① $$f(x) = \frac{x^2}{x+1} e^{-\frac{6x}{x+1}}$$

$$D(f) = \mathbb{R} \setminus \{-1\}$$

$$\lim_{x \to -\infty} \underbrace{\frac{x^2}{x+1}}_{\to -\infty} \cdot e^{\overbrace{-\frac{6x}{x+1}}^{\to -6}} = -\infty$$

Analogamente: $\lim_{x \to +\infty} f(x) = +\infty$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{\overbrace{\frac{x^2}{x+1}}^{\to -\infty}}{\underbrace{e^{\frac{6x}{x+1}}}_{+\infty}}$$

(applicando il teorema di De L'Hopital)

$$\overset{H}{=} \lim_{x \to -1^-} \frac{\frac{2x(x+1) - x^2}{\cancel{(x+1)^2}}}{e^{\frac{6x}{x+1}} \cdot \frac{6(x+1) - 6x}{\cancel{(x+1)^2}}} =$$

$$= \lim_{x \to -1^-} \frac{\overbrace{x^2 + 2x}^{-1}}{\underbrace{e^{\frac{6x}{x+1}} \cdot 6}_{\to +\infty}} = 0$$

$$\lim_{x \to -1^+} \underbrace{\frac{x^2}{x+1}}_{+\infty} \cdot e^{\overbrace{-\frac{6x}{x+1}}^{+\infty}} = +\infty$$

$$f'(x) = \left(\frac{2x(x+1) - x^2}{(x+1)^2} - \frac{6(x+1) - 6x}{(x+1)^2} \cdot \frac{x^2}{x+1} \right) e^{-\frac{6x}{x+1}}$$

$$= e^{-\frac{6x}{x+1}} \left(\frac{x^2 + 2x}{(x+1)^2} - \frac{6x^2}{(x+1)^3} \right) =$$

$$= e^{-\frac{6x}{x+1}} \cdot \frac{(x^2 + 2x)(x+1) - 6x^2}{(x+1)^3} =$$

$$= e^{-\frac{6x}{x+1}} \cdot \frac{x^3 + x^2 + 2x^2 + 2x - 6x^2}{(x+1)^3} =$$

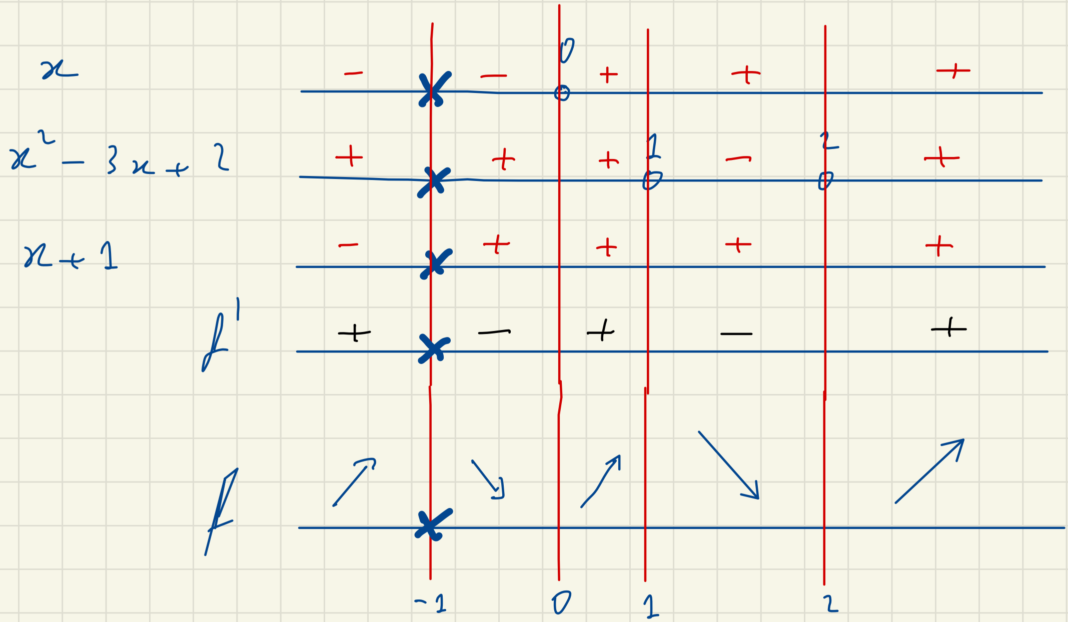
$$= e^{-\frac{6x}{x+1}} \cdot \frac{x^3 - 3x^2 + 2x}{(x+1)^3} =$$

$$= \underbrace{\frac{e^{-\frac{6x}{x+1}}}{(x+1)^2}}_{0} \cdot \frac{x(x^2 - 3x + 2)}{x+1}$$

$$x^2 - 3x + 2 = 0 \nearrow \quad x_{1,2} = \frac{3 \pm \sqrt{1}}{2} \left\langle \begin{array}{l} 1 \\ 2 \end{array} \right.$$

$$\frac{x(x^2-3x+2)}{x+1}$$

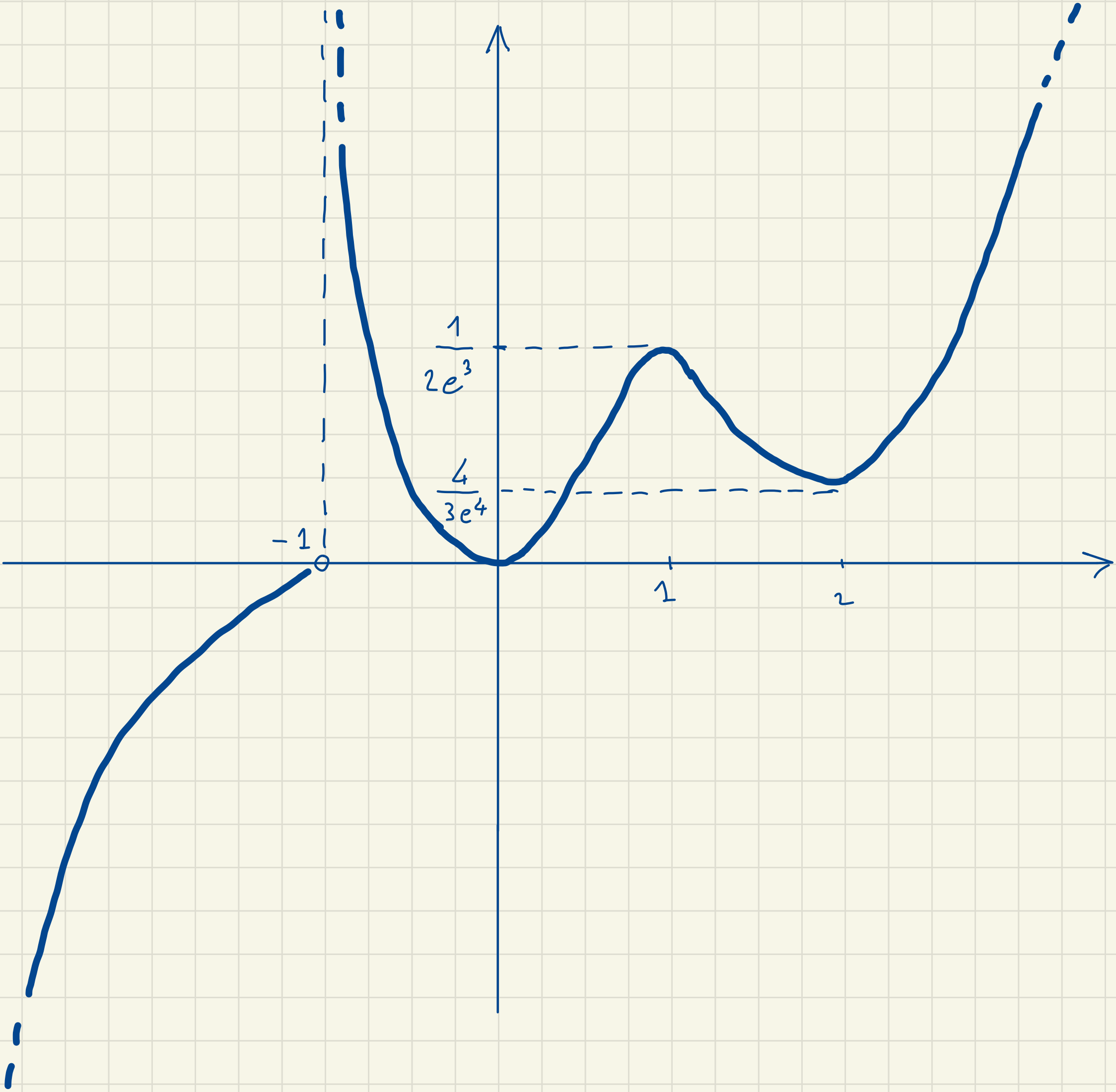
STUDIO DEL SEGNO



$f(0) = 0$ MINIMO REL.

$f(1) = \frac{1}{2} e^{-3}$ MAX REL.

$f(2) = \frac{4}{3} e^{-4}$ MIN. REL.



$\mathrm{Im}\, f = \mathbb{R}$

$f(x) = K$ ha 3 soluzioni : $K = \frac{4}{3e^4}, \frac{1}{2e^3}$

2

$$\lim_{x\to 0} \frac{\ln(1+x\cos x) - e^{x-\frac{x^3}{3}} + 1 + x^2}{e^{x^4}-1}$$

$$e^{x^4} - 1 = x^4 + o(x^4)$$

$$e^{x-\frac{x^3}{3}} = 1 + \left(x - \frac{x^3}{3}\right) + \frac{1}{2}\left(x - \frac{x^3}{3}\right)^2 + \frac{1}{3!}\left(x - \frac{x^3}{3}\right)^3$$

$$+ \frac{1}{4!}\left(x - \frac{x^3}{3}\right)^4 + o(x^4)$$

$$= 1 + \left(x - \frac{x^3}{3}\right) + \left(\frac{x^2}{2} - \frac{x^4}{3}\right) + \frac{x^3}{6} +$$

$$+ \frac{x^4}{24} + o(x^4)$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7}{24}x^4 + o(x^4)$$

$$x\cos x = x + o(x) \implies t = x\cos x$$

$$o(t^n) = o(x^n)$$

$n = 4$

$$\ln\left(1+\overbrace{x\cos x}^{=\,t}\right) =$$

$$= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + o(t^4)$$

$$= x\cos x - \frac{(x\cos x)^2}{2} + \frac{(x\cos x)^3}{3} -$$

$$- \frac{(x\cos x)^4}{4} + o(x^4)$$

$$= x\left(1 - \frac{x^2}{2} + o(x^3)\right) - \frac{\left(x - \frac{x^3}{2} + o(x^4)\right)^2}{2} +$$

$$+ \frac{(x + o(x^2))^3}{3} - \frac{(x + o(x^2))^4}{4} + o(x^4)$$

$$= x - \frac{x^3}{2} - \frac{1}{2}\left(x^2 - x^4\right) + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$$

$$= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{4} + o(x^4)$$

$$\lim_{x \to 0} \frac{\ln(1+x\cos x) - e^{x - \frac{x^3}{3}} + 1 + x^2}{e^{x^4} - 1} =$$

$$= \frac{\cancel{x} - \cancel{\frac{x^2}{2}} - \cancel{\frac{x^3}{6}} + \frac{x^4}{4} \cancel{-1} - \cancel{x} - \cancel{\frac{x^2}{2}} + \cancel{\frac{x^3}{6}} + \frac{7}{24}x^4 \cancel{+1} + \cancel{x^2} + o(x^4)}{x^4 + o(x^4)}$$

$$= \frac{\frac{13}{24}x^4 + o(x^4)}{x^4 + o(x^4)} \xrightarrow[x \to 0]{} \frac{13}{24}$$

(1) $$\lim_{x \to -3^+} f(x) = -\infty$$

$$\forall \varepsilon > 0 : \exists \delta = \delta(-3, \varepsilon) > 0 : \forall x \in \mathbb{R} : -3 < x < -3+\delta$$
$$\Rightarrow f(x) < -\varepsilon$$

(2) $f : \mathbb{R} \to \mathbb{R}$ $\qquad f(x) = x|x|$

Quindi:
$$f(x) = \begin{cases} x^2 & \text{se } x \geq 0 \\ -x^2 & \text{se } x < 0 \end{cases}$$

$\Rightarrow f$ è derivabile per $\mathbb{R} \setminus \{0\}$, in quanto coincide con un monomio.

Consideriamo $x = 0$

$$f'_+(0) = \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = \lim_{h \to 0^+} h = 0$$

$$f'_-(0) = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h^2}{h} = 0$$

$=$

f è derivabile in $x = 0 \Rightarrow f$ è derivabile su tutto $\mathbb{R}$