Corso di Laurea in Informatica
Prova scritta di Analisi Matematica
18 Giugno 2019
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Indicare quale modalità si intende scegliere per la prova orale:........................

Risultati

1.(pt.1)	
2.(pt.1)	
3.(pt.4)	
4.(pt.6)	
5.(pt.6)	
6.(pt.6)	
7.(pt.6)	

Risolvere gli esercizi seguenti, scrivendo e motivando dettagliatamente il procedimento seguito. Soluzioni prive di calcoli e spiegazioni non saranno valutate.

Esercizio 1(pt.1)
Cosa significa che la funzione $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ è differenziabile nel punto $(1, 2)$?
Risposta:

$$f(x,y) = f(1,2) + \langle \nabla f(1,2), \begin{pmatrix} x-1 \\ y-2 \end{pmatrix} \rangle + o(\|(x-1, y-2)\|)$$

Esercizio 2(pt.1)
Enunciare il teorema di Rolle.
Risposta:

Hp: $f : [a,b] \longrightarrow \mathbb{R}$

continua su $[a,b]$

derivabile su $]a,b[$

$f(a) = f(b)$

Th: $\exists\, c \in\,]a,b[\, : f'(c) = 0$

$$\alpha = \frac{1}{2} \qquad \frac{\alpha(\alpha-1)}{2} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2} = -\frac{1}{8}$$

$$\frac{\alpha(\alpha-1)(\alpha-2)}{3!} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6} = \frac{1}{16}$$

$$(1+t)^{\frac{1}{2}} = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + \frac{1}{16}t^3 + o(t^3)$$

Esercizio 3(pt. 4)
Sapendo che, per $t \to 0$,

- $(1+t)^\alpha = 1+\alpha t+\frac{\alpha(\alpha-1)}{2!}t^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}t^3+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}t^4+o(t^4)$

- $\ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \frac{1}{5}t^5 + o(t^5)$

Calcolare, INDICANDO SOLO IL RISULTATO FINALE:

$$\lim_{x\to 0} \frac{\sqrt{1+\ln(1+x)} - \sqrt{1+x} + x^2/4}{x^3} = \lim_{x\to 0} \frac{\cancel{-\frac{x^2}{4}} + \frac{7}{24}x^3 + o(x^3) + \cancel{\frac{x^2}{4}}}{x^3} = \frac{7}{24}$$

$$\ln(1+x) = x + o(x)$$

$$\sqrt{1+\ln(1+x)} = 1 + \frac{\ln(1+x)}{2} - \frac{1}{8}(\ln(1+x))^2 + \frac{1}{16}(\ln(1+x))^3 + o(x^3)$$

$$= 1 + \frac{1}{2}\left(x - \frac{1}{2}x^2 + \frac{x^3}{3}\right) - \frac{1}{8}\left(x - \frac{1}{2}x^2 + o(x^2)\right)^2 +$$

$$+ \frac{1}{16}x^3 + o(x^3)$$

$$= 1 + \frac{1}{2}x - \frac{x^2}{4} + \frac{x^3}{6} - \frac{1}{8}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^3 + o(x^3)$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)$$

$$\sqrt{1+\ln(1+x)} - \sqrt{1+x} = \cancel{1} + \cancel{\frac{x}{2}} - \frac{x^2}{4} + \frac{x^3}{6} \cancel{- \frac{1}{8}x^2} + \frac{1}{8}x^3 \cancel{+ \frac{1}{16}x^3} -$$

$$\cancel{-1} \cancel{- \frac{x}{2}} \cancel{+ \frac{x^2}{8}} \cancel{- \frac{x^3}{16}} + o(x^3)$$

$$= -\frac{x^2}{4} + \frac{4+3}{24}x^3 + o(x^3) = -\frac{x^2}{4} + \frac{7}{24}x^3 + o(x^3)$$

$$\mathcal{D}(\ln|x|) = \mathbb{R} \setminus \{0\}$$

$$\ln|x| - 2 = 0 \iff |x| = e^2 \iff x = \pm e^2$$

$$\mathcal{D}(f) = \mathbb{R} \setminus \{0, \pm e^2\}$$

Esercizio 4(pt. 6)
Sia data la funzione $\mathcal{D}(f) \to \mathbf{R}$

$$f(x) = \frac{2x^2}{\ln|x| - 2}$$

I. Disegnare il suo grafico.

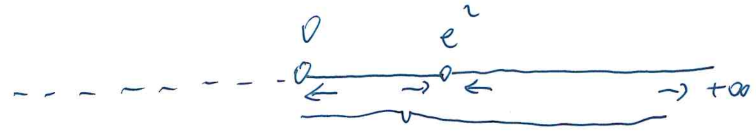
II. Calcolare l'immagine di f sul suo dominio naturale $\mathcal{D}(f)$.

III. Stabilire per quali $\lambda \in \mathbf{R}$ l'equazione $f(x) = \lambda$ ha 2 soluzioni reali distinte.

Risposta:

$$f(-x) = \frac{2(-x)^2}{\ln|-x| - 2} = \frac{2x^2}{\ln|x| - 2} = f(x) \Rightarrow f \text{ è PARI}$$

Studiamo f per $x \geq 0$.



$$f(x) = \frac{2x^2}{\ln x - 2}$$

$$\lim_{x \to 0^+} \frac{2x^2}{\ln x - 2} = \lim_{x \to 0^+} \underbrace{2x^2}_{\to 0} \cdot \frac{1}{\underbrace{\ln x - 2}_{\to -\infty}} = 0^-$$

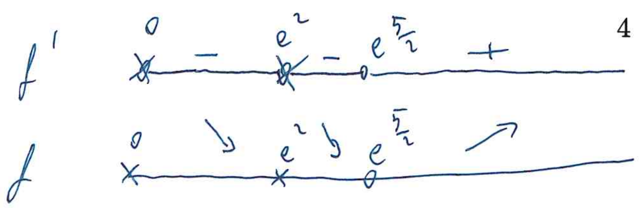
$$\lim_{x \to e^2 -} \frac{\overbrace{2x^2}^{\to 2e^4}}{\underbrace{\ln x - 2}_{\to 0^-}} = -\infty \qquad \lim_{x \to e^2 +} f(x) = +\infty$$

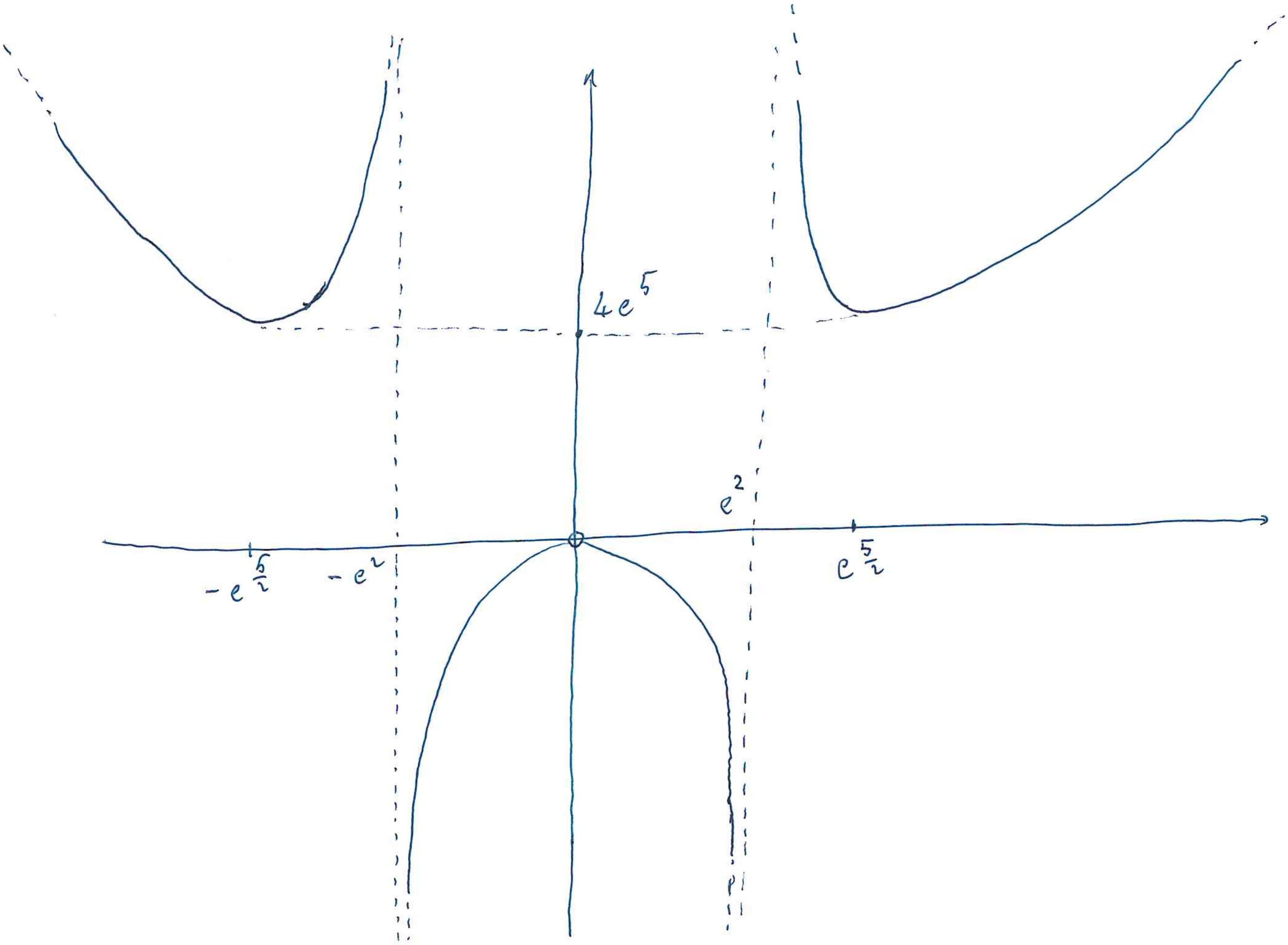
$$\lim_{x \to +\infty} \frac{2x^2}{\ln x - 2} \overset{H}{=} \lim_{x \to +\infty} \frac{4x}{\frac{1}{x}} = \lim_{x \to +\infty} 4x^2 = +\infty$$

$$f'(x) = \frac{4x(\ln x - 2) - \frac{1}{x} \cdot 2x^2}{(\ln x - 2)^2} = \frac{2x}{(\ln x - 2)^2} \cdot \underbrace{(2\ln x - 4 - 1)}_{2\ln x - 5}$$

$$2\ln x - 5 > 0 \iff x > e^{\frac{5}{2}}$$

$$f\left(e^{\frac{5}{2}}\right) = \frac{2 \cdot \left(e^{\frac{5}{2}}\right)^2}{\frac{5}{2} - 2} = \frac{2e^5}{\frac{1}{2}} = 4e^5$$





$$\operatorname{Im} f = \left]-\infty, 0\right[\cup \left[4e^5, +\infty\right[$$

$f(x) = \lambda$ ha 2 soluzioni $\iff \lambda < 0 \vee \lambda = 4e^5$

Esercizio 5(pt. 6)
Calcolare:
(I) una primitiva della seguente funzione:

$$\ln(x^2+4);$$

(II) il valore del seguente integrale

$$\int_{-2}^{0} \ln(x^2+4)\, dx.$$

Risposta:

$$\int \ln(x^2+4)\,dx = x\ln(x^2+4) - \int x \cdot \frac{2x}{x^2+4}\,dx$$

$$= x\ln(x^2+4) - 2\int \frac{x^2+4-4}{x^2+4}\,dx$$

$$= x\ln(x^2+4) - 2x + \boxed{8\int \frac{1}{x^2+4}\,dx}$$

$$= 2\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx = 4\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,d\left(\frac{x}{2}\right)$$

$$= 4\,\mathrm{arctg}\left(\frac{x}{2}\right)$$

$$= x\ln(x^2+4) - 2x + 4\,\mathrm{arctg}\,\frac{x}{2}$$

$$\int_{-2}^{0} \ln(x^2+4)\,dx = \Big[\quad\downarrow\quad\Big]_{-2}^{0} = -\Big(-2\ln 8 + 4 + 4\,\overbrace{\mathrm{arctg}(-1)}^{-\frac{\pi}{4}}\Big)$$

$$= 2\ln 8 - 4 + \pi$$

$$= 6\ln 2 - 4 + \pi$$

Esercizio 6(pt. 6)
Data $f : \mathbf{R}^2 \to \mathbf{R}$,

$$f(x,y) = xy(2x+y-4)$$

I) Determinare i suoi punti critici.
II) Stabilire quali siano i suoi eventuali punti di massimo, di minimo locali e di sella.
III) Calcolare il piano tangente al grafico di f nel punto $(1,2)$.
Risposta:

$$f_x = y(2x+y-4) + xy\cdot 2 = y(4x+y-4) = 4xy + y^2 - 4y$$

$$f_y = x(2x+y-4) + xy = x(2x+2y-4) = 2x^2 + 2xy - 4x$$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \longrightarrow \text{(I)} \begin{cases} y = 0 \\ x = 0 \end{cases} \quad \text{(II)} \begin{cases} y = 0 \\ 2x+2y-4=0 \end{cases} \begin{cases} y = 0 \\ 2x-4=0 \end{cases} \begin{cases} y = 0 \\ x = 2 \end{cases}$$

$$\text{(III)} \begin{cases} 4x+y-4=0 \\ x=0 \end{cases} \begin{cases} y = 4 \\ x = 0 \end{cases} \quad \text{(IV)} \begin{cases} 4x+y-4=0 \\ 2x+2y-4=0 \end{cases} \begin{cases} y = 4-4x \\ 2x+8-8x-4=0 \end{cases}$$

$$\begin{cases} y = 4-4x \\ -6x+4=0 \end{cases} \begin{cases} y = 4-\frac{8}{3} = \frac{4}{3} \\ x = \frac{4}{6} = \frac{2}{3} \end{cases}$$

$$(0,0)\,;\,(2,0)\,;\,(0,4)\,;\,\left(\tfrac{2}{3}, \tfrac{4}{3}\right)$$

$$H_f = \begin{pmatrix} 4y & 4x+2y-4 \\ 4x+2y-4 & 2x \end{pmatrix}$$

$$H_f(0,0) = \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} \quad \det < 0 \quad \text{SELLA}$$

$$H_f(2,0) = \begin{pmatrix} 0 & 4 \\ 4 & 4 \end{pmatrix} \quad \det = -16 < 0 \quad \text{SELLA}$$

$$H_f(0,4) = \begin{pmatrix} 16 & 4 \\ 4 & 0 \end{pmatrix} \quad \det = -16 < 0 \quad \text{SELLA}$$

$$H_f\left(\tfrac{2}{3}, \tfrac{4}{3}\right) = \begin{pmatrix} \overset{>0}{\frac{16}{3}} & \frac{4}{3} \\ \frac{4}{3} & \frac{4}{3} \end{pmatrix}$$

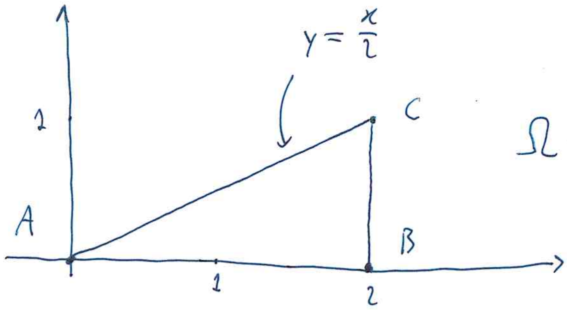
$$\det = \frac{16}{3}\cdot\frac{4}{3} - \left(\frac{4}{3}\right)^2 = \frac{4}{3}\left(\frac{16}{3} - \frac{4}{3}\right) > 0$$

P. DI MINIMO RELATIVO

$$z = \left\langle \nabla f(1,2), \begin{pmatrix} x-1 \\ y-2 \end{pmatrix} \right\rangle + f(1,2)$$

$$= \left\langle \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} x-1 \\ y-2 \end{pmatrix} \right\rangle + 0$$

$$= 4(x-1) + 2(y-2) = 4x + 2y - 8$$



$$\Omega = \{(x,y) \mid 0 \le y \le 1,\ 2y \le x \le 2\}$$

Esercizio 7(pt. 6)
Disegnare in $\mathbf{R}^2$ il triangolo Ω di vertici $A(0,0), B(2,0), C(2,1)$, descriverlo analiticamente come insieme semplice rispetto ad un asse, e calcolare

$$\iint_\Omega \frac{2y}{4+2y-x}\,dx\,dy.$$

Risposta:

$$\int_0^1 \left(\int_{2y}^2 \frac{2y}{4+2y-x}\,dx \right) dy = \int_0^1 2y \left(\int_{2y}^2 \frac{1}{4+2y-x}\,dx \right) dy$$

$$= \int_0^1 2y \left[-\ln|4+2y-x| \right]_{2y}^2 dy =$$

$$= \int_0^1 2y \left(\ln 4 - \ln(2+2y) \right) dy =$$

$$= \int_0^1 2y \cdot \ln 4\,dy - \int_0^1 2y \ln(2+2y)\,dy =$$

$$= \left[\ln 4 \cdot \frac{y^2}{2} \right]_0^1 - \left(\left[y^2 \ln(2+2y) \right]_0^1 - \underbrace{\int_0^1 y^2 \cdot \frac{1}{2+2y} \cdot 2\,dy}_{u} \right)$$

$$\int_0^1 \frac{y^2}{y+1}\,dy = \int_0^1 \frac{y^2-1+1}{y+1}\,dy =$$

$$= \int_0^1 \left(y-1+\frac{1}{y+1} \right) dy = \left[\frac{y^2}{2} - y + \ln(1+y) \right]_0^1$$

$$= \left[\ln 4 \cdot y^2 \right]_0^1 - \left[y^2 \ln(2+2y) \right]_0^1 + \left[\frac{y^2}{2} - y + \ln(1+y) \right]_0^1 =$$

$$= \ln 4 - \ln 4 + \frac{1}{2} - 1 + \ln 2 = \ln 2 - \frac{1}{2}$$