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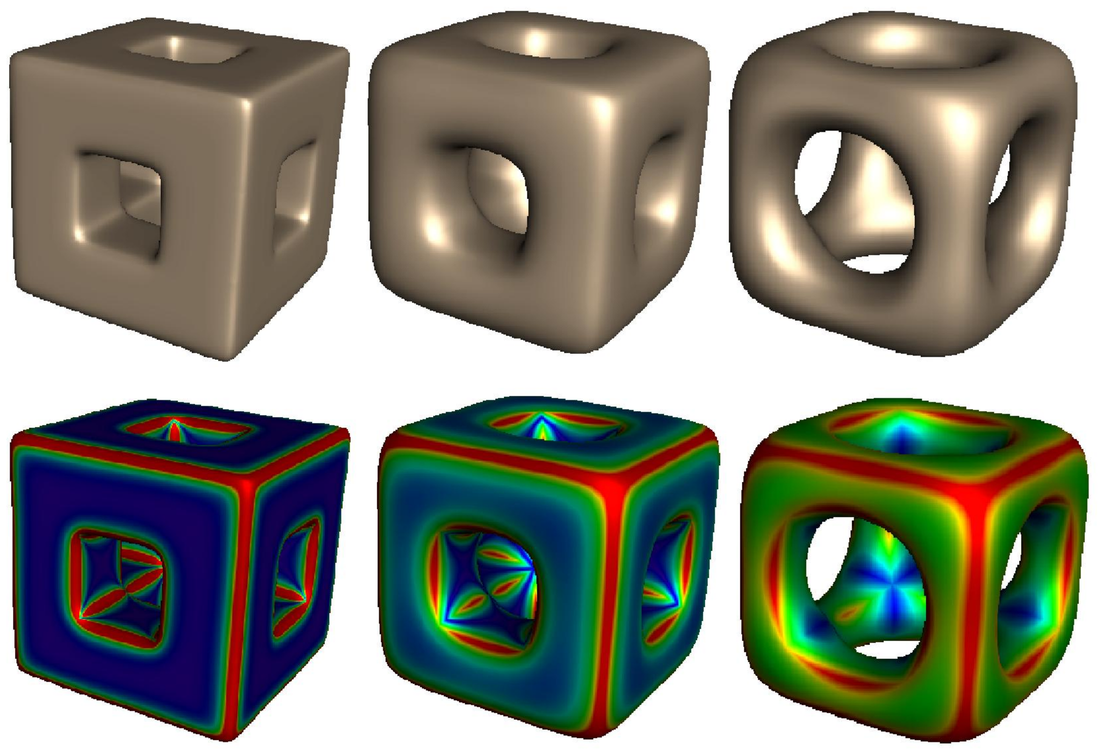
A Cascadic Geometric Filtering Approach to Subdivision

1 Introduction

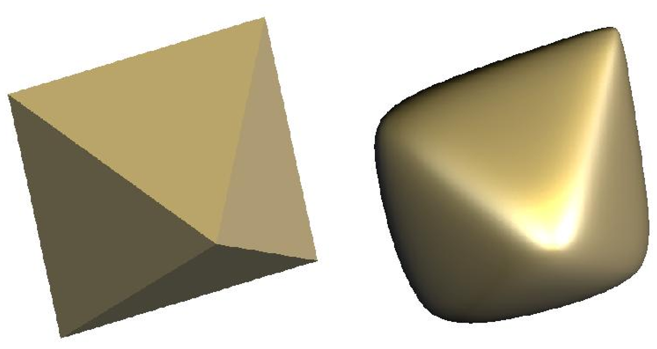
Given a coarse triangulation of some object, subdivision methods deal with the production of finer and finer triangulations of the object in every subdivision step. The aim is to produce a sequence of visually smoother and smoother representations of the object, which ideally converges to a smooth limit surface for infinitely many subdivision steps.

While classical subdivision methods are based on a discrete local averaging of the vertices of a triangulation, we present a new approach to subdivision which is based on *geometric filtering* and *curvature motion* methods.

The new approach naturally incorporates the possibility of choosing *different geometric filter widths* ranging from a smoothing of solely edges and corners of the object to a smoothing of the overall geometry as shown here (in the second line color coded curvature plots of the surfaces above are shown):



The curvature motion approach even allows for the choice of a *locally varying geometric filter width* resulting in a locally different pronounciation of edges and corners of the object during the subdivision process, here applied to the coarse inital surface on the left:



2 Surface Fairing and Curvature Motion

Curvature driven evolution processes such as Mean Curvature Motion (MCM) are well established methods for the fairing of initially noisy surfaces. If x denotes the coordinates of an embedded surface $\mathcal{M}$ in $\mathbb{R}^3$, the evolution by MCM is described by

$$\partial_t x(t) = -H(x)N(x)$$
$$\mathcal{M}(0) = \mathcal{M}_0\,,$$

where N is the surface normal and H is the mean curvature of $\mathcal{M}$. This equation can be reformulated equivalently by the partial differential equation

$$\partial_t x(t) - \Delta_{\mathcal{M}(t)} x(t) = 0\,,$$

where $\Delta_{\mathcal{M}(t)}$ is the Laplace-Beltrami Operator on $\mathcal{M}$.

Because in nD-image processing the last equation is equivalent to the application of a Gaussian filter with filter width

$$\sigma = \sqrt{2t}$$

it is instructive to regard σ as a *"geometric filter width"* also in the surface evolution case.

3 Subdivision filtering

To exploit the surface evolution equation

$$\partial_t x(t) - \Delta_{\mathcal{M}(t)} x(t) = 0$$

for a subdivision method we focus on one single fully implicit discrete time step of this equation, i. e.,

$$(x^* - x_0) - a\Delta_{\mathcal{M}^*} x^* = 0\,.$$

To approximate the solution x^* numerically, we first discretize the last equation in time by

$$(x^k - x_0) - a\,\Delta_{\mathcal{M}^{k-1}} x^k = 0\,,$$

and then in space by

$$(X^k - \mathcal{I}_h x_0) - a\,\Delta_{\mathcal{M}_h^{k-1}} X^k = 0\,.$$

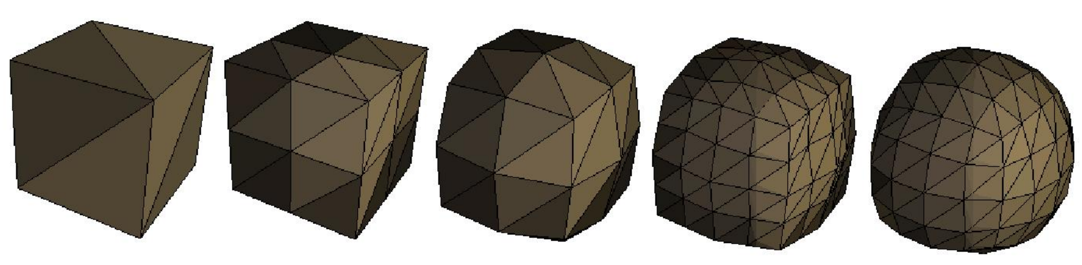
Here, $\mathcal{I}_h$ is the piecewise linear interpolation operator of the continuous data x_0 and X denotes the spatially discrete coordinate values of the surface $\mathcal{M}_h$.

In addition, we consider a sequence of nested grids $\mathcal{M}_{h_k}$ generated by any recursive and regular refinement rule, and refine the grid once after each iteration in the above equation.

Putting everything together we define our new subdivision scheme as $\mathcal{M}^k_{h_k} = \mathcal{S}_{h_k}(\mathcal{M}^{k-1}_{h_{k-1}})\mathcal{M}_0$, where the application of S_{h_k} is given by the solution of the operator equation

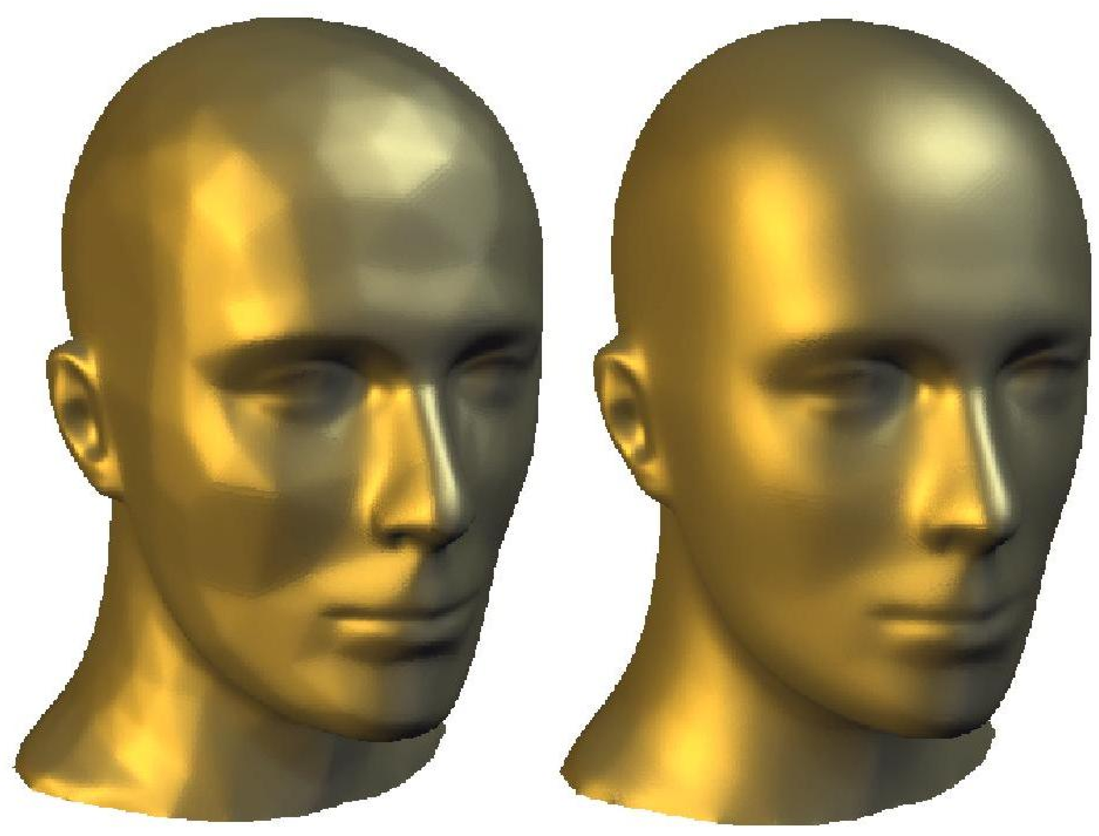
$$(X^k_{h_k} - \mathcal{I}_{h_k} x_0) - a\,\Delta_{\mathcal{M}^{k-1}_{h_{k-1}}} X^k_{h_k} = 0\,.$$

Thus, the resulting subdivision method consists of the application of the two basic operations *"refinement"* and *"smoothing"* in every subdivision step. In the following picture two subdivision steps applied to the coarse triangulation on the very left are shown. Each of the subdivision steps is split into the two basic operations *"refinement"* and *"smoothing"*:



4 Local Filter Width Expansion

In the case of triangulations with a considerable variation in the grid size, we locally adapt the geometric filter width to the grid size of the triangulation. The following figure shows two examples of limit surfaces without (left) and with (right) the local filter width expansion:



5 Cascadic Iterations

For the already mentioned discretization in space we use linear finite elements. The resulting linear systems are solved by CG- or Jacobi-iterations where we restrict the number of iterations in a multigrid fashion to

$$n_k = n_{k_{\max}} 2^{\frac{3}{2}(k_{\max}-k)}$$

in the case of CG-iterations and

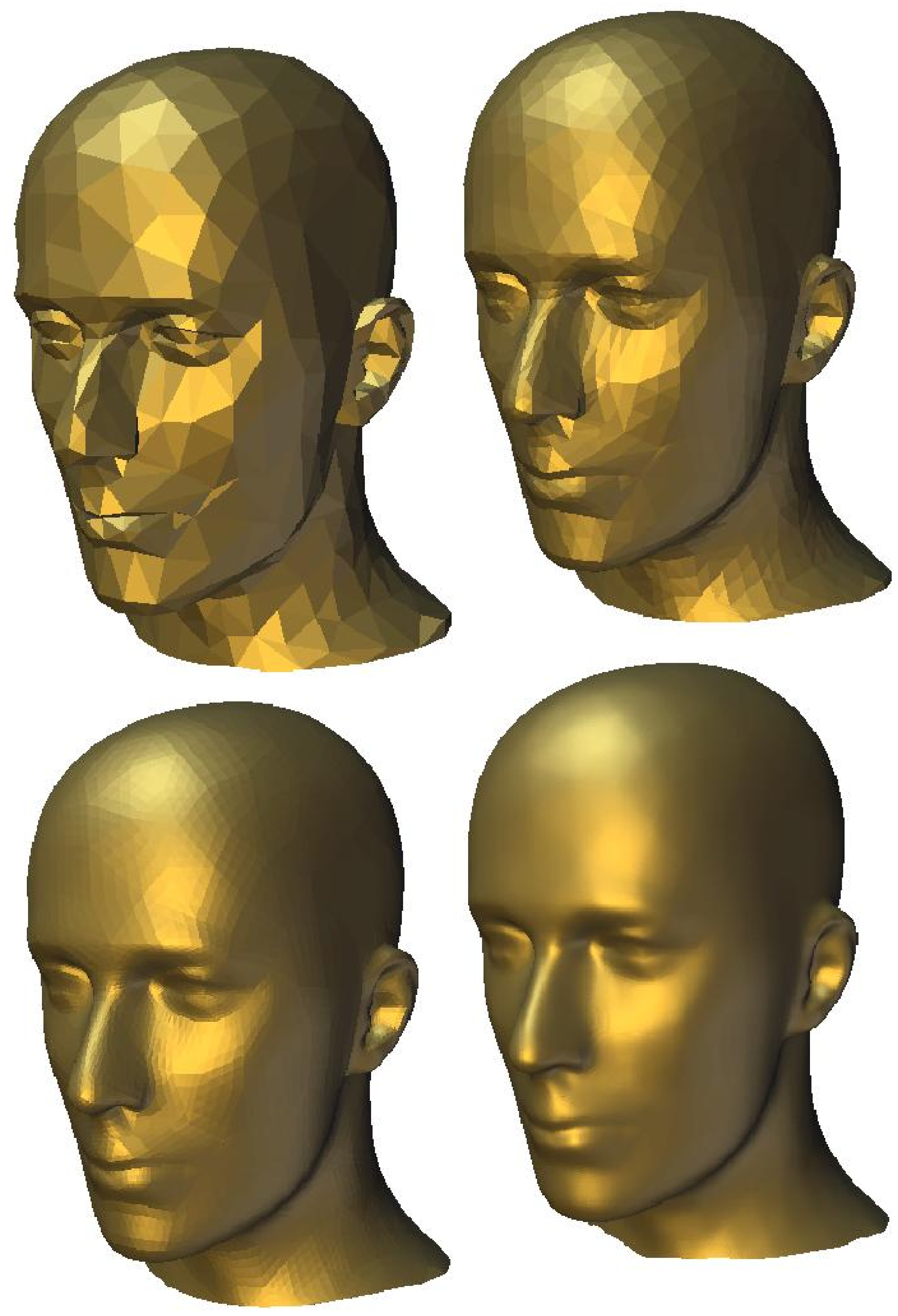
$$n_k = n_{k_{\max}} 2^{2(k_{\max}-k)}$$

in the case of Jacobi-iterations using the results of Bornemann and Deuflhard.

Thus, on the *finest level* $n_{k_{\max}}$ we restrict to a *single iteration* in the iterative solver, thus getting (nearly) optimal complexity of the proposed method.

6 Results

A sequence of flatshaded subdivision surfaces is shown using the proposed local filter width expansion. The starting surface comes along with a very irregular triangulation grid like different valences of the nodes, thin triangles and nonhomogeneous grid size. The proposed method is able to effectively deal with such surfaces:



We conjecture the limit surface of the proposed subdivision method to be in the class C^2. As a numerical indication for bounded second derivatives we depict the color coded modulus of the mean curvature of the last surface:

References

[1] U. Diewald, S. Morigi, and M. Rumpf, *A cascadic geometric filtering approach to subdivision*, CAGD, (to appear 2002).