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Strongly-cyclic branched coverings of $(1,1)$-knots and cyclic presentations of groups

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Abstract

We study the connections among the mapping class group of the twice punctured torus, the cyclic branched coverings of $(1,1)$-knots and the cyclic presentations of groups. We give the necessary and sufficient conditions for the existence and uniqueness of the n-fold strongly-cyclic branched coverings of $(1,1)$-knots, through the elements of the mapping class group. We prove that every n-fold strongly-cyclic branched covering of a $(1,1)$-knot admits a cyclic presentation for the fundamental group, arising from a Heegaard splitting of genus n. Moreover, we give an algorithm to produce the cyclic presentation and illustrate it in the case of cyclic branched coverings of torus knots of type $(k, hk \pm 1)$.

1. *Introduction and preliminaries*

The problem of determining whether a balanced presentation of a group is geometric (i.e. induced by a Heegaard diagram of a closed orientable 3-manifold) is of considerable interest in geometric topology and has already been examined by many authors (see [**9**, **22**, **24**–**27**, **30**]). Moreover, the connections between cyclic coverings of $\mathbf{S}^3$ branched over knots and cyclic presentations of their fundamental groups, induced by suitable Heegaard diagrams, have recently been discussed in several papers (see [**1**, **4**–**6**, **8**, **12**, **13**, **15**, **17**, **18**, **20**, **31**]).

We recall that a finite balanced presentation of a group $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n\rangle$ is said to be a *cyclic presentation* if there exists a word w in the free group F_n generated by $x_1, \ldots, x_n$ such that the relators of the presentation are $r_k = \theta_n^{k-1}(w)$, $k = 1, \ldots, n$, where $\theta_n: F_n \to F_n$ is the automorphism defined by $\theta_n(x_i) = x_{i+1}$ (subscripts mod n), $i = 1, \ldots, n$. This cyclic presentation and the related group will be denoted by $G_n(w)$, so

$$G_n(w) = \big\langle x_1, \ldots, x_n \mid w, \theta_n(w), \ldots, \theta_n^{n-1}(w)\big\rangle.$$

Obviously $G_n(w) \cong G_n(\theta_n^s(w))$ for every integer s. The *polynomial associated to the cyclic presentation* $G_n(w)$ is defined by

$$f_w(t) = \sum_{i=1}^{n} a_i t^{i-1},$$

where a_i is the exponent sum of x_i in w. For further details see [**14**].

Some interesting examples of cyclically presented groups which are the fundamental groups of cyclic branched coverings of $\mathbf{S}^3$ are the following:

(i) the Fibonacci group $F(2n) = G_{2n}(x_1x_2x_3^{-1}) = G_n(x_1^{-1}x_2^2x_3^{-1}x_2)$ is the fundamental group of the n-fold cyclic covering of $\mathbf{S}^3$ branched over the figure-eight knot, for all $n > 1$ (see [**13**]);

(ii) the Sieradski group $S(n) = G_n(x_1x_3x_2^{-1})$ is the fundamental group of the n-fold cyclic covering of $\mathbf{S}^3$ branched over the trefoil knot, for all $n > 1$ (see [**4**]);

(iii) the fractional Fibonacci group $\widetilde{F}_{l,k}(n) = G_n((x_1^{-l}x_2^l)^k x_2 (x_3^{-l}x_2^l)^k)$ is the fundamental group of the n-fold cyclic covering of $\mathbf{S}^3$ branched over the genus one two-bridge knot with Conway coefficients $[2l, -2k]$, for all $n > 1$ and $l, k > 0$ (see [**31**]).

Notice that all the above cyclic presentations are geometric (i.e., they arise from suitable Heegaard diagrams).

In order to investigate the relations between cyclic branched coverings of knots in $\mathbf{S}^3$ and cyclic presentations for their fundamental groups, Dunwoody introduced in [**8**] a class of Heegaard diagrams depending on six integers, having cyclic symmetry and defining cyclic presentations for the corresponding fundamental groups. In [**10**] it has been shown that the 3-manifolds represented by these diagrams – the so-called *Dunwoody manifolds* – are cyclic coverings of lens spaces, branched over $(1,1)$-knots. As a corollary, it has been proved that for some well-determined cases the manifolds turn out to be cyclic coverings of $\mathbf{S}^3$, branched over knots. This gives a positive answer to a conjecture formulated by Dunwoody in [**8**], which has also been independently proved in [**29**].

In what follows, we shall deal with $(1,1)$-knots, also called genus one 1-bridge knots. They are knots in lens spaces (possibly in $\mathbf{S}^3$) admitting the following decomposition, called $(1,1)$-*decomposition*. A knot $K \subset L(p,q)$ is said to be a $(1,1)$-*knot* if there exists a Heegaard splitting of genus one

$$(L(p,q), K) = (T, A) \cup_\varphi (T', A'),$$

where T and T' are solid tori, $A \subset T$ and $A' \subset T'$ are properly embedded trivial arcs[1] and $\varphi\colon (\partial T', \partial A') \to (\partial T, \partial A)$ is an orientation-reversing (attaching) homeomorphism.

Note that $(1,1)$-knots are just a particular case of the notion of (g,b)-links in closed orientable 3-manifolds (see [**7**] and [**11**]), which generalize the classical concept of bridge decomposition of links in $\mathbf{S}^3$. The class of $(1,1)$-knots is very important in the light of some results and conjectures involving Dehn surgery on knots (see references in [**11**]). It is well known that the subclass of $(1,1)$-knots in $\mathbf{S}^3$ contains all torus knots (trivially) and all 2-bridge knots [**16**].

In this paper we prove that every n-fold strongly-cyclic branched covering of a $(1,1)$-knot admits a Heegaard splitting of genus n which encodes a cyclic presentation for the fundamental group (see Section 4). The definition of strongly-cyclic branched coverings of $(1,1)$-knots will be given in Section 3, where we give necessary and sufficient conditions for their existence and uniqueness, using a representation of

[1] This means that there exists a disk $D \subset T$ (resp. $D' \subset T'$) with $A \cap D = A \cap \partial D = A$ and $\partial D - A \subset \partial T$ (resp. $A' \cap D' = A' \cap \partial D' = A'$ and $\partial D' - A' \subset \partial T'$).

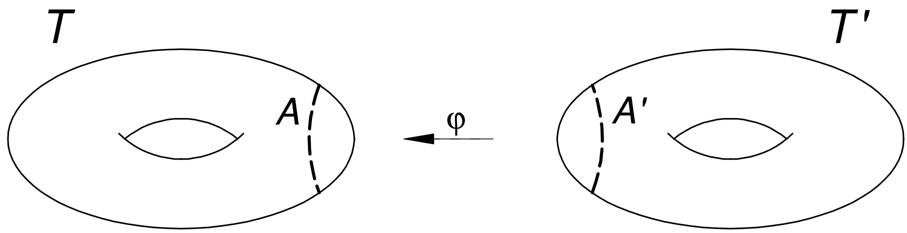

Fig. 1. A $(1, 1)$-knot decomposition.

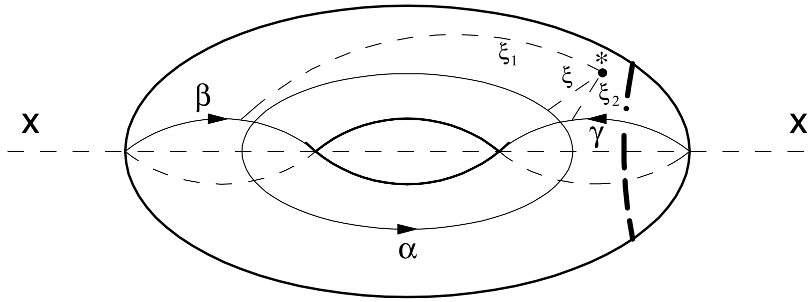

Fig. 2. Generators of $MCG(T_2)$.

$(1, 1)$-knots by elements of the mapping class group of the twice punctured torus (see Section 2). In Section 4 we give a constructive algorithm to obtain the word defining the cyclic presentation. Moreover, we show that the Alexander polynomial $\Delta_K(t)$ of a $(1, 1)$-knot $K \subset \mathbf{S}^3$ is equal to the polynomial associated to the cyclic presentation of the fundamental group of the n-fold strongly-cyclic branched covering of K, up to units of $\mathbb{Z}[t, t^{-1}]$, when n is greater than the degree of $\Delta_K(t)$.

2. $(1, 1)$-*knots and the mapping class group of the twice punctured torus*

Let $K \subset L(p, q)$ be a $(1, 1)$-knot with $(1, 1)$-decomposition $(L(p, q), K) = (T, A) \cup_{\varphi} (T', A')$ and let $\tau\colon (T, A) \to (T', A')$ be a fixed orientation-reversing homeomorphism, then $\psi = \varphi \circ \tau_{|\partial T}$ is an orientation-preserving homeomorphism of $(\partial T, \partial A)$. Moreover, since two isotopic attaching homeomorphisms produce equivalent $(1, 1)$-knots, we have a natural surjective map from the (orientation-preserving) mapping class group of the twice punctured torus $MCG(T_2)$ to the class $\mathcal{K}_{1,1}$ of all $(1, 1)$-knots

$$\psi \in MCG(T_2) \longmapsto K_{\psi} \in \mathcal{K}_{1,1}.$$

A standard set of generators for $MCG(T_2)$ is given by the rotation ρ of π radians around the x–x axis and the three right-hand Dehn twists $d_{\alpha}, d_{\beta}, d_{\gamma}$, respectively around the curves α, β, γ, as depicted in Fig. 2. Observe that $d_{\alpha}, d_{\beta}, d_{\gamma}$ fix the punctures, while ρ exchanges them.

The subgroup of $MCG(T_2)$ generated by $d_{\alpha}, d_{\beta}, d_{\gamma}$, i.e. the *pure mapping class group* of T_2, which contains only the elements fixing the punctures is denoted by $PMCG(T_2)$ [**19**] (a very simple presentation of $PMCG(T_2)$ can be found in [**28**]). Since ρ obviously commutes with the other generators, we have $MCG(T_2) \cong PMCG(T_2) \oplus \mathbb{Z}_2$ and every element ψ of $MCG(T_2)$ can be written as $\psi = \psi' \rho^k$, where $\psi' \in PMCG(T_2)$. Since ρ can be extended to a homeomorphism of the pair

(T,A), the $(1,1)$-knots K_ψ and $K_{\psi'}$ are equivalent. So, for our discussion it suffices to consider only the elements of $PMCG(T_2)$.

The group $PMCG(T_2)$ naturally maps by an epimorphism to the mapping class group of the torus (no punctures) $MCG(T_0)$, which is generated by d_α and $d_\beta = d_\gamma$. As is well known, $MCG(T_0)$ is isomorphic to $SL(2,\mathbb{Z})$ via the map which associates to any element $\psi \in MCG(T_0)$ the matrix representing the isomorphism $\psi_\#$ of $H_1(T_0) \cong \mathbb{Z} \oplus \mathbb{Z}$, with respect to a fixed ordered basis $\mathcal{B}$. Assuming $\mathcal{B} = (\beta, \alpha)$, we have the epimorphism

$$\mathbf{\Omega}\colon PMCG(T_2) \longrightarrow SL(2,\mathbb{Z}) \qquad (2\cdot1)$$

given by

$$d_\beta, d_\gamma \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \text{ and } d_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

From this assumption, if

$$\mathbf{\Omega}(\psi) = \begin{pmatrix} q & s \\ p & r \end{pmatrix}$$

then K_ψ is a $(1,1)$-knot in the lens space $L(|p|, q)$ [**3**], and therefore it is a knot in $\mathbf{S}^3$ if and only if $p = \pm 1$.

It is easy to see that if either $\psi = 1$ or $\psi = d_\beta$ or $\psi = d_\gamma$, then K_ψ is the trivial knot in $\mathbf{S}^1 \times \mathbf{S}^2$, and if $\psi = d_\alpha$ then K_ψ is the trivial knot in $\mathbf{S}^3$.

Now we compute the fundamental group of the complement of a $(1,1)$-knot, by applying the Seifert–Van Kampen theorem to its $(1,1)$-decomposition. In order to do that, we fix the base point $*$ and define the loops $\bar\alpha = \xi \cdot \alpha \cdot \xi^{-1}$, $\bar\beta = \xi_1 \cdot \beta \cdot \xi_1^{-1}$ and $\bar\gamma = \xi_2 \cdot \gamma \cdot \xi_2^{-1}$, where ξ, ξ_1, ξ_2 are the paths connecting $*$ to α, β and γ respectively, as depicted in Fig. 2. It is easy to see that $\bar\alpha, \bar\beta$ and $\bar\gamma$ are homologous to α, β and γ respectively. Moreover, we set $\bar\alpha' = \tau(\bar\alpha)$, $\bar\beta' = \tau(\bar\beta)$, $\bar\gamma' = \tau(\bar\gamma)$ and $*' = \tau(*)$. Observe that $*' = \varphi^{-1}(*)$, up to isotopy. The homotopy classes of $\bar\alpha, \bar\beta, \bar\gamma$ generate $\pi_1(\partial T - \partial A, *)$ and the homotopy classes of $\bar\alpha', \bar\beta', \bar\gamma'$ generate $\pi_1(\partial T' - \partial A', *')$. In order to simplify the notation, in the following we use the same symbol for loops (resp. for cycles) and for corresponding homotopy classes (resp. homology classes).

PROPOSITION 1. *The fundamental group of the complement of a* $(1,1)$-*knot* $K_\psi \subset L(p,q)$ *admits the presentation*

$$\pi_1(L(p,q) - K, *) = \langle \bar\alpha, \bar\gamma \mid r(\bar\alpha, \bar\gamma) \rangle,$$

where $r(\bar\alpha, \bar\gamma)$ *is obtained by erasing all* $\bar\beta$*'s terms from the word representing the homotopy class of* $\psi(\bar\beta)$.

Proof. We have $\pi_1(T - A, *) = \langle \bar\alpha, \bar\beta, \bar\gamma \mid \bar\beta \rangle$ and $\pi_1(T' - A', *') = \langle \bar\alpha', \bar\beta', \bar\gamma' \mid \bar\beta' \rangle$. Applying the Seifert–Van Kampen theorem we get

$$\begin{aligned}\pi_1(L(p,q) - K, *) &= \langle \bar\alpha, \bar\beta, \bar\gamma, \bar\alpha', \bar\beta', \bar\gamma' \mid \bar\beta, \bar\beta', \varphi(\bar\alpha')\bar\alpha'^{-1}, \varphi(\bar\beta')\bar\beta'^{-1}, \varphi(\bar\gamma')\bar\gamma'^{-1} \rangle \\ &= \langle \bar\alpha, \bar\beta, \bar\gamma \mid \bar\beta, \psi(\bar\beta) \rangle = \langle \bar\alpha, \bar\gamma \mid r(\bar\alpha, \bar\gamma) \rangle,\end{aligned}$$

and the proof is obtained.

From this result the computation of the first homology group is straightforward, observing that $\{\alpha, \beta, \gamma\}$ is a set of free generators for $H_1(\partial T - \partial A)$ and that, by (2·1), the homology relation $\psi(\beta) = p\alpha + q'\beta + q''\gamma$ (where $q' + q'' = q$) holds for a $(1,1)$-knot $K_\psi \subset L(p,q)$.

COROLLARY 2. *Let* K_ψ *be a* $(1,1)$-*knot in* $L(p,q)$ *then*

$$H_1(L(p,q)-K_\psi)=\langle \alpha,\gamma \mid p\alpha+q''\gamma\rangle \cong \mathbb{Z}\oplus\mathbb{Z}_{\gcd(p,q'')},$$

where q'' *is uniquely determined by the homology relation*:

$$\psi(\beta)=p\alpha+q'\beta+q''\gamma.$$

In order to compute the homotopy (and homology) classes of $\psi(\bar\beta)$, we list the actions of the homomorphisms $d_\alpha, d_\beta, d_\gamma$ and their inverses on the homotopy classes $\bar\alpha,\bar\beta,\bar\gamma$.

$$\begin{array}{lll}
d_\alpha(\bar\alpha)=\bar\alpha & d_\alpha(\bar\beta)=\bar\alpha\bar\beta & d_\alpha(\bar\gamma)=\bar\alpha\bar\gamma,\\
d_\alpha^{-1}(\bar\alpha)=\bar\alpha & d_\alpha^{-1}(\bar\beta)=\bar\alpha^{-1}\bar\beta & d_\alpha^{-1}(\bar\gamma)=\bar\alpha^{-1}\bar\gamma,\\
d_\beta(\bar\alpha)=\bar\beta^{-1}\bar\alpha & d_\beta(\bar\beta)=\bar\beta & d_\beta(\bar\gamma)=\bar\gamma,\\
d_\beta^{-1}(\bar\alpha)=\bar\beta\bar\alpha & d_\beta^{-1}(\bar\beta)=\bar\beta & d_\beta^{-1}(\bar\gamma)=\bar\gamma,\\
d_\gamma(\bar\alpha)=\bar\alpha\bar\gamma^{-1} & d_\gamma(\bar\beta)=\bar\beta & d_\gamma(\bar\gamma)=\bar\gamma,\\
d_\gamma^{-1}(\bar\alpha)=\bar\alpha\bar\gamma & d_\gamma^{-1}(\bar\beta)=\bar\beta & d_\gamma^{-1}(\bar\gamma)=\bar\gamma.
\end{array} \tag{2·2}$$

The next proposition describes a class of non-trivial $(1,1)$-knots in $\mathbf{S}^3$.

PROPOSITION 3. *If* $\psi = d_\alpha^{\pm h} d_\gamma^{-k} d_\beta^{1+k} d_\alpha$, *with* $h,k>0$, *then* K_ψ *is the torus knot* $\mathbf{t}(k,hk\mp 1)$ *in* $\mathbf{S}^3$. *In particular, for* $h=1$ *and* $k=2$, *the knot* K_ψ *is the trefoil knot.*

Proof. K_ψ is always a knot in $\mathbf{S}^3$, since

$$\boldsymbol{\Omega}(\psi)=\begin{pmatrix}0 & -1\\ 1 & 1\mp h\end{pmatrix}.$$

Then, using (2·2), we obtain $\psi(\bar\beta)=(\bar\beta^{-1}\bar\alpha^{\mp h})^{1+k}\bar\alpha(\bar\alpha^{\pm h}\bar\gamma)^k\bar\alpha^{\pm h}\bar\beta$ and therefore $r(\bar\alpha,\bar\gamma)=\bar\alpha^{1\mp hk}(\bar\gamma\bar\alpha^{\pm h})^k$. We have

$$\pi_1(\mathbf{S}^3-K_\psi,*)=\langle \bar\alpha,\bar\gamma \mid \bar\alpha^{1\mp hk}(\bar\gamma\bar\alpha^{\pm h})^k\rangle \cong \langle \bar\alpha,u \mid \bar\alpha^{1\mp hk}u^k\rangle,$$

which is isomorphic to the group of the torus knot $\mathbf{t}(k,hk\mp 1)$. From a result of [**2**], we can conclude that in fact $K_\psi=\mathbf{t}(k,hk\mp 1)$.

3. *Strongly-cyclic branched coverings of* $(1,1)$-*knots*

An n-fold cyclic branched covering between two orientable closed manifolds $f\colon M\to N$, with branching set L, is completely defined by an epimorphism (called *monodromy*) $\omega_f\colon H_1(N-L)\to\mathbb{Z}_n$, where $\mathbb{Z}_n$ is the cyclic group of order n. Moreover, two n-fold cyclic coverings $f'\colon M'\to N$ and $f''\colon M''\to N$, branched over L, with associated monodromies $\omega_{f'}\colon H_1(N-L)\to\mathbb{Z}_n$ and $\omega_{f''}\colon H_1(N-L)\to\mathbb{Z}_n$ respectively, are equivalent if and only if there exists $u\in\mathbb{Z}_n$, with $\gcd(u,n)=1$, such that $\omega_{f''}=u\omega_{f'}$, where $u\omega_{f'}$ is the multiplication of $\omega_{f'}$ by u.

Our aim is to obtain cyclic branched coverings with cyclically presented fundamental groups. In order to achieve this, we will select branched cyclic coverings of $(1,1)$-knots of "special type", naturally generalizing the case of knots and links in $\mathbf{S}^3$.

An n-fold cyclic covering f of $L(p,q)$ branched over a $(1,1)$-knot $K\subset L(p,q)$ will be called *strongly-cyclic* (and denoted by $C_n(K)$) if the branching index of K is n. This means that the fiber $f^{-1}(x)$ of each point $x\in K$ contains a single point. In this case the homology class of a meridian loop m around K is mapped by ω_f in a generator of $\mathbb{Z}_n$ (up to equivalence we can always suppose $\omega_f(m)=1$). In the case of

$(1,1)$-knots in $\mathbf{S}^3$, strongly-cyclic branched coverings and cyclic branched coverings are equivalent notions.

This type of covering appears to be a suitable tool for producing 3-manifolds with fundamental group admitting cyclic presentation, as shown in the next section. For example, it is easy to see that all Dunwoody manifolds are coverings of this type.

As is well known, an n-fold cyclic branched covering of a knot K in $\mathbf{S}^3$ always exists and is unique (up to equivalence) for all $n > 1$, since $H_1(\mathbf{S}^3 - K) \cong \mathbb{Z}$ and the homology class m of a meridian loop around the knot is mapped by ω_f in a generator of $\mathbb{Z}_n$, which can be chosen equal to one, up to equivalence. Obviously, this property is no longer true when the ambient space is not a 3-sphere. For a $(1,1)$-knot $K \subset L(p,q)$ the homology of $L(p,q) - K$ has the structure described in Corollary 2. As a consequence, the n-fold strongly-cyclic branched covering of a $(1,1)$-knot not belonging in $\mathbf{S}^3$ may not exist, and when it exists it may not be unique.

The next theorem gives necessary and sufficient conditions for the existence of these coverings, establishing how many of these coverings exist, up to equivalence.

Theorem 4. *Let K_ψ be a $(1,1)$-knot in $L(p,q)$. Then K_ψ admits an n-fold strongly-cyclic branched covering if and only if d divides q'', where $d = \gcd(p,n)$ and q'' is uniquely determined by the homology relation $\psi(\beta) = p\alpha + q'\beta + q''\gamma$. In this case there exists exactly d of such coverings, up to equivalence.*

Proof. By definition, the existence of the n-fold strongly-cyclic branched covering f means that $\omega_f(\gamma) = 1$, up to equivalence. We have $H_1(L(p,q) - K_\psi) = \langle \alpha, \gamma \mid p\alpha + q''\gamma \rangle$, from Corollary 2, where q'' is uniquely determined by the homology relation $\psi(\beta) = p\alpha + q'\beta + q''\gamma$. So, the covering exists if and only if $\omega_f(p\alpha + q''\gamma) \equiv 0 \bmod n$ or, in other words, if and only if there exists an element $x \in \mathbb{Z}_n$ such that $px + q'' \equiv 0 \bmod n$. This equation is solvable if and only if d divides q'', where $d = \gcd(p,n)$ and in this case it has exactly d solutions. Since two different solutions give non-equivalent coverings, the statement is proved.

In particular, for $(1,1)$-knots in $\mathbf{S}^3$ we have $p = \pm 1$ and the existence and uniqueness of the n-fold cyclic branched covering immediately follows.

Example 1. Let $\psi = d_\alpha^2 d_\gamma d_\alpha^{-4}$, then K_ψ is a $(1,1)$-knot in $L(6,5)$. Applying (2·2), we obtain $\psi(\bar\beta) = (\bar\alpha^2\bar\gamma\bar\alpha^{-1})^4\bar\alpha^2\bar\beta$. So we have

$$H_1(L(6,5) - K_\psi) = \langle \alpha, \gamma \mid 6\alpha + 4\gamma \rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

In this case no 6-fold strongly-cyclic branched coverings of K_ψ exist, because $d = 6$ does not divide $q'' = 4$. Observe that there exists a 6-fold cyclic branched covering of K_ψ: take for example $\omega_f(\alpha) = 1$ and $\omega_f(\gamma) = 3$; but obviously it is not strongly-cyclic because the index around the knot is two.

Example 2. Let $\psi = d_\alpha^{-2} d_\gamma^{-2} d_\alpha^{-2}$, then K_ψ is a $(1,1)$-knot in $L(4,1)$. We have $\psi(\bar\beta) = ((\bar\gamma^{-1}\bar\alpha^2)^2\bar\alpha^{-1})^2\bar\alpha^{-2}\bar\beta$ and therefore

$$H_1(L(4,1) - K_\psi) = \langle \alpha, \gamma \mid 4\alpha - 4\gamma \rangle \cong \mathbb{Z} \oplus \mathbb{Z}_4.$$

In this case there are exactly four non-equivalent 4-fold strongly-cyclic branched coverings of K_ψ, depending on the choice of $\omega_f(\alpha) \in \mathbb{Z}_4$.

Non-equivalent n-fold strongly-cyclic branched coverings of the same $(1,1)$-knot may be effectively non-homeomorphic, as shown in Remark 1.

4. *Connections with cyclic presentations of groups*

In this section we study the connections between strongly-cyclic branched coverings of $(1,1)$-knots and cyclic presentations of their fundamental groups.

Theorem 5 [**23**]. *Every n-fold strongly-cyclic branched covering of a $(1,1)$-knot admits a cyclic presentation for the fundamental group induced by a Heegaard splitting of genus n.*

Proof. Let $f\colon (M, f^{-1}(K)) \to (L(p,q), K) = (T, A) \cup_{\varphi} (T', A')$ be an n-fold strongly-cyclic branched covering of the $(1,1)$-knot K. Then $Y_n = f^{-1}(T)$ and $Y'_n = f^{-1}(T')$ are both handlebodies of genus n. Moreover, $f^{-1}(A)$ and $f^{-1}(A')$ are both properly embedded trivial arcs in Y_n and Y'_n respectively. We get a genus n Heegaard splitting $(M, f^{-1}(K)) = (Y_n, f^{-1}(A)) \cup_{\Phi} (Y'_n, f^{-1}(A'))$, where $\Phi\colon \partial Y'_n \to \partial Y_n$ is the lifting of φ with respect to f. Let m be a meridian loop around A and $\widehat{\alpha} \subset T - A$ be a generator of $\pi_1(T, *)$, such that $\omega_f(\widehat{\alpha}) = 0$. It exists: take a generator $\eta \subset T - A$ of $\pi_1(T, *)$; if $\omega_f(\eta) = k$ then choose a loop $\widehat{\alpha}$ homotopic to ηm^{-k}. The set $f^{-1}(\widehat{\alpha})$ has exactly n components $\tilde{\alpha}_1, \ldots \tilde{\alpha}_n$ and it is a set of generators for $\pi_1(Y_n)$. A generator $\mathbf{\Psi}$ of the group of covering transformations cyclically permutes these components. Let E' be a meridian disk for the torus T' such that $E' \cap A' = \emptyset$, then $f^{-1}(E')$ is a system of meridian disks $\{\tilde{E}'_1, \ldots, \tilde{E}'_n\}$ for the handlebody Y'_n, and they are cyclically permuted by $\mathbf{\Psi}$. The curves $\Phi(\partial\tilde{E}'_1), \ldots, \Phi(\partial\tilde{E}'_n)$ give the relators for the presentation of $\pi_1(M)$ induced by the Heegaard splitting. Since both generator and relator curves are cyclically permuted by $\mathbf{\Psi}$, we get the statement.

Since 2-bridge knots and torus knots are $(1,1)$-knots in $\mathbf{S}^3$ we have the following consequence:

Corollary 6. *Every n-fold cyclic branched covering of a 2-bridge knot and of a torus knot admits a geometric cyclic presentation for the fundamental group with n generators.*

We remark that the previous result for 2-bridge knots has also been obtained (by another technique) in [**10**] and the result for torus knots largely generalizes a result obtained in [**4**] and [**5**].

Let K be a $(1,1)$-knot in $L(p,q)$ and let ψ be an element of $PMCG(T_2)$ such that $K = K_{\psi}$. Then $\pi_1(L(p,q) - K, *) = \langle \bar{\alpha}, \bar{\gamma} \mid r(\bar{\alpha}, \bar{\gamma}) \rangle$, as shown in Proposition 1. Now, let ω_f be the monodromy of an n-fold strongly-cyclic branched covering of K. Following the proof of Theorem 5 [**23**], we choose a new generator $\widehat{\alpha} = \bar{\alpha}\bar{\gamma}^{-\omega_f(\bar{\alpha})}$ and we obtain $\pi_1(L(p,q) - K, *) = \langle \widehat{\alpha}, \bar{\gamma} \mid \bar{r}(\widehat{\alpha}, \bar{\gamma}) \rangle$, with $\bar{r}(\widehat{\alpha}, \bar{\gamma}) = r(\widehat{\alpha}\bar{\gamma}^{\omega_f(\bar{\alpha})}, \bar{\gamma})$. We have $\bar{r}(\widehat{\alpha}, \bar{\gamma}) = \widehat{\alpha}^{\varepsilon_1}\bar{\gamma}^{\delta_1} \cdots \widehat{\alpha}^{\varepsilon_s}\bar{\gamma}^{\delta_s}$ for some $\varepsilon_1, \ldots, \varepsilon_s, \delta_1, \ldots, \delta_s \in \mathbb{Z}$.

Theorem 7. *According to the assumptions listed in this section, let $\bar{r}(\widehat{\alpha}, \bar{\gamma}) = \widehat{\alpha}^{\varepsilon_1}\bar{\gamma}^{\delta_1} \cdots \widehat{\alpha}^{\varepsilon_s}\bar{\gamma}^{\delta_s}$. Then the fundamental group of the n-fold strongly-cyclic branched covering of K, with monodromy ω_f, admits the cyclic presentation $G_n(w)$, with:*

$$w = x_{i_1}^{\varepsilon_1} \cdots x_{i_s}^{\varepsilon_s}$$

(subscripts mod n), where $i_k \equiv 1 + \sum_{j=1}^{k-1} \delta_j \mod n$, for $k = 1, \ldots, s$.

Proof. From the proof of Theorem 5 [**23**], the fundamental group of $C_n(K)$ has generators corresponding to the components $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$ of the lifting of $\widehat{\alpha}$. The relators are (homotopic to) $\Phi(\partial\tilde{E}'_1), \ldots, \Phi(\partial\tilde{E}'_n)$, where Φ is the lifting of φ with respect to f and each $\partial\tilde{E}'_i$ is a component of the lifting of a meridian disc E' of T' such that $T' \cap A' = \emptyset$. So we can choose E' such that $\partial E' = \tau(\beta)$ and therefore the relators are (homotopic to) the components of the lifting of $r(\bar{\alpha}, \bar{\gamma})$, or equivalently $\bar{r}(\widehat{\alpha}, \bar{\gamma})$, which arise from the relation $\bar{\beta}' = \varphi(\bar{\beta}')$. Now, since $\omega_f(\gamma) = 1$, a factor $\bar{\gamma}^k$ lifts to a path connecting the point of $f^{-1}(*)$ in the sheet i with the corresponding point in the sheet $i + k$ (mod n). The result immediately follows.

We resume the algorithm for finding the word w of the cyclic presentation of $\pi_1(C_n(K))$, defined by the monodromy ω_f, starting from an element $\psi \in PMCG(T_2)$ such that $K = K_\psi$:

(i) use (2·2) to calculate the homotopy class $\psi(\bar{\beta})$;
(ii) obtain $r(\bar{\alpha}, \bar{\gamma})$ by erasing all $\bar{\beta}$'s from $\psi(\bar{\beta})$;
(iii) compute $\bar{r}(\widehat{\alpha}, \bar{\gamma})$ by replacing $\bar{\alpha}$ with $\widehat{\alpha}\bar{\gamma}^{\omega_f(\bar{\alpha})}$ in $r(\bar{\alpha}, \bar{\gamma})$;
(iiii) get w by applying Theorem 7.

As an application we give an explicit cyclic presentation for the fundamental group of the strongly-cyclic branched coverings of the torus knots $\mathbf{t}(k, hk \pm 1)$, with $h, k > 0$.

PROPOSITION 8. *The fundamental group of the n-fold cyclic branched covering of the torus knot* $\mathbf{t}(k, hk + 1)$*, with* $h, k > 0$*, admits the cyclic presentation* $G_n(w)$*, where*

$$w = \left(\prod_{j=0}^{h(k-1)} x_{1-jk}\right)\left(\prod_{i=0}^{k-2}\left(\prod_{l=1}^{h} x^{-1}_{2+i-(h(k-1-i)+1-l)k}\right)\right),$$

(*subscripts mod n*).
The fundamental group of the n-fold cyclic branched covering of the torus knot $\mathbf{t}(k, hk-1)$*, with* $h, k > 0$*, admits the cyclic presentation* $G_n(w)$*, where*

$$w = \left(\prod_{j=1}^{h(k-1)-1} x^{-1}_{1+jk}\right)\left(\prod_{i=0}^{k-2}\left(\prod_{l=0}^{h-1} x_{2+i+(h(k-1-i)-1-l)k}\right)\right),$$

(*subscripts mod n*).

Proof. First we consider the case $\mathbf{t}(k, hk+1)$. From Proposition 3 we have $\mathbf{t}(k, hk+1) = K_\psi$ with $\psi = d_\alpha^{-h} d_\gamma^{-k} d_\beta^{k+1} d_\alpha$ and $\pi_1(\mathbf{S}^3 - \mathbf{t}(k, hk+1)) = \langle \bar{\alpha}, \bar{\gamma} \mid r(\bar{\alpha}, \bar{\gamma})\rangle$, with $r(\bar{\alpha}, \bar{\gamma}) = \bar{\alpha}^{hk+1}(\bar{\gamma}\bar{\alpha}^{-h})^k$ (see the proof of Proposition 3). Therefore we obtain $H_1(\mathbf{S}^3 - \mathbf{t}(k, hk+1)) = \langle \alpha, \gamma \mid \alpha + k\gamma\rangle$. Since $\omega_f(\bar{\gamma}) = 1$, then $\omega_f(\bar{\alpha}) = -k$ and $\widehat{\alpha} = \bar{\alpha}\bar{\gamma}^k$. We get $\pi_1(\mathbf{S}^3 - \mathbf{t}(k, hk+1)) = \langle \widehat{\alpha}, \bar{\gamma} \mid \bar{r}(\widehat{\alpha}, \bar{\gamma})\rangle$, with $\bar{r}(\widehat{\alpha}, \bar{\gamma}) = (\widehat{\alpha}\bar{\gamma}^{-k})^{1+hk}(\bar{\gamma}(\bar{\gamma}^k\widehat{\alpha}^{-1})^h)^k$. So the fundamental group of the n-fold strongly-cyclic branched covering $\pi_1(C_n(\mathbf{t}(k, h)))$ admits a cyclic presentation $G_n(w)$, where $w = (\prod_{j=0}^{hk} x_{1-jk})(\prod_{i=0}^{k-1}(\prod_{l=0}^{h-1} x^{-1}_{2+i-(h(k-i)-l)k}))$ (subscripts mod n). Since the last h letters are the inverse of the first h (in the opposite order), we can remove them from the word and, shifting all the indexes of the letters by hk, the statement holds. The case $\mathbf{t}(k, hk-1)$ can be obtained in a perfectly analogous way.

In particular, for $h = 1$ and $k = 2$ we have that $\mathbf{t}(k, hk+1) = \mathbf{t}(2, 3)$ is the trefoil knot and the fundamental group of its n-fold cyclic branched covering is $G_n(x_3 x_1 x_2^{-1})$, which is clearly isomorphic to the Sieradski group $S(n)$ described in Section 1.

The above proposition clarifies the group presentations discussed in the main theorem of [**5**].

Remark 1. Using the above algorithm it is easy to show that non-equivalent n-fold strongly-cyclic branched coverings of the same $(1,1)$-knot may be effectively non-homeomorphic. For example, the fundamental groups of the four non-equivalent 4-fold strongly cyclic coverings of Example 2 admit cyclic presentations $G_4(w_i)$, $i=0,1,2,3$, defined by the words:

$$w_0 = x_4^2 x_3 x_2^2 x_1^{-1}, \qquad w_1 = x_4 x_1^3 x_2 x_1^{-1},$$
$$w_2 = x_4 x_2 x_3 x_4 x_2 x_1^{-1}, \qquad w_3 = x_4 x_3 x_1 x_3 x_2 x_1^{-1}$$

for $\omega_f(\alpha) = 0,1,2,3$ respectively. The first homology groups of the strongly-cyclic branched coverings $C_4(K_\psi)$ are:

$$H_1(C_4(K_\psi)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_4 & \text{if} \quad \omega_f(\alpha) = 1,3, \\ \mathbb{Z}_8 \oplus \mathbb{Z}_8 & \text{if} \quad \omega_f(\alpha) = 0,2. \end{cases}$$

So at least two of these coverings are not homeomorphic.

There is a strict relation between the Alexander polynomial of a $(1,1)$-knot in $\mathbf{S}^3$ and the polynomial associated with the cyclic presentation constructed according to Theorem 7.

Proposition 9. *Let* $K \subset \mathbf{S}^3$ *be a* $(1,1)$*-knot. If* $\Delta_K(t)$ *is the Alexander polynomial of* K *and* $f_w(t)$ *is the polynomial associated to the cyclic presentation of the* n*-fold cyclic branched covering of* K*, obtained by applying Theorem* 7*, then* $\Delta_K(t) = f_w(t)$*, up to units of* $\mathbb{Z}[t,t^{-1}]$*, when* $n > \deg \Delta_K(t)$.

Proof. The statement follows from the arguments of [**21**, remarks 3–4].

Since Dunwoody manifolds are strongly-cyclic branched coverings of $(1,1)$-knots, the above result gives a positive answer to a question posed in [**8**]. Moreover, it could be a possible starting point to extend the notion of Alexander polynomial to all $(1,1)$-knots.

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