3-manifolds with cyclic symmetry and $(1,1)$-knots *

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Abstract

The connections between the 3-manifolds with cyclically presented fundamental groups and the strongly-cyclic branched coverings of $(1,1)$-knots are studied. A list of recent results on the topics is shown, including some important examples.

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1 Introduction

In the last decade many examples of cyclic branched coverings of knots in $\mathbf{S}^3$ admitting cyclic presentations for the fundamental groups have been described (see [1, 5, 6, 7, 9, 17, 18, 20, 21, 22, 24, 31]).

In the last three years our attention has been focused on the more general setting of 3-manifolds which are cyclic coverings of lens spaces, branched over $(1,1)$-knots.

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The first result in this context has been obtained in [13], where it is proved that all Dunwoody manifolds [9], which are manifolds with cyclically presented fundamental groups, are strongly-cyclic branched coverings of $(1,1)$-knots. After that, it has been shown in [28] that every n-fold strongly-cyclic branched covering of a $(1,1)$-knot admits a Heegaard diagram of genus n which encodes a cyclic presentation for the fundamental group. In [3] this result has been improved, obtaining a constructive algorithm which explicitly gives the cyclic presentations, starting from a representation of $(1,1)$-knots through the elements of the mapping class group of the twice punctured torus (see [4] for further details on this representation).

All these results, as well as others on the same topic, will be described in Section 3, where some examples are also presented.

The necessary basic notions concerning strongly-cyclic branched coverings of (1,1)-knots, cyclically presented groups and Dunwoody manifolds are described in Section 2.

2 Preliminaries

2.1 Strongly-cyclic branched coverings of (1,1)-knots

An n-fold cyclic covering of a 3-manifold N^3 branched over a knot $K \subset N^3$ will be called *strongly-cyclic* if the branching index of K is n. This means that the homology class of a meridian loop m around K is mapped by the monodromy map $\omega : H_1(N^3 - K) \to \mathbf{Z}_n$ in a generator of $\mathbf{Z}_n$ (up to equivalence we can always suppose $\omega[m] = 1$).

Observe that a cyclic branched covering of a knot K in $\mathbf{S}^3$ is always strongly-cyclic and is uniquely determined, up to equivalence, since $H_1(\mathbf{S}^3 - K) \cong \mathbf{Z}$. Obviously, this property is no longer true for a knot in a more general 3-manifold.

In this paper we deal with strongly-cyclic branched coverings of $(1,1)$-knots, which are knots in lens spaces (possibly in $\mathbf{S}^3$).

A knot K in a 3-manifold N^3 is called a $(1,1)$-*knot* if there exists a Heegaard splitting of genus one

$$(N^3, K) = (T, A) \cup_\phi (T', A'),$$

where T and T' are solid tori, $A \subset T$ and $A' \subset T'$ are properly embedded trivial arcs[1], and $\phi : (\partial T, \partial A) \to (\partial T', \partial A')$ is an attaching homeomorphism (see Figure 1). Obviously, N^3 turns out to be a lens space $L(p,q)$ (including $\mathbf{S}^3 = L(1,0)$).

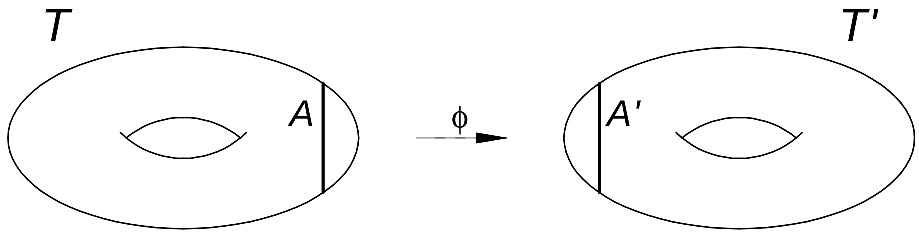

Figure 1: A $(1,1)$-knot decomposition.

It is well known that the family of $(1,1)$-knots contains all torus knots (trivially) and all two-bridge knots in $\mathbf{S}^3$ (see [23]). Several topological properties of $(1,1)$-knots have recently been investigated in [2, 3, 4, 8, 10, 11, 12, 13, 14, 15, 16, 25, 26, 27, 28, 32, 33, 34].

An algebraic representation of $(1,1)$-knots have been developed in [3, 4], where it is shown that there is a natural surjective map from the pure mapping class group of the twice punctured torus $PMCG_2(\partial T)$ to the class $\mathcal{K}_{1,1}$ of all $(1,1)$-knots

$$\varphi \in PMCG_2(\partial T) \mapsto K_\varphi \in \mathcal{K}_{1,1}.$$

A standard set of generators for $PMCG_2(\partial T)$ is given by the three right-hand Dehn twists $\delta_\alpha, \delta_\beta, \delta_\gamma$, respectively around the curves α, β, γ depicted in Figure 2.

The fundamental group and the first integer homology group of the exterior of a $(1,1)$-knot K_φ can be explicitly obtained from its representation φ. Referring to Figure 2, we define the loops $\bar{\alpha} = \xi \cdot \alpha \cdot \xi^{-1}$, $\bar{\beta} = \xi_1 \cdot \beta \cdot \xi_1^{-1}$ and $\bar{\gamma} = \xi_2 \cdot \gamma \cdot \xi_2^{-1}$, where ξ, ξ_1, ξ_2 are paths connecting $*$ to α, β and γ respectively, as depicted in Figure 2.

Lemma 1 [3] *Let K_φ be a $(1,1)$-knot in $L(p,q)$. Then:*

i) $\pi_1(L(p,q) - K_\varphi, *) = \langle\, \bar{\alpha}, \bar{\gamma} \,|\, i_\# \varphi_\# (\bar{\beta}) \,\rangle$, *where* $i : (\partial T, \partial A) \to (T, A)$ *is the inclusion map;*

[1] This means that there exists a disk $D \subset T$ (resp. $D' \subset T'$) with $A \cap D = A \cap \partial D = A$ and $\partial D - A \subset \partial T$ (resp. $A' \cap D' = A' \cap \partial D' = A'$ and $\partial D' - A' \subset \partial T'$).

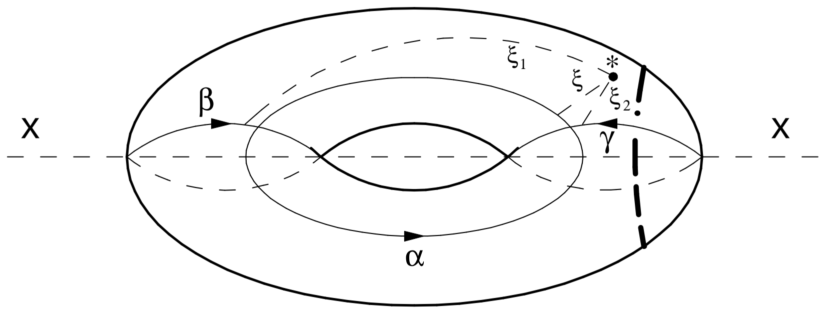

Figure 2: Generators of $PMCG_2(\partial T)$.

ii) $H_1(L(p,q) - K_\varphi) = \langle \alpha, \gamma \,|\, p\alpha + q''\gamma \rangle \cong \mathbf{Z} \oplus \mathbf{Z}_{\gcd(p,q'')}$, *where* q'' *is uniquely determined by the homology relation:* $\varphi_*(\beta) = p\alpha + q'\beta + q''\gamma$.

As a consequence, an n-fold strongly-cyclic branched covering of a $(1,1)$-knot does not necessarily exist and, if it exists, may not be unique, up to equivalence.

Proposition 2 [3] *Let* K_φ *be a* $(1,1)$*-knot in* $L(p,q)$*. Then* K_φ *admits an* n*-fold strongly-cyclic branched covering if and only if* d *divides* q''*, where* $d = \gcd(p,n)$ *and* q'' *is uniquely determined by the homology relation* $\varphi_*(\beta) = p\alpha + q'\beta + q''\gamma$*. In this case, there exist exactly* d *of such coverings, up to equivalence.*

2.2 Cyclically presented groups

A finite balanced presentation of a group

$$< x_1, \ldots, x_n \,|\, r_1, \ldots, r_n >$$

is said to be a *cyclic presentation* if there exists a word w in the free group F_n generated by $x_1, \ldots, x_n$ such that $r_k = \theta_n^{k-1}(w)$, $k = 1, \ldots, n$, where $\theta_n : F_n \to F_n$ denotes the automorphism defined by $\theta_n(x_i) = x_{i+1}$ (subscripts mod n), $i = 1, \ldots, n$. This presentation (and the related group) will be denoted by $G_n(w)$. For further details see [19].

The following examples show that these groups are strictly connected with 3-manifolds with cyclic symmetry:

- the Fibonacci group $F(2n) = G_{2n}(x_1x_2x_3^{-1}) = G_n(x_1^{-1}x_2^2x_3^{-1}x_2)$ is the fundamental group of the n-fold cyclic covering of $\mathbf{S}^3$ branched over the figure-eight knot, for all $n > 1$ (see [18]);

- the Sieradsky group $S(n) = G_n(x_1x_3x_2^{-1})$ is the fundamental group of the n-fold cyclic covering of $\mathbf{S}^3$ branched over the trefoil knot, for all $n > 1$ (see [5]);

- the fractional Fibonacci group $\widetilde{F}_{l,k}(n) = G_n((x_1^{-l}x_2^l)^k x_2 (x_3^{-l}x_2^l)^k)$ is the fundamental group of the n-fold cyclic covering of $\mathbf{S}^3$ branched over the genus one two-bridge knot with Conway coefficients $[2l, -2k]$, for all $n > 1$ and $l, k > 0$ (see [31]).

Moreover, all the above cyclic presentations are geometric [29] (i.e. they arise from suitable Heegaard diagrams).

2.3 Dunwoody manifolds

In order to investigate the relations between cyclic branched coverings of knots and cyclic presentation of groups, M.J. Dunwoody introduced [9] a class of Heegaard diagrams with cyclic symmetry, depending on six integers a, b, c, n, r, s, such that $n > 0$, $a, b, c \geq 0$ and $a + b + c > 0$. This construction gave rise to a wide class of closed orientable 3-manifolds $D(a, b, c, n, r, s)$, called *Dunwoody manifolds*, with fundamental groups admitting geometric cyclic presentations.

For particular values of the parameters, called admissible, a Dunwoody diagram is an (open) Heegaard diagram of genus n (see Figure 3), which contains n upper cycles $C'_1, \ldots, C'_n$, and n lower cycles $C''_1, \ldots, C''_n$, each having $d = 2a + b + c$ vertices. For every $i = 1, \ldots, n$, the cycle C'_i (resp. C''_i) is connected to the cycle C'_{i+1} (resp. C''_{i+1}) by a parallel arcs, to the cycle C''_i by c parallel arcs and to the cycle C''_{i+1} by b parallel arcs. The cycle C'_i is glued to the cycle C''_{i-s} (subscripts mod n) so that equally labelled vertices are identified together (the labelling of the vertices is pointed out in Figure 4). It is evident that the identification rule, as well as the diagram, is invariant with respect to a cyclic action of order n.

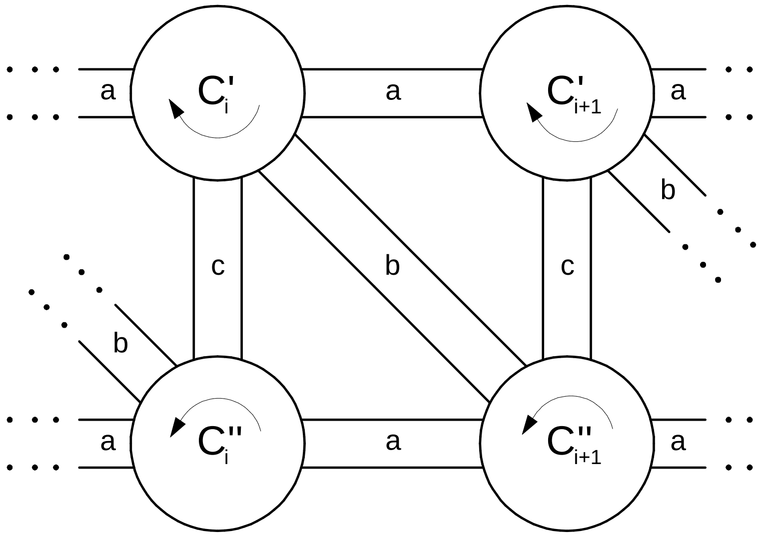

Figure 3: Heegaard diagram of Dunwoody type.

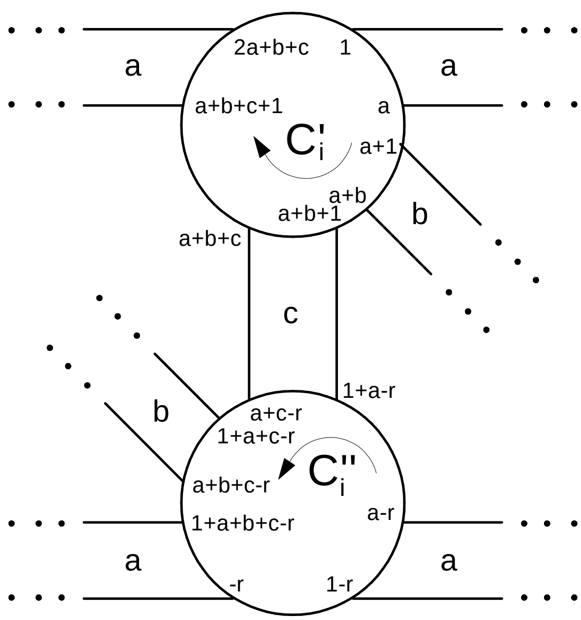

Figure 4:

3 Main results

The next theorem gives a characterization of Dunwoody manifolds as strongly-cyclic branched coverings of $(1,1)$-knots.

Theorem 3 [13] *The Dunwoody manifold $D(a,b,c,n,r,s)$ is the n-fold strongly-cyclic covering of a lens space (possibly $\mathbf{S}^3$) branched over a $(1,1)$-knot K, which only depends on the integers a,b,c,r.*

As a consequence, for certain particular values of the parameters, a Dunwoody manifold turns out to be a cyclic covering of $\mathbf{S}^3$ branched over a knot. This gives a positive answer to a conjecture formulated by Dunwoody in [9], which has also been independently proved in [30].

Corollary 4 [13] *Each Dunwoody manifold $D(a,b,c,n,r,s)$ of Section 3 of [9] is an n-fold cyclic covering of $\mathbf{S^3}$, branched over a $(1,1)$-knot $K \subset \mathbf{S}^3$, which only depends on a,b,c,r.*

An interesting problem, which arises naturally, is that of characterizing the set $\mathcal{D}$ of branching knots in $\mathbf{S^3}$ involved in Corollary 4. The next proposition shows that $\mathcal{D}$ contains all 2-bridge knots. Note that in the statement $\bar{s}$ is an integer only depending on a and r (see details in Section 3 of [13]).

Proposition 5 [13] *For all $a,r>0$ and $n>1$, the n-fold cyclic covering of $\mathbf{S}^3$ branched over $\mathbf{b}(2a+1,2r)$ is the Dunwoody manifold $D(a,0,1,n,r,\bar{s})$. Thus, all cyclic branched coverings of 2-bridge knots are Dunwoody manifolds.*

The computational results listed in [30] suggest that $\mathcal{D}$ could contain all torus knots. The following recent result involves a large class of torus knots.

Proposition 6 *For all $c>0$ and $k,n>1$, the n-fold cyclic covering of $\mathbf{S}^3$ branched over the torus knot $\mathbf{t}(k,ck+1)$ (resp. $\mathbf{t}(k,ck-1)$) is the Dunwoody manifold $D(1,k-2,(k-1)(2c-1),n,k,k)$ (resp. $D(1,k-2,(k-1)(2c-1)-2,n,(k-1)(2c-3),-k)$). Thus, all cyclic branched coverings of torus knots of bridge number ≤ 4 are Dunwoody manifolds.*

The next theorem shows that $(1,1)$-knots are suitable objects to produce manifolds with cyclically presented fundamental groups.

Theorem 7 [28] *Every n-fold strongly-cyclic branched covering of a $(1,1)$-knot admits a Heegaard diagram of genus n, which induces a cyclic presentation for the fundamental group of the manifold.*

Necessary and sufficient conditions for the existence and uniqueness of the n-fold strongly-cyclic branched covering of a $(1,1)$-knot are contained in Proposition 2.

The following result gives an explicit way of finding a cyclic presentation for the fundamental group of any strongly-cyclic branched covering of a $(1,1)$-knot.

Theorem 8 [3] *Let $C_{n,\omega}(K_\varphi)$ be the n-fold strongly-cyclic branched covering of the $(1,1)$-knot K_φ, with associated monodromy ω. Let $\bar{r}(\bar{x},\bar{\gamma}) = r(\bar{x}\bar{\gamma}^{\omega(\alpha)},\bar{\gamma})$, where $r(\bar{\alpha},\bar{\gamma}) = i_\#\varphi_\#(\bar{\beta})$. If $\bar{r}(\bar{x},\bar{\gamma}) = \bar{x}^{\epsilon_1}\bar{\gamma}^{\delta_1}\cdots\bar{x}^{\epsilon_s}\bar{\gamma}^{\delta_s}$, then the fundamental group of $C_{n,\omega}(K_\varphi)$ admits the cyclic presentation $G_n(w)$, with:*

$$w = x_{i_1}^{\epsilon_1}\cdots x_{i_s}^{\epsilon_s}$$

(subscripts mod n), where $i_k \equiv 1 + \sum_{j=1}^{k-1}\delta_j \mod n$, for $k = 1,\ldots,s$.

The following examples have been obtained in [3] and [4] by applying Theorem 8.

(i) The fundamental group of the n-fold cyclic branched covering of a genus one two-bridge knot, with Conway parameters $[2a_1, 2b_1]$, admits the cyclic presentation $G_n(w)$, where

$$w = (x_1^{-b_1}x_2^{b_1})^{a_1}x_2(x_3^{-b_1}x_2^{b_1})^{a_1}.$$

(ii) The fundamental group of the n-fold cyclic branched covering of the torus knot $\mathbf{t}(k, ck+1)$, with $c, k > 0$, admits the cyclic presentation $G_n(w)$, where

$$w \quad = \quad \prod_{j=0}^{c(k-1)} x_{1+jk}\prod_{i=0}^{k-2}\prod_{l=1}^{c} x^{-1}_{k(c(k-1)+1)-i(ck+1)-lk}$$

(subscripts mod n).

(iii) The fundamental group of the n-fold cyclic branched covering of the torus knot $\mathbf{t}(k, ck+2)$, with $c, k > 0$ and k odd, admits the cyclic presentation $G_n(w)$, where w is equal to

$$\prod_{i=0}^{(k-3)/2} \Big(\prod_{j=0}^{c(k-1)/2} x_{1-i(ck+2)+jk} \prod_{l=0}^{c(k+1)/2} x^{-1}_{ck(k-1)/2-i(ck+2)-lk} \Big) \prod_{m=0}^{c(k-1)/2} x_{1-(k-1)(ck+2)/2+mk}$$

(subscripts mod n).

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