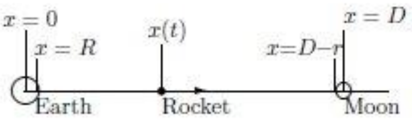

Figure 2.2: 1D Moon Rocket

2.10 Case study: Rocket to the moon

Now we have a reasonably powerful apparatus for integration of initial-value problems and systems, including the automatic regulation of step size and built-in error estimation. In order to try out this software on a problem that will use all of its capability, in this section we are going to derive the differential equations that govern the flight of a rocket to the moon. We do this first in a one-dimensional model, and then in two dimensions. It will be very useful to have these equations available for testing various proposed integration methods. Great accuracy will be needed, and the ability to chenge the step size, both to increase it and the devrease it, will be essential, or else the computation will become intolerably long. The variety of solutions that can be obtained is quite remarkable.

First, in the one-dimensional simplified model, we place the center of the earth at the origin of the x-axis, and let R denote the earth's radius. At the point $x = D$ we place the moon, and we let its radius be r. Finally, at a position $x = x(t)$ is our rocket, making its way towards the moon.

We will use Newton's law of gravitation to find the net gravitational force on the rocket,and equate it to the mass of the rocket times its acceleration (Newton's second law of motion). According to Newton's law of gravitation, the gravitational force exerted by one body on another is proportional to the product of their masses and inversely proportional to the square of the distance between them. If we use K for the constant of proportionality, then the force on the rocket due to the earth is

$$-K\frac{M_E m}{x^2}, \tag{2.10.1}$$

whereas the force on the rocket due to the moon's gravity is

$$K\frac{M_M m}{(D-x)^2} \tag{2.10.2}$$

where M_E, M_M and m are, respectively, the masses of the earth, the moon and the rocket.

The acceleration of the rocket is of course $x''(t)$, and so the assertion that the net force is equal to mass times acceleration takes the form:

$$mx'' = -K\frac{M_E m}{x^2} + K\frac{M_M m}{(D-x)^2}. \tag{2.10.3}$$

This is a (nasty) differential equation of second order in the unknown function $x(t)$, the position of the rocket at time t. Note the nonlinear way in which this unknown function appears on the right-hand side.

A second-order differential equations deserves two initial values, and we will oblige. First. let's agree that at time $t = 0$ the rocket was on the surface of the earth, and second, that the rocket was fired at the moon with a certain initial velocity V. Hence, the initial conditions that go with (2.10.3) are

$$x(0) = R; \qquad x'(0) = V. \tag{2.10.4}$$

Now, just a quick glance at (2.10.3) shows that m cancels out, so let's remove it, but not before pointing out the immense significance of that fact. It implies that the motion of the rocket is independent of its mass. For performing a now-legendary experiment with rocks of different sizes dropping from the Tower of Pisa, Galileo demonstrated that fact to an incredulous world.

At any rate, (2.10.3) now reads as

$$x'' = -\frac{KM_E}{x} + \frac{KM_M}{(D-x)^2}. \tag{2.10.5}$$

We can make this equation a good bit prettier by changing the units of distance and time from miles and seconds (or whatever) to a set of more natural units for the problem.

For our unit of distance we choose R, the radius of the earth. If we divide (2.10.5) through by R, we can write the result as

$$\left(\frac{x}{R}\right)'' = -\frac{\frac{KM_E}{R^3}}{\left(\frac{X}{R}\right)^2} + \frac{\frac{KM_M}{R^3}}{\left(\frac{D}{R} - \frac{x}{R}\right)^2}. \tag{2.10.6}$$

Now instead of the unknown function $x(t)$, we define $y(t) = x(t)/R$. Then $y(t)$ is the position of the rocket, expressed in earth radii, at time t. Further, the ratio D/R that occurs in (2.10.6) is a dimensionless quantity, whose numerical value is about 60. Hence (2.10.6) has now been transformed to

$$y'' = -\frac{\frac{KM_E}{R^3}}{y^2} + \frac{\frac{KM_M}{R^3}}{(60-y)^2}. \tag{2.10.7}$$

Next we tackle the new time units. Since y is now dimensionless, the dimension of the left side of the equation is the reciprocal of the square of a time. If we look next at the first term on the right, which of course has the same dimension, we see that the quantity R^3/KM_E is the square of a time, so

$$T_0 = \sqrt{\frac{R^3}{KM_E}} \tag{2.10.8}$$

is a time. Its numerical value is easier to calculate if we change the formula first, as follows.

Consider a body of mass m on the surface of the earth. Its weight is the magnitude of the force exerted on it by the earth's gravity, namely KM_Em/R^2. Its weight is also equal to m times the acceleration of the body, namely the acceleration due to gravity, usually denoted by g, and having the value 32.2 feet/sec^2.

It follows that

$$\frac{KM_em}{R^2} = mg, \tag{2.10.9}$$

and if we substitute into (2.10.8) we find that our time unit is

$$T_0 = \sqrt{\frac{R}{g}}. \tag{2.10.10}$$

We take $R = 4000$ miles, and find T_0 is about 13 minutes and 30 seconds. We propose to measure time in units of T_0. To that end, we multiply through equation (2.10.7) by T_0 and get

$$T_0^2 y'' = -\frac{1}{y^2} + \frac{\frac{M_M}{M_E}}{(60-y)^2}. \tag{2.10.11}$$

The ratio of the mass M_M of the moon to the mass M_E of the earth is about 0.012. Furthermore, we will now introduce a new independent variable τ and a new dependent variable $u = u(\tau)$ by the relations

$$u(\tau) = y(\tau T_0); \qquad t = \tau T_0. \tag{2.10.12}$$

Thus, $u(\tau)$ represents the position of the rocket, measured in units of the radius of the earth, at a time τ that is measured in units of T_0, *i.e.*, in units of 13.5 minutes.

The substitution of (2.10.12) into (2.10.11) yields the differential equation for the scaled distance $u(\tau)$ as a function of the scaled time τ in the form

$$u'' = -\frac{1}{u^2} + \frac{0.012}{(60-u)^2}. \tag{2.10.13}$$

Finally we must translate the initial conditions (2.10.4) into conditions on the new variables. The first condition is easy: $u(0) = 1$. Next, if we differentiate (2.10.12) and set $\tau = 0$ we get

$$u'(0) = \frac{T_0 V}{R} = \frac{V}{R/T_0}. \tag{2.10.14}$$

This is a ratio of two velocities. In the numerator is the velocity with which the rocket is launched. What is the significance of the velocity R/T_0?

We claim that it is, aside from a numerical factor, the escape velocity from the earth, if there were no moon. Perhaps the quickest way to see this is to go back to equation (2.10.11) and drop the second term on the right-hand side (the one that comes from the moon). Then we will be looking at the differential equation that would govern the motion if the moon were absent. This equation can be solved. Multiply bth sides by $2y'$, and it becomes

$$T_0^2\left((y')^2\right)' = \left(\frac{2}{y}\right)', \tag{2.10.15}$$

and integration yields

$$T_0^2(y')^2 = \frac{2}{y} + C. \tag{2.10.16}$$

Now let $t = 0$ and find that $C = T_0^2V^2/R^2 - 2$, so

$$T_0^2(y')^2 = \frac{2}{y} - \left(\frac{T_0^2V^2}{R^2} - 2\right). \tag{2.10.17}$$

Suppose the rocket is launched with sufficient initial velocity to escape from the earth. Then the function $y(t)$ will grow without bound. Hence let $y \to \infty$ on the right side of (2.10.17). For all values of y, the left side is a square, and therefore a nonnegative quantity. Hence the right side, which approaches the constant C, must also be nonnegative. Thus $C \geq 0$ or, equivalently

$$V \geq \sqrt{2}\frac{R}{T_0}. \tag{2.10.18}$$

Thus, if the rocket escapes, then (2.10.18) is true, and the converse is easy to show also. Hence the quantity $\sqrt{2}\,R/T_0$ is the *escape velocity* from the earth. We shall denote it by V_{esc}. Its numerical value is approximately 25, 145 miles per hour.

Now we can return to (2.10.12) to translate the initial conditions on $x'(t)$ into initial conditions on $u'(\tau)$. In terms of the escape velocity, it becomes $u'(0) = \sqrt{2}\,V/V_{esc}$. We might say that if we choose to measure distance in units of earth radii, and time in units of T_0, then velocities turn out to be measured in units of escape velocity, aside from the $\sqrt{2}$.

In summary, the differential equation and the initial conditions have the final form

$$\begin{aligned} u'' &= -\frac{1}{u^2} + \frac{0.012}{(60-u)^2} \\ u(0) &= 1 \\ u'(0) &= \sqrt{2}\frac{V}{V_{esc}} \end{aligned} \tag{2.10.19}$$

Since that was all so easy, let's try the two-dimensional case next. Here, the earth is centered at the origin of the xy-plane, and the moon is moving. Let the coordinates of the moon at time t be $(x_m(t), y_m(t))$. For example, if we take the orbit of the moon to be a circle of radius D, then we would have $x_m = D\cos(\omega t)$ and $y_m(t) = D\sin(\omega t)$.

If we put the rocket at a generic position $(x(t), y(t))$ on the way to the moon, then we have the configuration shown in figure 1.16.2.

Consider the net force on the rocket in the x direction. It is given by

$$F_x = -\frac{KM_Em\cos\theta}{x^2+y^2} + \frac{KM_Mm\cos\psi}{(x-x_m)^2+(y-y_m)^2}, \tag{2.10.20}$$

where the angles θ and ψ are shown in figure 1.16.2. From that figure, we see that

$$\cos\theta = x\sqrt{x^2+y^2} \tag{2.10.21}$$

and

$$\cos\psi = \frac{x_m - x}{\sqrt{(x_m-x)^2+(y_m-y)^2}}. \tag{2.10.22}$$

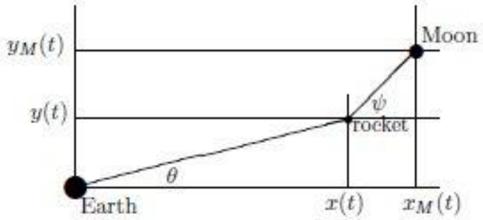

Figure 2.3: THE 2D MOON ROCKET

Now we substitute into (2.10.20), and equate the force in the x direction to $mx''(t)$, to obtain the differential equation

$$mx''(t) = -\frac{KM_E mx}{(x^2+y^2)^{3/2}} + \frac{KM_M m(x_m - x)}{((x_m-x)^2+(y_m-y)^2)^{3/2}}. \tag{2.10.23}$$

If we carry out a similar analysis for the y-component of the force on the rocket, we get

$$my''(t) = -\frac{KM_E my}{(x^2+y^2)^{3/2}} + \frac{KM_M m(y_m - y)}{((x_m-x)^2+)y_m-y)^2)^{3/2}}. \tag{2.10.24}$$

We are now looking at two (even nastier) simultaneous differential equations of the second order in the two unknown functions $x(t)$, $y(t)$ that describe the position of the rocket. To go with these equations, we need four initial conditions. We will suppose that at time $t = 0$, the rocket is on the earth's surface, at the point $(R, 0)$. Further, at time $t = 0$, it will be fired with an initial speed of V, in a direction that makes an angle α with the positive x-axis. Thus, our initial conditions are

$$\begin{cases} x(0) = R; & y(0) = 0 \\ x'(0) = V\cos\alpha; & y'(0) = V\sin\alpha \end{cases} . \tag{2.10.25}$$

The problem has now been completely defined. It remains to change the units into the same natural dimensions of distance and time that were used in the one-dimensional problem. This time we leave the details to the reader, and give only the results. If $u(\tau)$ and $v(\tau)$ denote the x and y coordinates of the rocket, measured in units of earth radii, at a time τ measured in units of T_0 (see (2.10.10)), then it turns out the u and v satisfy the differential equations

$$\begin{aligned} u'' &= -\frac{u}{(u^2+v^2)^{3/2}} + \frac{0.012(u_m-u)}{((u_m-u)^2+(v_m-v)^2)^{3/2}} \\ v'' &= -\frac{v}{(u^2+v^2)^{3/2}} + \frac{0.012(v_m-v)}{((u_m-u)^2+(v_m-v)^2)^{3/2}}. \end{aligned} \tag{2.10.26}$$

Furthermore, the initial data (2.10.25) take the form

$$\begin{cases} u(0) = 1; & v(0) = 0 \\ u'(0) = \sqrt{2}\dfrac{V\cos\alpha}{V_{esc}}; & v'(0) = \sqrt{2}\dfrac{V\sin\alpha}{V_{esc}}. \end{cases} \tag{2.10.27}$$

In these equations, the functions $u_m(\tau)$ and $v_m(\tau)$ are the x and y coordinates of the moon, in units of R, at the time τ. Just to be specific, let's decree that the moon is in a circular orbit of radius $60R$, and that it completes a revolution every twenty eight days. Then, after a brief session with a hand calculator or a computer, we discover that the equations

$$\begin{aligned} u_m(\tau) &= 60\cos(0.002103745\tau) \\ v_m(\tau) &= 60\sin(0.002103745\tau) \end{aligned} \tag{2.10.28}$$

represent the position of the moon.

2.11 Maple programs for the trapezoidal rule

In this section we will first display a complete Maple program that can numerically solve a system of ordinary differential equations of the first order together with given initial values. After discussing those programs, we will illustrate their operation by doing the numerical solution of the one dimensional moon rocket problem.

We will employ Euler's method to predict the values of the unknowns at the next point $x+h$ from their values at x, and then we will apply the trapezoidal rule to correct these predicted values until sufficient convergence has occurred.

First, here is the program that does the Euler method prediction.

```
> eulermethod:=proc(yin,x,h,f)
> local yout,ll,i:
> # Given the array yin of unknowns at x, uses Euler method to return
> # the array of values of the unknowns at x+h. The function f(x,y) is
> # the array-valued right hand side of the given system of ODE's.
> ll:=nops(yin):
> yout:=[]:
> for i from 1 to ll do
>   yout:=[op(yout),yin[i]+h*f(x,yin,i)];
>                  od:
> RETURN(yout):
> end:
```

Next, here is the program that takes as input an array of guessed values of the unknowns at $x+h$ and refines the guess to convergence using the trapezoidal rule.

```
> traprule:=proc(yin,x,h,eps,f)
> local ynew,yfirst,ll,toofar,yguess,i,allnear,dist;
> # Input is the array yin of values of the unknowns at x. The program
> # first calls eulermethod to obtain the array ynew of guessed values
> # of y at x+h. It then refines the guess repeatedly, using the trapezoidal
> # rule, until the previous guess, yguess, and the refined guess, ynew, agree
> # within a tolerance of eps in all components. Program then computes dist,
> # which is the largest deviation of any component of the final converged
> # solution from the initial Euler method guess. If dist is too large
```