

ON p -SYMMETRIC HEEGAARD SPLITTINGS

MICHELE MULAZZANI

*Department of Mathematics and C.I.R.A.M.,
University of Bologna, I-40127 Bologna, ITALY.
mulazza@dm.unibo.it*

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ABSTRACT

We show that every p -fold strictly-cyclic branched covering of a b -bridge link in \mathbf{S}^3 admits a p -symmetric Heegaard splitting – in the sense of Birman and Hilden – of genus $g = (b - 1)(p - 1)$. This gives a complete converse of one of the results of the two authors. Moreover, we introduce the concept of weakly p -symmetric Heegaard splittings and prove similar results in this more general context.

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1. Introduction

Let M be a closed, orientable 3-manifold, Y_g and Y'_g be handlebodies of genus $g \geq 0$ and $\phi : \partial Y_g \rightarrow \partial Y'_g$ be an attaching homeomorphism. A Heegaard splitting $M = Y_g \cup_\phi Y'_g$ is called in [2] *p -symmetric*, with $p > 1$, if there is a disjoint embedding of Y_g and Y'_g into \mathbf{E}^3 such that $Y'_g = \tau(Y_g)$, for a translation τ of \mathbf{E}^3 , and an orientation-preserving homeomorphism $\mathcal{P} : \mathbf{E}^3 \rightarrow \mathbf{E}^3$ of period p , such that $\mathcal{P}(Y_g) = Y_g$ and, if \mathcal{G} is the cyclic group of order p generated by \mathcal{P} and $\Phi : \partial Y_g \rightarrow \partial Y_g$ is the orientation-preserving homeomorphism $\Phi = \tau_{|\partial Y'_g}^{-1} \phi$, the following conditions are fulfilled:

- i) Y_g/\mathcal{G} is homeomorphic to a 3-ball;
- ii) $\text{Fix}(\mathcal{P}_{|Y_g}^h) = \text{Fix}(\mathcal{P}_{|Y_g})$, for each $1 \leq h \leq p - 1$;
- iii) $\text{Fix}(\mathcal{P}_{|Y_g})/\mathcal{G}$ is an unknotted set of arcs* in the ball Y_g/\mathcal{G} ;

*A set of mutually disjoint arcs $\{t_1, \dots, t_n\}$ properly embedded in a handlebody Y is *unknotted* if there is a set of mutually disjoint discs $D = \{D_1, \dots, D_n\}$ properly embedded in Y such that $t_i \cap D_i = t_i \cap \partial D_i = t_i$, $t_i \cap D_j = \emptyset$ and $\partial D_i - t_i \subset \partial Y$ for $1 \leq i, j \leq n$ and $i \neq j$.

iv) there exists an integer p_0 such that $\Phi\mathcal{P}|_{\partial Y_g}\Phi^{-1} = (\mathcal{P}|_{\partial Y_g})^{p_0}$.[†]

Observe that the map $\mathcal{P}' = \tau\mathcal{P}\tau^{-1}$ is obviously an orientation-preserving homeomorphism of period p of \mathbf{E}^3 with the same properties as \mathcal{P} , regarding Y'_g , and the relation $\phi\mathcal{P}'|_{\partial Y'_g}\phi^{-1} = (\mathcal{P}'|_{\partial Y'_g})^{p_0}$ easily holds.

The p -symmetric Heegaard genus $g_p(M)$ of a 3-manifold M is the smallest integer g such that M admits a p -symmetric Heegaard splitting of genus g . The following results have been established in [2]:

Proposition 1. ([2], Theorem 2) *Every closed, orientable 3-manifold of p -symmetric Heegaard genus g admits a representation as a p -fold cyclic covering of \mathbf{S}^3 , branched over a link L of bridge number*

$$b(L) \leq 1 + \frac{g}{p-1}.$$

Proposition 2. ([2], Theorem 3) *The p -fold cyclic covering of \mathbf{S}^3 branched over a knot of braid number b is a closed, orientable 3-manifold M of p -symmetric Heegaard genus*

$$g_p(M) \leq (b-1)(p-1). \tag{*}$$

As a consequence, Birman and Hilden got the interesting result that every closed, orientable 3-manifold of Heegaard genus $g \leq 2$ is a 2-fold covering of \mathbf{S}^3 branched over a link of bridge number $g+1$ and, conversely, the 2-fold covering of \mathbf{S}^3 branched over a link of bridge number $b \leq 3$ is a closed, orientable 3-manifold of Heegaard genus $b-1$ (compare also [7]).

Note that Proposition is not a complete converse of Proposition , since it only concerns knots and, moreover, the relation (*) connects the genus with the braid number, which is greater than or equal to (often greater) the bridge number. In this paper we fill this gap, giving a complete converse of Proposition . Since the coverings involved in Proposition are strictly-cyclic (see next chapter for details on strictly-cyclic branched coverings of links), our statement will concern these kind of coverings. More precisely, we shall prove in Theorem 2 that a p -fold strictly-cyclic covering of \mathbf{S}^3 , branched over a link of bridge number b , is a closed, orientable 3-manifold M which admits a p -symmetric Heegaard splitting of genus $g = (b-1)(p-1)$, and therefore has p -symmetric Heegaard genus $g_p(M) \leq (b-1)(p-1)$. The proof of this fact will be achieved using the concept of special plat presentation of a link.

In the last section we introduce a more intrinsic notion of p -symmetric Heegaard splitting of a 3-manifold – which will be called weakly p -symmetric Heegaard splitting – generalizing the one given by Birman and Hilden. Proposition will translate

[†]By the positive solution of the Smith Conjecture [5] it is easy to see that necessarily $p_0 \equiv \pm 1 \pmod p$.

to this new context and a complete converse of it will be given, for all cases of branched cyclic coverings of links.

2. Special plat presentations of links

Let $\beta = \{(p_k(t), t) \mid 1 \leq k \leq 2n, t \in [0, 1]\} \subset \mathbf{E}^2 \times [0, 1]$ be a geometric $2n$ -string braid of \mathbf{E}^3 [1], where $p_1, \dots, p_{2n} : [0, 1] \rightarrow \mathbf{E}^2$ are continuous maps such that $p_k(t) \neq p_{k'}(t)$, for every $k \neq k'$ and $t \in [0, 1]$, and such that $\{p_1(0), \dots, p_{2n}(0)\} = \{p_1(1), \dots, p_{2n}(1)\}$. We set $P_k = p_k(0)$, for each $k = 1, \dots, 2n$, and $A_i = (P_{2i-1}, 0), B_i = (P_{2i}, 0), A'_i = (P_{2i-1}, 1), B'_i = (P_{2i}, 1)$, for each $i = 1, \dots, n$ (see Figure 1). Moreover, we set $\mathcal{F} = \{P_1, \dots, P_{2n}\}$, $\mathcal{F}_1 = \{P_1, P_3, \dots, P_{2n-1}\}$ and $\mathcal{F}_2 = \{P_2, P_4, \dots, P_{2n}\}$.

The braid β is realized through an ambient isotopy $\hat{\beta} : \mathbf{E}^2 \times [0, 1] \rightarrow \mathbf{E}^2 \times [0, 1]$, $\hat{\beta}(x, t) = (\beta_t(x), t)$, where β_t is an homeomorphism of \mathbf{E}^2 such that $\beta_0 = \text{Id}_{\mathbf{E}^2}$ and $\beta_t(P_i) = p_i(t)$, for every $t \in [0, 1]$. Therefore, the braid β naturally defines an orientation-preserving homeomorphism $\tilde{\beta} = \beta_1 : \mathbf{E}^2 \rightarrow \mathbf{E}^2$, which fixes the set \mathcal{F} . Note that β uniquely defines $\tilde{\beta}$, up to isotopy of $\mathbf{E}^2 \text{ mod } \mathcal{F}$.

Connecting the point A_i with B_i by a circular arc α_i (called *top arc*) and the point A'_i with B'_i by a circular arc α'_i (called *bottom arc*), as in Figure 1, for each $i = 1, \dots, n$, we obtain a $2n$ -plat presentation of a link L in \mathbf{E}^3 , or equivalently in \mathbf{S}^3 . As is well known, every link admits plat presentations and, moreover, a $2n$ -plat presentation corresponds to an n -bridge presentation of the link. So, the bridge number $b(L)$ of a link L is the smallest positive integer n such that L admits a representation by a $2n$ -plat. For further details on braids, plat and bridge presentations of links we refer to [1].

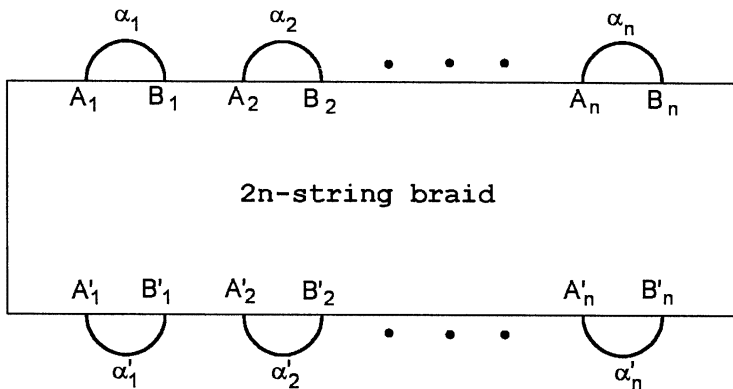


Fig. 1. A $2n$ -plat presentation of a link.

Remark 1 A $2n$ -plat presentation of a link $L \subset \mathbf{E}^3 \subset \mathbf{S}^3 = \mathbf{E}^3 \cup \{\infty\}$ furnishes a $(0, n)$ -decomposition [6] $(\mathbf{S}^3, L) = (D, A_n) \cup_{\phi'} (D', A'_n)$ of the link, where D and D' are the 3-balls $D = (\mathbf{E}^2 \times]-\infty, 0]) \cup \{\infty\}$ and $D' = (\mathbf{E}^2 \times [1, +\infty[) \cup \{\infty\}$,

$A_n = \alpha_1 \cup \dots \cup \alpha_n$, $A'_n = \alpha'_1 \cup \dots \cup \alpha'_n$ and $\phi' : \partial D \rightarrow \partial D'$ is defined by $\phi'(\infty) = \infty$ and $\phi'(x, 0) = (\tilde{\beta}(x), 1)$, for each $x \in \mathbb{E}^2$.

If a $2n$ -plat presentation of a μ -component link $L = \bigcup_{j=1}^{\mu} L_j$ is given, each component L_j of L contains n_j top arcs and n_j bottom arcs. Obviously, $\sum_{j=1}^{\mu} n_j = n$. A $2n$ -plat presentation of a link L will be called *special* if:

- (1) the top arcs and the bottom arcs belonging to L_1 are $\alpha_1, \dots, \alpha_{n_1}$ and $\alpha'_1, \dots, \alpha'_{n_1}$, the top arcs and the bottom arcs belonging to L_2 are $\alpha_{n_1+1}, \dots, \alpha_{n_1+n_2}$ and $\alpha'_{n_1+1}, \dots, \alpha'_{n_1+n_2}$, ..., the top arcs and the bottom arcs belonging on L_{μ} are $\alpha_{n_1+\dots+n_{\mu-1}+1}, \dots, \alpha_{n_1+\dots+n_{\mu}} = \alpha_n$ and $\alpha'_{n_1+\dots+n_{\mu-1}+1}, \dots, \alpha'_{n_1+\dots+n_{\mu}} = \alpha'_n$;
- (2) $p_{2i-1}(1) \in \mathcal{F}_1$ and $p_{2i}(1) \in \mathcal{F}_2$, for each $i = 1, \dots, n$.

It is clear that, because of (2), the homeomorphism $\tilde{\beta}$, associated to a $2n$ -string braid β defining a special plat presentation, keeps fixed both the sets \mathcal{F}_1 and \mathcal{F}_2 . Although a special plat presentation of a link is a very particular case, we shall prove that every link admits such kind of presentation.

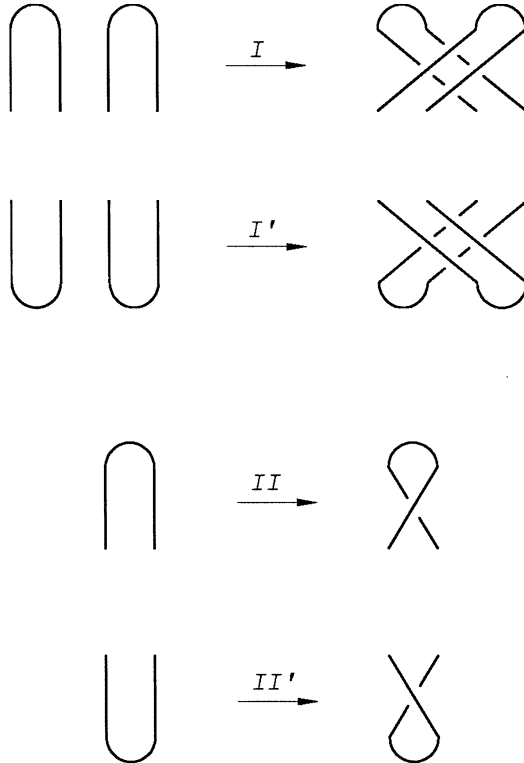


Fig. 2. Moves on plat presentations.

Proposition 3. *Every link L admits a special $2n$ -plat presentation, for each $n \geq b(L)$.*

Proof. Let L be presented by a $2n$ -plat. We show that this presentation is equivalent to a special one, by using a finite sequence of moves on the plat presentation which changes neither the link type nor the number of plats. The moves are of the four types I, I', II and II' depicted in Figure 2. First of all, it is straightforward that condition (1) can be satisfied by applying a suitable sequence of moves of type I and I' . Furthermore, condition (2) is equivalent to the following: (2') there exists an orientation of L such that, for each $i = 1, \dots, n$, the top arc α_i is oriented from A_i to B_i and the bottom arc α'_i is oriented from B'_i to A'_i . Therefore, choose any orientation on L and apply moves of type II (resp. moves of type II') to the top arcs (resp. bottom arcs) which are oriented from B_i to A_i (resp. from A'_i to B'_i) \square .

A p -fold branched cyclic covering of an oriented μ -component link $L = \bigcup_{j=1}^{\mu} L_j \subset \mathbf{S}^3$ is completely determined (up to equivalence) by assigning to each component L_j an integer $c_j \in \mathbf{Z}_p - \{0\}$, such that the set $\{c_1, \dots, c_{\mu}\}$ generates the group \mathbf{Z}_p . The monodromy associated to the covering sends each meridian of L_j , coherently oriented with the chosen orientations of L and \mathbf{S}^3 , to the permutation $(12 \dots p)^{c_j} \in \Sigma_p$. By multiplying each c_j by the same invertible elements c of \mathbf{Z}_p , we get an equivalent covering.

Following [4] we shall call a branched cyclic covering:

- a) *strictly-cyclic* if $c_{j'} = c_{j''}$, for every $j', j'' \in \{1, \dots, \mu\}$,
- b) *almost-strictly-cyclic* if $c_{j'} = \pm c_{j''}$, for every $j', j'' \in \{1, \dots, \mu\}$,[‡]
- c) *meridian-cyclic* if $\gcd(b, c_j) = 1$, for every $j \in \{1, \dots, \mu\}$,
- d) *singly-cyclic* if $\gcd(b, c_j) = 1$, for some $j \in \{1, \dots, \mu\}$,
- e) *monodromy-cyclic* if it is cyclic.

The following implications are straightforward:

$$a) \Rightarrow b) \Rightarrow c) \Rightarrow d) \Rightarrow e) .$$

Moreover, the five definitions are equivalent when L is a knot. Similar definitions and properties also hold for a p -fold cyclic covering of a 3-ball, branched over a set of properly embedded (oriented) arcs.

Note that every branched cyclic covering of a link arising from a p -symmetric Heegaard splitting – according to Birman-Hilden construction – is strictly-cyclic. We show that, conversely, every p -fold branched strictly-cyclic covering of a link admits a p -symmetric Heegaard splitting.

[‡]By a suitable reorientation of the link, an almost-strictly-cyclic covering becomes a strictly-cyclic one.

Theorem 4. *A p -fold strictly-cyclic covering of S^3 branched over a link L of bridge number b is a closed, orientable 3-manifold M of p -symmetric Heegaard genus*

$$g_p(M) \leq (b - 1)(p - 1).$$

Proof. Let L be presented by a special $2b$ -plat arising from a braid β , and let $(S^3, L) = (D, A_b) \cup_{\phi'} (D', A'_b)$ be the $(0, b)$ -decomposition described in Remark 1. Now, all arguments of the proofs of Theorem 3 of [2] entirely apply and the condition of Lemma 4 of [2] is satisfied, since the homeomorphism $\tilde{\beta}$ associated to β fixes both the sets \mathcal{F}_1 and \mathcal{F}_2 \square .

As a consequence of Theorem 2 and Birman-Hilden results, there is a natural one-to-one correspondence between p -symmetric Heegaard splittings and p -fold branched strictly-cyclic coverings of links.

3. Weakly p -symmetric Heegaard splittings

A Heegaard splitting $M = Y_g \cup_{\psi} Y'_g$ of a 3-manifold M will be called *weakly p -symmetric*, with $p > 1$, if there is an orientation-preserving homeomorphism $\mathcal{R} : Y_g \rightarrow Y_g$ of period p and a homeomorphism $\rho : Y_g \rightarrow Y'_g$ such that, if \mathcal{H} is the cyclic group of order p generated by \mathcal{R} and $\Psi : \partial Y_g \rightarrow \partial Y'_g$ is the orientation-preserving homeomorphism $\Psi = \rho|_{\partial Y_g}^{-1} \psi$, the following conditions are fulfilled:

- i') Y_g/\mathcal{H} is homeomorphic to a 3-ball;
- ii') $\bigcup_{h=1}^{p-1} \text{Fix}(\mathcal{R}^h)/\mathcal{H}$ is an unknotted set of arcs in the ball Y_g/\mathcal{H} ;
- iii') there exists an integer p_0 such that $\Psi \mathcal{R}|_{\partial Y_g} \Psi^{-1} = (\mathcal{R}|_{\partial Y_g})^{p_0}$.

Observe that the map $\mathcal{R}' = \rho \mathcal{R} \rho^{-1}$ is obviously an orientation-preserving homeomorphism of period p of Y'_g with the same properties as \mathcal{R} , and the relation $\psi \mathcal{R}|_{\partial Y_g} \psi^{-1} = (\mathcal{R}'|_{\partial Y'_g})^{p_0}$ easily holds.

The *weakly p -symmetric Heegaard genus* $\tilde{g}_p(M)$ of a 3-manifold M is the smallest integer g such that M admits a weakly p -symmetric Heegaard splitting of genus g . Observe that a p -symmetric Heegaard splitting, as defined in Section 1, is weakly p -symmetric (take $\mathcal{R} = \mathcal{P}|_{Y_g}$ and $\rho = \tau$).

In order to obtain results analogous to the ones for p -symmetric Heegaard splittings, for weakly p -symmetric Heegaard splittings, we make use of some classical results on branched cyclic coverings of S^2 . A p -fold cyclic covering $\gamma : M^2 \rightarrow S^2$, branched over a set of N points $B_\gamma = \{P_1, \dots, P_N\}$ is completely determined, up to equivalence, by a map $\omega'_\gamma : B_\gamma \rightarrow \mathbf{Z}_p - \{0\}$, $\omega'_\gamma(P_k) = c_k$ such that (A) $\{c_1, \dots, c_N\}$ generates the group \mathbf{Z}_p and (B) $\sum_{k=1}^N c_k = 0$. The monodromy map $\omega_\gamma : \pi_1(S^2 - B_\gamma) \rightarrow \Sigma_p$ associated to the covering sends a loop m_k represented by a small circle around P_k , coherently oriented with a chosen orientation of S^2 , to the permutation $(1\ 2 \cdots p)^{c_k}$, for all $k = 1, \dots, N$. Condition (A) guarantees that M^2

is connected and condition (B) depends on the fact that $\pi_1(\mathbb{S}^2 - B_\gamma)$ admits the finite presentation

$$\pi_1(\mathbb{S}^2 - B_\gamma) = \langle m_1, \dots, m_N \mid \prod_{k=1}^N m_k = 1 \rangle.$$

The set $\gamma^{-1}(P_k)$ has cardinality $\gcd(p, c_k)$ for all $k = 1, \dots, N$ and therefore, by standard calculations, M^2 has Euler characteristic:

$$\chi(M^2) = 2p - Np + \sum_{k=1}^N \gcd(p, c_k). \tag{**}$$

Theorem 5. *Every closed, orientable 3-manifold of weakly p -symmetric Heegaard genus g admits a representation as a p -fold cyclic covering of \mathbb{S}^3 , branched over a link L of bridge number*

$$b(L) \leq \frac{p - 1 + g}{p - p^*},$$

where $p^* = \max\{d \mid 1 \leq d < p, d \text{ divides } p\}$.

Proof. The proof is similar to the one of Theorem 2 of [2]. The definition of weakly p -symmetric splitting implies that the canonical projections $\pi : Y_g \rightarrow Y_g/\mathcal{H}$ and $\pi' : Y'_g \rightarrow Y_g/\mathcal{H}'$, where \mathcal{H}' is the cyclic group of order p generated by \mathcal{R}' , are p -fold branched cyclic coverings. By iii'), there is a map $\psi' : \partial Y_g/\mathcal{H} \rightarrow \partial Y'_g/\mathcal{H}'$ such that $\pi' \psi' = \psi \pi$. Therefore, the map $\pi \cup \pi' : Y_g \cup_\psi Y'_g \rightarrow (Y_g/\mathcal{H}) \cup_{\psi'} (Y'_g/\mathcal{H}') \cong \mathbb{S}^3$ is a p -fold branched cyclic covering. The restriction map $\gamma = \pi|_{\partial Y_g}$ turns out to be a p -fold cyclic covering of $\partial Y_g/\mathcal{H} \cong \mathbb{S}^2$, branched over $2n$ points P_1, \dots, P_{2n} , with $\omega'_\gamma(P_k) = c_k$ such that $c_{2h} = -c_{2h-1}$ for each $h = 1, \dots, n$. By (**) we get $2 - 2g = 2p - 2np + 2 \sum_{h=1}^n \gcd(p, c_{2h-1})$. Since $c_k \neq 0$, we have $(p, c_k) \leq p^*$ and therefore $p - 1 + g = np - \sum_{h=1}^n \gcd(p, c_{2h-1}) \geq np - np^*$. Thus, $b(L) \leq n \leq (p - 1 + g)/(p - p^*)$ and the statement is achieved \square .

Corollary 6. *If p is prime, then every closed, orientable 3-manifold of weakly p -symmetric Heegaard genus g admits a representation as a p -fold cyclic covering of \mathbb{S}^3 , branched over a link L of bridge number*

$$b(L) \leq 1 + \frac{g}{p - 1}.$$

Proof. Straightforward, since $p^* = 1$ when p is prime. \square .

In order to prove the converse of Theorem 2, we need the following result (compare Lemma 4 of [2]):

Lemma 7. *Let $\gamma : M^2 \rightarrow \mathbb{S}^2$ be a p -fold cyclic covering, branched over the set $B_\gamma = \{P_1, \dots, P_N\}$. If $\Psi' : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is an orientation-preserving homeomorphism such that $\Psi'(B_\gamma) = B_\gamma$ and $\omega'_\gamma(\Psi'(P_k)) = \omega'_\gamma(P_k)$, for every $k = 1, \dots, N$, then*

Ψ' lifts to an orientation-preserving homeomorphism $\Psi : M^2 \rightarrow M^2$ such that $\Psi'\gamma = \gamma\Psi$.

Proof. It is well known that Ψ' lifts if and only if the induced homomorphism Ψ'_* on the fundamental group $\pi_1(\mathbf{S}^2 - B_\gamma)$ leaves the subgroup H of the covering invariant [3]. The subgroup H is the kernel of the homomorphism $\omega : \pi_1(\mathbf{S}^2 - B_\gamma) = \langle m_1, \dots, m_N \mid \prod_{k=1}^N m_k = 1 \rangle \rightarrow \mathbf{Z}_p$, defined by $\omega(m_k) = c_k$, for each $k = 1, \dots, N$. Then we have $\omega\Psi'_*(m_k) = \omega'_\gamma(\Psi'(P_k)) = \omega'_\gamma(P_k) = \omega(m_k)$ and therefore $\omega\Psi'_* = \omega \square$.

Theorem 8. *A p -fold cyclic covering of \mathbf{S}^3 , branched over a link L of bridge number b , is a closed, orientable 3-manifold M of weakly p -symmetric Heegaard genus*

$$\tilde{g}_p(M) \leq (b - 1)(p - 1).$$

Proof. Let $q : M \rightarrow \mathbf{S}^3$ be a p -fold cyclic covering, branched over L , and let $L = \bigcup_{j=1}^\mu L_j$ be presented by a special $2b$ -plat associated to a braid β . If $(\mathbf{S}^3, L) = (D, A_b) \cup_{\psi'} (D', A'_b)$ is the $(0, b)$ -decomposition of L described in Remark 1, the map $\rho' : D \rightarrow D'$ defined by $\rho'(\infty) = \infty$ and $\rho'(x, t) = (x, 1 - t)$, for every $x \in \mathbf{E}^2$ and $t \leq 0$, is an homeomorphism which sends α_i onto α'_i , for each $i = 1, \dots, b$. Now, orient L in such a way that, for each $i = 1, \dots, b$, the arc α_i is oriented from A_i to B_i and the arc α'_i is oriented from B'_i to A'_i (this is possible since the plat is special). Let $c_1, \dots, c_\mu \in \mathbf{Z}_p - \{0\}$ be the integers associated to the components of L , according to the chosen orientations, and defining the covering. For each $j = 1, \dots, \mu$, the component L_j contains the b_j top arcs $\alpha_{b_1+\dots+b_{j-1}+1}, \dots, \alpha_{b_1+\dots+b_j}$ and the b_j bottom arcs $\alpha'_{b_1+\dots+b_{j-1}+1}, \dots, \alpha'_{b_1+\dots+b_j}$. Let $\pi : Y_g \rightarrow D$ be the p -fold cyclic covering of the 3-ball D , branched over the set of arcs A_b , with associated integers c_{j_1}, \dots, c_{j_b} , where L_{j_i} is the component of L containing α_i , for $i = 1, \dots, b$. If Y'_g is another handlebody of genus g and $\rho : Y_g \rightarrow Y'_g$ is a fixed homeomorphism, then the map $\pi' = \rho'\pi\rho^{-1} : Y'_g \rightarrow D'$ is a p -fold cyclic covering of D' , branched over A'_b , with associated integer list c_{j_1}, \dots, c_{j_b} . Now let $\gamma = \pi|_{\partial Y_g} : \partial Y_g \rightarrow \partial D$ be the restrictions of π to the surface ∂Y_g . The map γ is a p -fold cyclic covering of $\partial D \cong \mathbf{S}^2$, branched over the $2b$ points $A_i, B_i \in \partial D$, for $i = 1, \dots, b$, such that $\omega'_\gamma(A_i) = -\omega'_\gamma(B_i) = c_{j_i}$. Because of properties (1) and (2) of special plats, the homeomorphism $\Psi' = (\rho'_{|\partial D})^{-1}\psi'$ of ∂D satisfies the condition of Lemma 2. Hence, Ψ' lifts to a homeomorphism Ψ of ∂Y_g such that $\Psi'\gamma = \gamma\Psi$. If we define the homeomorphism $\psi = \rho|_{\partial Y_g} \Psi : \partial Y_g \rightarrow \partial Y'_g$, the manifold $Y_g \cup_\psi Y'_g$ turns out to be, by construction, a p -fold branched cyclic covering of L , with the same monodromy as the covering q and therefore $Y_g \cup_\psi Y'_g$ is homeomorphic to M . Moreover, $Y_g \cup_\psi Y'_g$ is a p -symmetric Heegaard splitting of genus g . Indeed, let \mathcal{H} be the group of covering transformations of π and \mathcal{R} be a generator of \mathcal{H} . Then condition i') easily holds. As regards condition ii') we have: $x \in \bigcup_{h=1}^{p-1} \text{Fix}(\mathcal{R}^h)/\mathcal{H} \Leftrightarrow |[x]_{\mathcal{H}}| < p \Leftrightarrow \pi(x) \in B_\pi$. As for condition iii'), the map $\Psi\mathcal{R}|_{\partial Y_g}\Psi^{-1}$ is the lifting, with respect to γ , of $\Psi'\Psi'^{-1} = \text{Id}_{\partial D}$ and therefore it is a covering transformation of π . This proves

point iii'). As far as the genus of the splitting is concerned, we have by (**): $2 - 2g = 2p - 2bp + 2 \sum_{h=1}^b \gcd(p, c_{2h-1})$. Therefore $\tilde{g}_p(M) \leq g = 1 - p + bp - \sum_{h=1}^b \gcd(p, c_{2h-1}) \leq 1 - p + bp - b = (b-1)(p-1) \square$.

The case of branched cyclic coverings of two-bridge knots or links is of particular interest.

Corollary 9. *A p -fold branched cyclic covering of a two-bridge knot or link is a closed, orientable 3-manifold M of weakly p -symmetric Heegaard genus*

$$\tilde{g}_p(M) \leq p - 1.$$

Proof. Follows from previous theorem, with $b = 2$. \square .

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References

- [1] J.S. Birman, *Braids, links and mapping class groups*, Ann. of Math. Studies, vol. 82, Princeton Univ. Press, Princeton, N. J., 1975.
- [2] J.S. Birman and H.M. Hilden, *Heegaard splittings of branched coverings of S^3* , Trans. Am. Math. Soc. **213** (1975) 315–352.
- [3] W.S. Massey, *Algebraic topology: An introduction*, Harcourt, Brace & World, New York, 1967.
- [4] J. Mayberry and K. Murasugi, *Torsion groups of abelian coverings of links*, Trans. Amer. Math. Soc. **271** (1982) 143–173.
- [5] J.W. Morgan and H. Bass, *The Smith conjecture*, Academic Press, Inc., 1984.
- [6] K. Morimoto and M. Sakuma, *On unknotting tunnels for knots*, Math. Ann. **289** (1991) 143–167.
- [7] O.Ja. Viro, *Linkings, two-sheeted branched coverings and braids*, Math. USSR, Sbornik **16** (1972) 223–236.