① $f(x) = \dfrac{x^2}{1-x} e^{-\frac{6x}{x-1}}$

$D(f) = \mathbb{R} \setminus \{+1\}$

$$\lim_{x \to -\infty} \underbrace{\frac{x^2}{1-x}}_{+\infty} \cdot e^{\overbrace{-\frac{6x}{x-1}}^{\to -6}} = +\infty$$

Analogamente: $\lim_{x \to +\infty} f(x) = -\infty$

$$\lim_{x \to +1^+} f(x) = \lim_{x \to +1^+} \frac{\overbrace{\frac{x^2}{1-x}}^{\to -\infty}}{\underbrace{e^{\frac{6x}{x-1}}}_{+\infty}}$$

(applicando il teorema di De L'Hopital)

$$\overset{H}{=} \lim_{x \to +1^+} \frac{\frac{2x(1-x)+x^2}{\cancel{(1-x)^2}}}{e^{\frac{6x}{x-1}} \cdot \frac{6(x-1)-6x}{\cancel{(x-1)^2}}} =$$

$$= \lim_{x \to +1^+} \frac{\overbrace{-x^2+2x}^{+1}}{\underbrace{e^{\frac{6x}{x-1}} \cdot (-6)}_{\to -\infty}} = 0$$

$$\lim_{x\to 1^-} \frac{x^2}{1-x} \cdot e^{-\frac{6x}{x-1}} = +\infty$$

$$\frac{x^2}{1-x} \to +\infty \qquad -\frac{6x}{x-1} \to +\infty$$

$$f'(x) = \left(\frac{2x(1-x)+x^2}{(1-x)^2} - \frac{6(x-1)-6x}{(x-1)^2}\cdot\frac{x^2}{1-x}\right) e^{-\frac{6x}{x-1}}$$

$$= e^{-\frac{6x}{x-1}} \left(\frac{-x^2+2x}{(1-x)^2} + \frac{6x^2}{(1-x)^3}\right) =$$

$$= e^{-\frac{6x}{x-1}} \cdot \frac{(-x^2+2x)(1-x)+6x^2}{(1-x)^3} =$$

$$= e^{-\frac{6x}{x-1}} \cdot \frac{+x^3-x^2-2x^2+2x+6x^2}{(1-x)^3} =$$

$$= e^{-\frac{6x}{x-1}} \cdot \frac{x^3+3x^2+2x}{(1-x)^3} =$$

$$= \frac{e^{-\frac{6x}{x-1}}}{(1-x)^2} \cdot \frac{x(x^2+3x+2)}{(1-x)^3}$$

$$\frac{e^{-\frac{6x}{x-1}}}{(1-x)^2} \to 0$$

$$x^2+3x+2=0 \nearrow \quad x_{1,2} = \frac{-3\pm\sqrt{1}}{2} \begin{cases} -2 \\ -1 \end{cases}$$

$$\frac{x(x^2+3x+2)}{1-x}$$

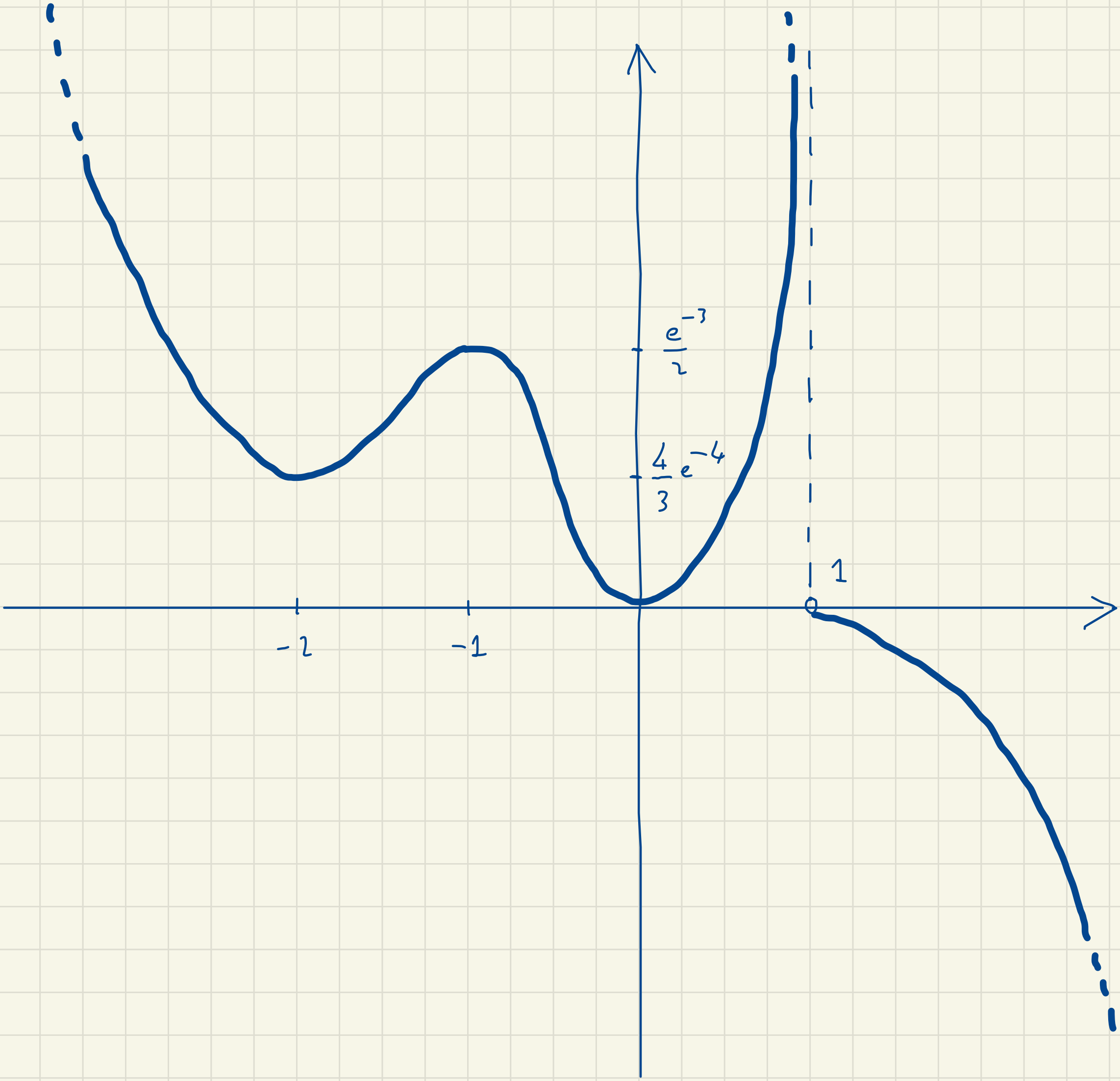
STUDIO DEL SEGNO

		-2		-1		0		1	
x	$-$		$-$		$-$	0	$+$	$\times$	$+$
x^2+3x+2	$+$	0	$-$	0	$+$		$+$	$\times$	$+$
$1-x$	$+$		$+$		$+$		$+$	$\times$	$-$
f'	$-$		$+$		$-$		$+$	$\times$	$-$
f	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$	$\times$	$\searrow$

$f(0) = 0$ MINIMO REL.

$f(-1) = \frac{1}{2} e^{-3}$ MAX REL.

$f(-2) = \frac{4}{3} e^{-4}$ MIN. REL.



$\operatorname{Im} f = \mathbb{R}$

$f(x) = K$ ha 4 soluzioni : $\frac{4}{3e^4} < K < \frac{1}{2e^3}$

2

$$\lim_{x\to 0} \frac{\ln(1-x\cos x) - e^{-x+\frac{x^3}{3}} + 1 + x^2}{1-\cos x^2}$$

$$1-\cos(x^2) = 1 - \left(1 - \frac{x^4}{2} + o(x^4)\right) = \frac{x^4}{2} + o(x^4)$$

$$e^{-x+\frac{x^3}{3}} = 1 + \left(-x+\frac{x^3}{3}\right) + \frac{1}{2}\left(-x+\frac{x^3}{3}\right)^2 + \frac{1}{3!}\left(-x+\frac{x^3}{3}\right)^3$$

$$+ \frac{1}{4!}\left(-x+\frac{x^3}{3}\right)^4 + o(x^4)$$

$$= 1 + \left(-x+\frac{x^3}{3}\right) + \left(\frac{x^2}{2} - \frac{x^4}{3}\right) - \frac{x^3}{6} +$$

$$+ \frac{x^4}{24} + o(x^4)$$

$$= 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{7}{24}x^4 + o(x^4)$$

$$x\cos x = x + o(x) \implies t = x\cos x$$

$$o(t^n) = o(x^n)$$

$$n = 4$$

$$\ln\big(1+\overbrace{-x\cos x}^{t}\big) =$$

$$= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + o(t^4)$$

$$= -x\cos x - \frac{(-x\cos x)^2}{2} + \frac{(-x\cos x)^3}{3} -$$

$$- \frac{(-x\cos x)^4}{4} + o(x^4)$$

$$= -x\left(1 - \frac{x^2}{2} + o(x^3)\right) - \frac{\left(x - \frac{x^3}{2} + o(x^4)\right)^2}{2} +$$

$$- \frac{(x + o(x^2))^3}{3} - \frac{(x + o(x^2))^4}{4} + o(x^4)$$

$$= -x + \frac{x^3}{2} - \frac{1}{2}(x^2 - x^4) - \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$$

$$= -x - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4} + o(x^4)$$

$$\lim_{x\to 0} \frac{\ln(1-x\cos x) - e^{-x+\frac{x^3}{3}} + 1 + x^2}{\frac{x^4}{2} + o(x^4)} =$$

$$= \frac{\cancel{-x} - \cancel{\frac{x^2}{2}} + \cancel{\frac{x^3}{6}} + \frac{x^4}{4} \cancel{-1} + \cancel{x} - \cancel{\frac{x^2}{2}} - \cancel{\frac{x^3}{6}} + \frac{7}{24}x^4 \cancel{+1} + \cancel{x^2} + o(x^4)}{\frac{x^4}{2} + o(x^4)}$$

$$= \frac{\frac{13}{24}x^4 + o(x^4)}{\frac{x^4}{2} + o(x^4)} \xrightarrow[x\to 0]{} \frac{13}{24}\cdot 2 = \frac{13}{12}$$

1

$$\lim_{x \to -3^-} f(x) = -7$$

$\forall \varepsilon > 0 : \exists \delta = \delta(-3, \varepsilon) > 0 : \forall x \in \mathbb{R} : -3-\delta < x < -3$

$\implies |f(x) + 7| < \varepsilon$

2

$f : \mathbb{R} \to \mathbb{R}$ $\qquad f(x) = x|x|$

Quindi:

$$f(x) = \begin{cases} x^2 & \text{se } x \geq 0 \\ -x^2 & \text{se } x < 0 \end{cases}$$

$\implies f$ è derivabile per $\mathbb{R} \setminus \{0\}$,

in quanto coincide con un monomio.

Consideriamo $x = 0$

$$f'_+(0) = \lim_{h \to 0+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0+} \frac{h^2}{h} = \lim_{h \to 0+} h = 0$$

$=$

$$f'_-(0) = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h^2}{h} = 0$$

f è derivabile in $x = 0 \implies f$ è derivabile su tutto $\mathbb{R}$