$(1, 1)$-knots and Dunwoody manifolds

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Abstract

A list of recent results on 3-manifolds with cyclically presented fundamental groups is given. In particular, it is shown that the class of strongly-cyclic branched coverings of $(1, 1)$-knots coincides with the class of Dunwoody manifolds.

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1 Introduction

In the last decade many examples of cyclic branched coverings of knots in $\mathbf{S}^3$ admitting cyclic presentations for the fundamental groups have been described (see [2, 6, 7, 8, 11, 13, 14, 15, 16, 18]).

In order to investigate this subject, M. J. Dunwoody introduced in [9] a class of manifolds depending on six integers, called Dunwoody manifolds, with cyclically presented fundamental groups. It is proved in [10] that all Dunwoody manifolds are strongly-cyclic coverings of lens spaces (possibly $\mathbf{S}^3$), branched over $(1, 1)$-knots.

After that, the attention has been focused on the relations between strongly-cyclic branched coverings of $(1, 1)$-knots, and manifolds with cyclically presented fundamental groups. It has been shown in [17] that every n-fold strongly-cyclic branched covering of a $(1, 1)$-knot admits a Heegaard diagram of genus n which encodes a cyclic presentation for the fundamental group. In [3] this result has been improved, obtaining a constructive algorithm which explicitly gives the cyclic presentations, starting from a representation of $(1,1)$-knots through the elements of the mapping class group of the twice punctured torus (see [4] for further details on this representation). Finally, it is proved in [5] that all strongly-cyclic branched coverings of $(1,1)$-knots are Dunwoody manifolds.

All these results, as well as others on the same topic, are described in Section 3, where some examples are also listed.

2 Basic notions

2.1 Strongly-cyclic branched coverings of (1,1)-knots

An n-fold cyclic covering of a 3-manifold N^3 branched over a knot $K \subset N^3$ will be called *strongly-cyclic* if the branching index of K is n. This means that the homology class of a meridian loop m around K is mapped by the monodromy map $\omega : H_1(N^3 - K) \to \mathbf{Z}_n$ in a generator of $\mathbf{Z}_n$ (up to equivalence we can always suppose $\omega[m] = 1$).

Observe that a cyclic branched covering of a knot K in $\mathbf{S}^3$ is always strongly-cyclic and is uniquely determined, up to equivalence, since $H_1(\mathbf{S}^3 - K) \cong \mathbf{Z}$. Obviously, this property is no longer true for a knot in a more general 3-manifold.

In this paper we deal with strongly-cyclic branched coverings of $(1,1)$-knots, which are knots in lens spaces (possibly in $\mathbf{S}^3$).

A knot K in a 3-manifold N^3 is called a $(1,1)$-*knot* if there exists a Heegaard splitting of genus one $(N^3, K) = (T, A) \cup_\phi (T', A')$, where T and T' are solid tori, $A \subset T$ and $A' \subset T'$ are properly embedded trivial arcs[1], and $\phi : (\partial T, \partial A) \to (\partial T', \partial A')$ is an attaching homeomorphism (see Figure 1). Obviously, N^3 turns out to be a lens space $L(p,q)$ (including $\mathbf{S}^3 = L(1,0)$).

It is well known that the family of $(1,1)$-knots contains all torus knots and all two-bridge knots in $\mathbf{S}^3$. Several topological properties of $(1,1)$-knots have recently been investigated (see references in [4]).

An algebraic representation of $(1,1)$-knots have been developed in [3, 4], where it is shown that there is a natural surjective map $\varphi \in PMCG_2(\partial T) \mapsto K_\varphi \in \mathcal{K}_{1,1}$ from the pure mapping class group of the twice punctured torus $PMCG_2(\partial T)$ to the class $\mathcal{K}_{1,1}$ of all $(1,1)$-knots.

[1]This means that there exists a disk $D \subset T$ (resp. $D' \subset T'$) with $A \cap D = A \cap \partial D = A$ and $\partial D - A \subset \partial T$ (resp. $A' \cap D' = A' \cap \partial D' = A'$ and $\partial D' - A' \subset \partial T'$).

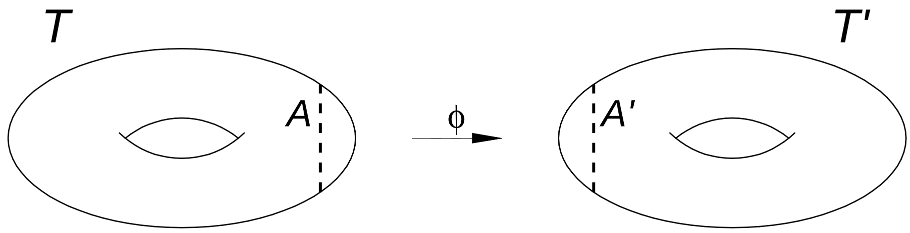

Figure 1: A $(1,1)$-knot decomposition.

The fundamental group and the first integer homology group of the exterior E_φ of a $(1,1)$-knot K_φ can be explicitly obtained from its representation φ (see [3]). Since $H_1(E_\varphi)$ is in general not homeomorphic to $\mathbf{Z}$, an n-fold strongly-cyclic branched covering of K_φ does not necessarily exist and, if it exists, may not be unique, up to equivalence. The existence and uniqueness conditions have been obtained in [3].

2.2 Cyclically presented groups

A finite balanced presentation of a group $< x_1, \ldots, x_n \mid r_1, \ldots, r_n >$ is said to be a *cyclic presentation* if there exists a word w in the free group F_n generated by $x_1, \ldots, x_n$ such that $r_k = \theta_n^{k-1}(w)$, $k = 1, \ldots, n$, where $\theta_n : F_n \to F_n$ denotes the automorphism defined by $\theta_n(x_i) = x_{i+1}$ (subscripts mod n), $i = 1, \ldots, n$. This presentation (and the related group) will be denoted by $G_n(w)$. For further details see [12].

The following examples show that these groups are strictly connected with 3-manifolds with cyclic symmetry:

- the Fibonacci group $F(2n) = G_{2n}(x_1 x_2 x_3^{-1}) = G_n(x_1^{-1} x_2^2 x_3^{-1} x_2)$ is the fundamental group of the n-fold cyclic covering of $\mathbf{S}^3$ branched over the figure-eight knot, for all $n > 1$ (see [11]);

- the Sieradsky group $S(n) = G_n(x_1 x_3 x_2^{-1})$ is the fundamental group of the n-fold cyclic covering of $\mathbf{S}^3$ branched over the trefoil knot, for all $n > 1$ (see [6]);

- the fractional Fibonacci group $\widetilde{F}_{l,k}(n) = G_n((x_1^{-l} x_2^l)^k x_2 (x_3^{-l} x_2^l)^k)$ is the fundamental group of the n-fold cyclic covering of $\mathbf{S}^3$ branched over the genus one two-bridge knot with Conway coefficients $[2l, -2k]$, for all $n > 1$ and $l, k > 0$ (see [18]).

2.3 Dunwoody manifolds

M. J. Dunwoody introduced in [9] a class of diagrams with cyclic symmetry, depending on six integers a, b, c, n, r, s, such that $n > 0$, $a, b, c \geq 0$. The diagrams define a wide class of closed orientable 3-manifolds $D(a, b, c, n, r, s)$, called *Dunwoody manifolds*, with fundamental groups admitting cyclic presentations.

More precisely, for particular values of the parameters, called *admissible*, a Dunwoody diagram is an (open) Heegaard diagram of genus n (see Figure 2), which contains n internal circles $C'_1, \ldots, C'_n$, and n external circles $C''_1, \ldots, C''_n$, each having $d = 2a + b + c$ vertices. For every $i = 1, \ldots, n$, the circle C'_i (resp. C''_i) is connected to the circle C'_{i+1} (resp. C''_{i+1}) by a parallel arcs, to the circle C''_i by c parallel arcs and to the circle C''_{i-1} by b parallel arcs, for every $i = 1, \ldots, n$ (subscripts mod n). The cycle C'_i is glued to the cycle C''_{i+s} (subscripts mod n) so that equally labelled vertices are identified together.

Observe that the parameters r and s can be considered mod d and n, respectively. Since the identification rule and the diagram are invariant with respect to an obvious cyclic action of order n, the Dunwoody manifold $D(a, b, c, r, n, s)$ admits a cyclic symmetry of order n. Of course, $D(a, b, c, 1, r, 0)$ is homeomorphic to a lens space or to $\mathbf{S}^3$, since it admits a genus one Heegaard splitting.

3 Main results

The next theorem gives a characterization of Dunwoody manifolds as strongly-cyclic branched coverings of $(1, 1)$-knots.

Theorem 1 [10] *The Dunwoody manifold $D(a, b, c, n, r, s)$ is the n-fold strongly-cyclic covering of a lens space (possibly $\mathbf{S}^3$) branched over a $(1, 1)$-knot K, which only depends on the integers a, b, c, r.*

The next theorem shows that $(1, 1)$-knots are suitable objects to produce manifolds with cyclically presented fundamental groups.

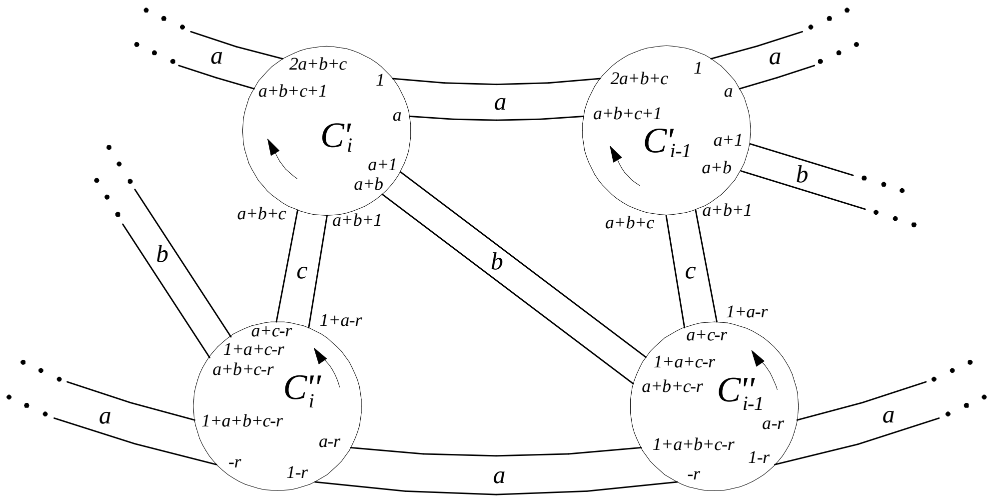

Figure 2: Heegaard diagram of Dunwoody type.

Theorem 2 [17] *Every n-fold strongly-cyclic branched covering of a* $(1,1)$*-knot admits a Heegaard diagram of genus n, which induces a cyclic presentation for the fundamental group of the manifold.*

An explicit way of finding a cyclic presentation for the fundamental group of any strongly-cyclic branched covering of a $(1,1)$-knot is given in [3].

A complete converse to Theorem 1 is the following:

Theorem 3 [5] *Every strongly-cyclic branched covering of a* $(1,1)$*-knot is a Dunwoody manifold.*

Corollary 4 [5] *The class $\mathcal{D}$ of Dunwoody manifolds coincides with the class $\mathcal{S}$ of strongly-cyclic branched coverings of* $(1,1)$*-knots.*

Now we give some important examples of Dunwoody manifolds.

- For all p, q, the n-fold strongly-cyclic branched covering of the trivial knot in the lens space $L(p,q)$ (including $L(1,0) \cong \mathbf{S}^3$), which is homeomorphic to the connected sum of n copies of $L(p,q)$, is the Dunwoody manifold $D(0,0,p,n,q,0)$.

- The 3-torus $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ is the Dunwoody manifold $D(1,1,1,3,2,1)$. It is well known that this manifold cannot be a cyclic branched covering of any knot in $\mathbf{S}^3$, but turns out to be a 3-fold cyclic covering of $\mathbf{S}^2 \times \mathbf{S}^1 \cong D(1,1,1,1,2,0)$, branched over a $(1,1)$-knot.

- [10] For all $n > 1$ and $a, r > 0$ such that $\gcd(2a+1, 2r) = 1$, the n-fold cyclic covering of $\mathbf{S}^3$ branched over the two-bridge knot of type $(2a+1, 2r)$ is the Dunwoody manifold $D(a, 0, 1, n, r, \bar{s})$, where $\bar{s}$ is an integer only depending on a and r.

- [1] For all $c > 0$ and $k, n > 1$, the n-fold cyclic covering of $\mathbf{S}^3$ branched over the torus knot $\mathbf{t}(k, ck+1)$ is the Dunwoody manifold $D(1, k-2, (k-1)(2c-1), n, k, k)$. For all $c > 1$ and $k, n > 1$, the n-fold cyclic covering of $\mathbf{S}^3$ branched over the torus knot $\mathbf{t}(k, ck-1)$ is the Dunwoody manifold $D(1, k-2, (k-1)(2c-1)-2, n, (k-1)(2c-3), -k)$.

As far as we know, no one has found an example of a 3-manifold with cyclically presented fundamental group that is not a Dunwoody manifold.

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