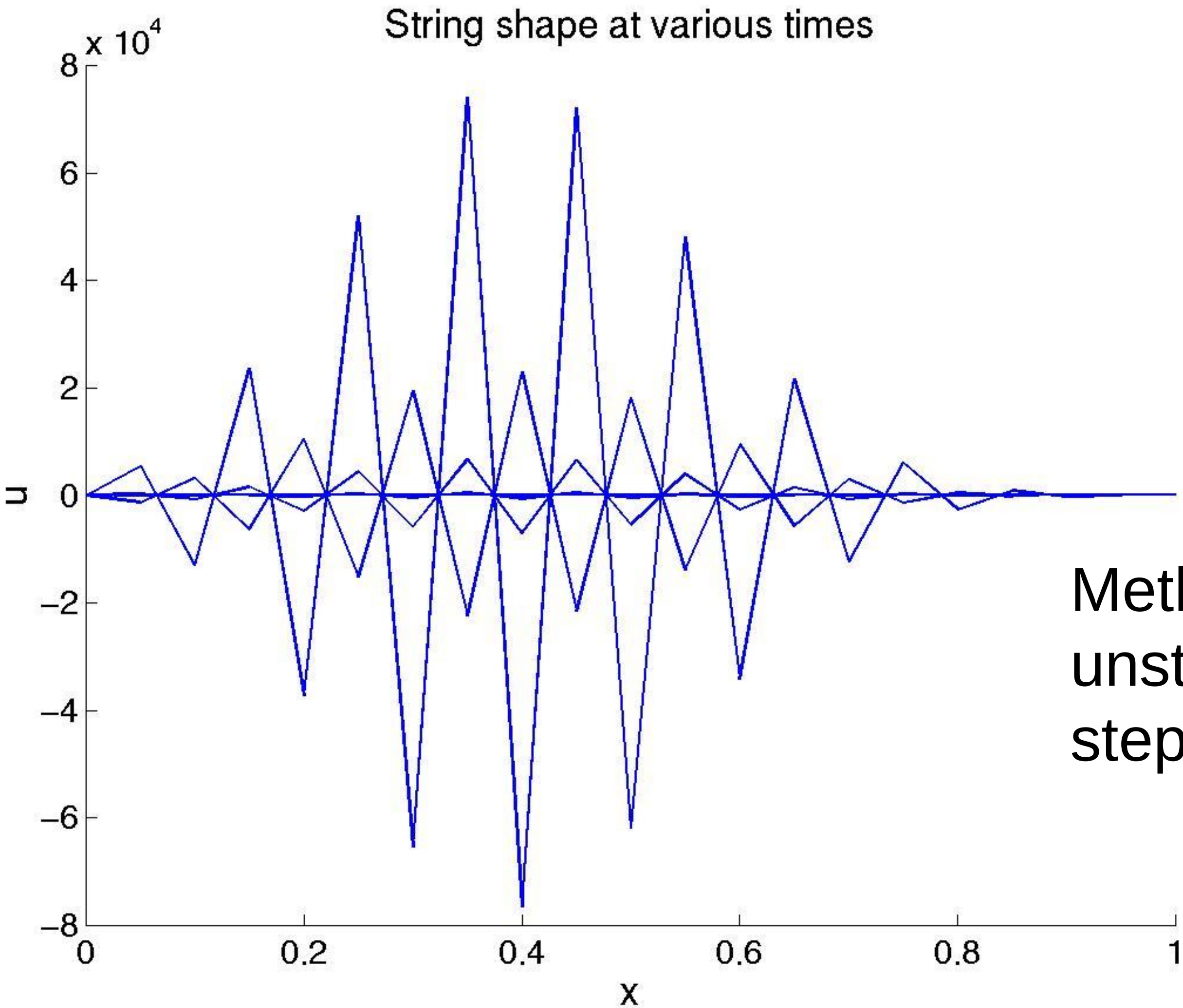
dx=1/30

dt=.01

Wave Equation Results



Poor results when dt too big

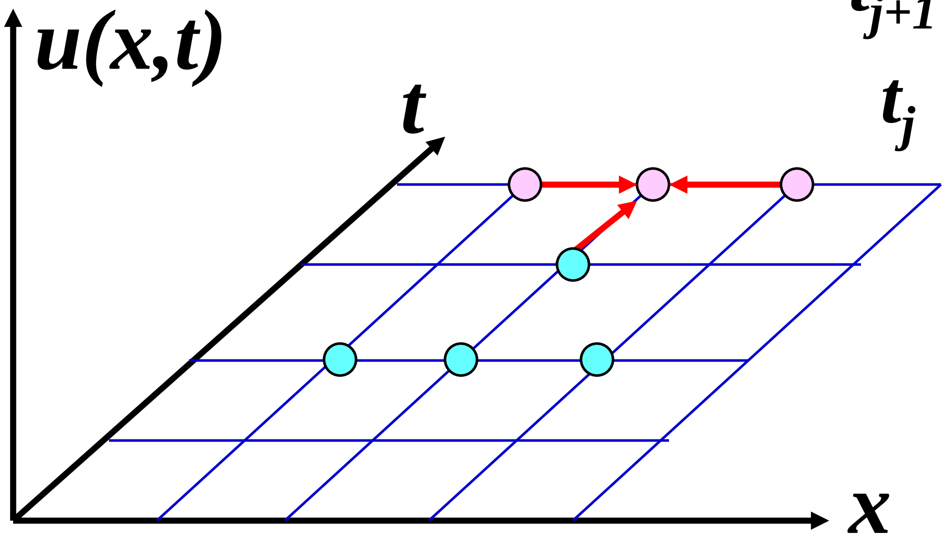


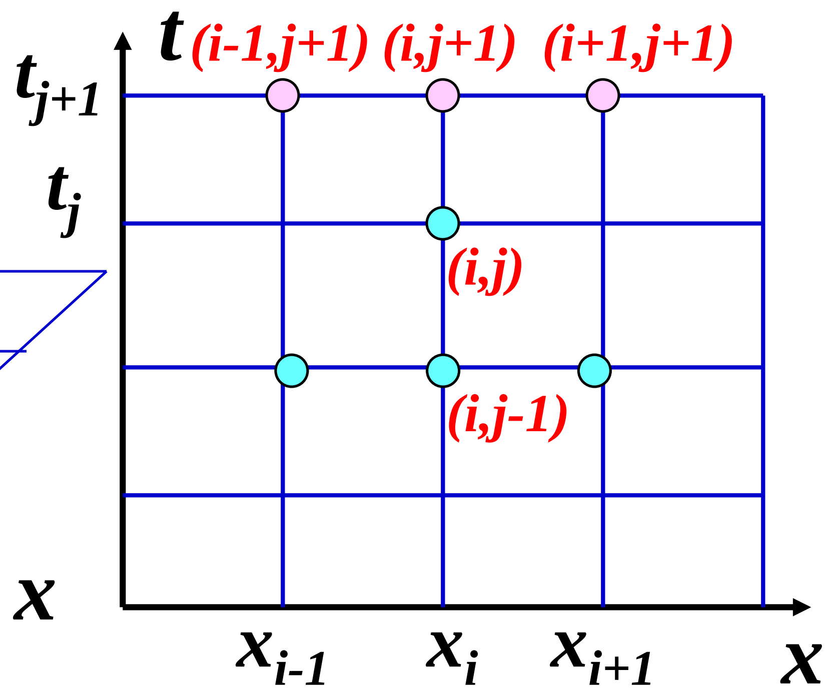
dx=.05

dt=.06

Method unstable when step too large

Implicit Method





CT centered differences in time

$$u_{tt} = \frac{1}{k^2}(u_{i,j-1} - 2u_{i,j} + u_{i,j+1})$$

CS average of the CS at time j+1 and j-1

$$c^2 u_{xx} = \frac{c^2}{2h^2}(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1} + u_{i-1,j-1} - 2u_{i,j-1} + u_{i+1,j-1})$$

Implicit Method

$$
\begin{cases}
2(1+r^2)u_{1,j+1} - r^2u_{2,j+1} = 4u_{1,j} - 2(1+r^2)u_{1,j-1} + r^2u_{2,j-1} + r^2u_{0,j+1} + r^2u_{0,j-1} \\
-r^2u_{1,j+1} + 2(1+r^2)u_{2,j+1} - r^2u_{3,j+1} = \\
\qquad 4u_{2,j} + r^2u_{1,j-1} - 2(1+r^2)u_{2,j-1} + r^2u_{3,j-1} \\
\ldots\ldots \\
-r^2u_{i-1,j+1} + 2(1+r^2)u_{i,j+1} - r^2u_{i+1,j+1} = \\
\qquad 4u_{i,j} + r^2u_{i-1,j-1} - 2(1+r^2)u_{i,j-1} + r^2u_{i+1,j-1} \\
\ldots\ldots \\
-r^2u_{n-1,j+1} + 2(1+r^2)u_{n,j+1} = \\
\qquad 4u_{n,j} + r^2u_{n-1,j-1} - 2(1+r^2)u_{n,j-1} + r^2u_{n+1,j-1} + r^2u_{n+1,j+1}
\end{cases}
$$

j=1,2,....

Linear System at each time step j with Tridiagonal Matrix (Thomas's algorithm)

$$[-r^2 \quad 2(1+r^2) \quad -r^2]$$

- **Unconditional stability**

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