

Lins–Mandel graphs representing 3-manifolds[☆]

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Abstract

In this paper we find necessary and sufficient conditions for a Lins–Mandel 4-coloured graph to represent a 3-manifold.

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0. Introduction

Throughout this paper we only consider compact connected manifolds. Graphs are always finite and connected; multiple edges are allowed but loops are forbidden. If I is a set and $i \in I$ we denote with \hat{i} the subset $I - \{i\}$.

An n -coloured graph is an n -regular graph $\Gamma = (V(\Gamma), A(\Gamma))$ together with an edge coloration $\gamma: A(\Gamma) \rightarrow \mathcal{C}$ such that $\#\mathcal{C} = n$ and $\gamma(a) \neq \gamma(b)$ if a and b are adjacent edges. For each $\mathcal{F} \subseteq \mathcal{C}$ we call \mathcal{F} -residue each connected component of the subgraph $\Gamma_{\mathcal{F}} = (V(\Gamma), \gamma^{-1}(\mathcal{F}))$. An m -residue is a \mathcal{F} -residue such that $\#\mathcal{F} = m$. The number of the $\{i, j\}$ -residues will be indicated with $g_{i,j}$ while the number of the \hat{i} -residues with r_i .

In [11] is explained how to associate 3-manifolds to a class of 4-coloured graphs called 3-gems. Let Γ be a 4-coloured graphs and v, e, g, r be, respectively, the number of its 0-residues (vertices), 1-residues (edges), 2-residues (bicoloured cycles) and 3-residues; then Γ is a 3-gem iff $v + r = g$. Every 3-manifold admits a representation by 3-gems [11, Theorems 1 and 2]. A 3-gem Γ such that $r = 4$ is called a *crystallization* (see [8] for a survey on this concept). Every 3-manifold admits a representation by crystallizations ([12]).

Lins and Mandel [11] introduced an interesting class of 4-coloured graphs $S(b, l, t, c)$ depending on four positive integers b, l, t, c . It was extensively studied by

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several researchers as Cavicchioli [1–4], Casali and Grasselli [5,6], Donati [7], Johnson and Thomas [10].

Lins and Mandel showed in [11] that if the following conditions hold: (i) $(b, c) = 1$, (ii) $(l, t) = 1$, (iii) $c = (-1)^l$ if l is odd, then $S(b, l, t, c)$ is a crystallization and hence it represents a 3-manifold. Besides Casali and Grasselli [5] proved that if $S(b, l, t, c)$ is a crystallization then it satisfies the conditions (i)–(iii).

In what follows, we find necessary and sufficient conditions for $S(b, l, t, c)$ to be a 3-gem and hence to represent a 3-manifold.

1. The family of Lins–Mandel graphs

The family of Lins–Mandel 4-coloured graphs $S = \{S(b, l, t, c) \mid b, l, t, c \in \mathbb{Z}^+\}$ was defined in [11] by the following rules: the set of vertices is $V(S) = \mathbb{Z}_b \times \mathbb{Z}_{2l}$ and the coloured edges are obtained by these fixed-point-free involutions on $V(S)$

$$\alpha(i, j) = (i + c\mu(j - t), 1 - j + 2t),$$

$$\pi(i, j) = (i, j - (-1)^j),$$

$$\varepsilon(i, j) = (i, j + (-1)^j),$$

$$\nu(i, j) = (i + \mu(j), 1 - j),$$

where the arithmetic is mod b in the first and mod $2l$ in the second coordinate and $\mu: \mathbb{Z}_{2l} \rightarrow \{-1, 1\}$ is the function

$$\mu(j) = \begin{cases} 1 & \text{if } 1 \leq j \leq l, \\ -1 & \text{if } l+1 \leq j \leq 2l. \end{cases}$$

For each $\kappa \in \{\alpha, \pi, \varepsilon, \nu\}$ we link the vertices $v, w \in V(S)$ by an edge coloured κ iff $\kappa(v) = w$. Note that, without loss of generality, we may always suppose $1 \leq t \leq 2l$ and $1 \leq c \leq b$. Lemma 1.3 will show that it is sufficient to consider $1 \leq t \leq l$. Further it is easy to prove that $\mu(1 - j) = -\mu(j)$ for every $j \in \mathbb{Z}_{2l}$ and this will be really useful later.

Lemma 1.1. *Each graph $S(b, l, t, c)$ is bipartite.*

Proof. Let $A = \{(i, j) \in V(S) \mid j \text{ is even}\}$, $B = \{(i, j) \in V(S) \mid j \text{ is odd}\}$. Then it is not difficult to see that $\kappa(A) = B$ and $\kappa(B) = A$ for each $\kappa \in \{\alpha, \pi, \varepsilon, \nu\}$. \square

Lemma 1.2. *If $S(b, l, t, c)$ is a 3-gem then it represents an orientable 3-manifold.*

Proof. A 3-gem represents an orientable manifold iff it is bipartite [11]. \square

Lemma 1.3 (Lins and Mandel [11]). *The graphs $S(b, l, t, c)$ and $S(b, l, t - l, b - c)$ are equal.*

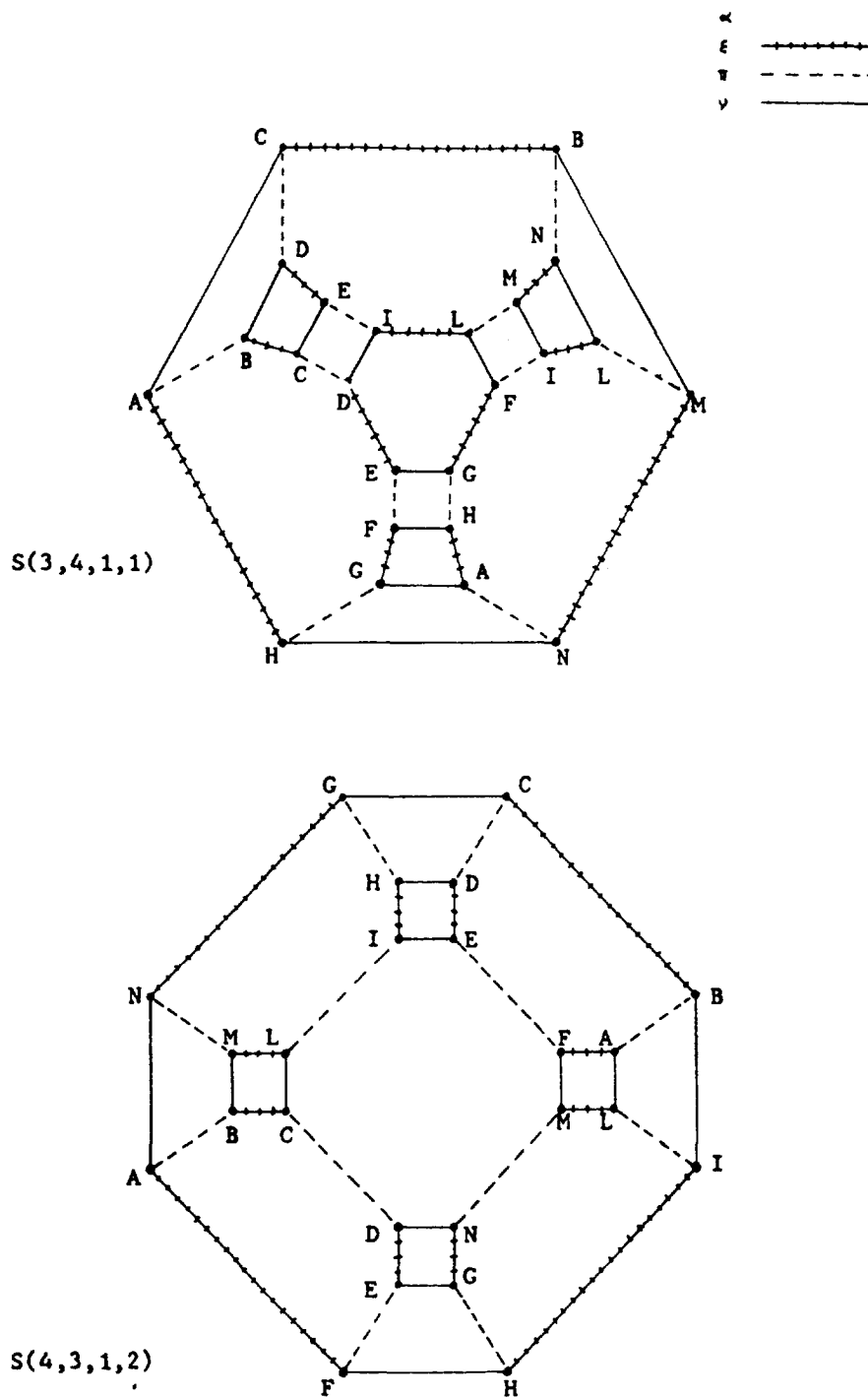


Fig. 1. $S(3,4,1,1)$ and $S(4,3,1,2)$.

Proof. Since π, ε, ν depends only on b and l , we just have to compare the first involution of the graphs. We have: $\alpha'(i, j) = (i + (b - c)\mu(j - t + l), 1 - j + 2t - 2l) = (i + c\mu(j - t), 1 - j + 2t) = \alpha(i, j)$.

2. Computation of the 2-residues

Since a 4-coloured graph is a 3-gem iff $v + r = g$, we need to compute the number of 2- and 3-residues of graphs $S(b, l, t, c)$. In this paragraph we are going to find the 2-residues that obviously are alternatively bicoloured cycles of even length. We will use the simplified notation (n, m) to indicate $\text{GCD}(n, m)$.

Lemma 2.1. For every $S(b, l, t, c)$: $g_{\varepsilon, \pi} = b$.

Proof. Since $(\pi\varepsilon)^n(i, j) = (i, j + (-1)^j 2n)$ each $\{\varepsilon, \pi\}$ -residue has length $2l$ and contains all the vertices with the same first coordinate. \square

We will call C_i ($1 \leq i \leq b$) those 2-residues.

Lemma 2.2. For every $S(b, l, t, c)$:

$$g_{\pi, \nu} = \begin{cases} bl/2 & \text{if } l \text{ is even,} \\ 1 + b(l-1)/2 & \text{if } l \text{ is odd.} \end{cases}$$

Proof. We have

$$\nu\pi(i, j) = (i + \mu(j - (-1)^j), 1 - j + (-1)^j),$$

$$(\nu\pi)^2(i, j) = (i + \mu(j - (-1)^j) + \mu(1 - j), j) = (i + \mu(j - (-1)^j) - \mu(j), j).$$

If l is even it is easy to see that we always obtain

$$1 - j + (-1)^j \not\equiv j \pmod{2l}, \tag{1}$$

$$\mu(j - (-1)^j) = \mu(j). \tag{2}$$

Therefore, all the π, ν -cycles are of length 4 and $g_{\pi, \nu} = 2bl/4 = bl/2$. If l is odd and $j \neq l, l + 1$ then (1) and (2) are still true. Besides we have $\pi(i, l) = (i, l + 1), \nu(i, l + 1) = (i - 1, l)$; hence there is a π, ν -cycle of length $2b$ and all the others are of length 4. In this case $g_{\pi, \nu} = 1 + (2bl - 2b)/4 = 1 + b(l - 1)/2$. \square

Lemma 2.3. For every $S(b, l, t, c)$:

$$g_{\varepsilon, \nu} = \begin{cases} 2 + b \frac{l-2}{2} & \text{if } l \text{ is even,} \\ 1 + b \frac{l-1}{2} & \text{if } l \text{ is odd.} \end{cases}$$

Proof. We have

$$v\varepsilon(i, j) = (i + \mu(j + (-1)^j), 1 - j - (-1)^j),$$

$$(v\varepsilon)^2(i, j) = (i + \mu(j + (-1)^j) + \mu(1 - j), j) = (i + \mu(j + (-1)^j) - \mu(j), j).$$

Let us consider the following conditions:

$$1 - j - (-1)^j \not\equiv j \pmod{2l}, \tag{3}$$

$$\mu(j + (-1)^j) = \mu(j). \tag{4}$$

If l is even then (3) and (4) hold when $j \neq 0, 1, l, l+1$; whereas if l is odd they are verified when $j \neq 0, 1$. Then we have $\varepsilon(i, 0) = (i, 1)$, $v(i, 1) = (i+1, 0)$ and, if l is odd, $\varepsilon(i, l) = (i, l+1)$, $v(i, l+1) = (i-1, l)$. Hence there is one (resp. two) ε, v -cycle of length $2b$ if l is odd (resp. even) and all the others are of length 4. Finally,

$$g_{\varepsilon, v} = 2 + (2bl - 4b)/4 = 2 + b(l-2)/2 \text{ if } l \text{ is even,}$$

$$g_{\varepsilon, v} = 1 + (2bl - 2b)/4 = 1 + b(l-1)/2 \text{ if } l \text{ is odd.} \quad \square$$

Corollary 2.1. For every $S(b, l, t, c)$: $g_{\varepsilon, v} + g_{\alpha, v} = 2 + b(l-1)$.

Lemma 2.4. For every $S(b, l, t, c)$:

$$g_{\varepsilon, \alpha} = \begin{cases} 2(b, c) + b(l-2)/2 & \text{if } t, l \text{ are even,} \\ bl/2 & \text{if } t \text{ is odd, } l \text{ is even,} \\ (b, c) + b(l-1)/2 & \text{if } l \text{ is odd.} \end{cases}$$

Proof. Set $d = (b, c)$, $b = db'$ and $c = dc'$. We have:

$$\alpha\varepsilon(i, j) = (i + c\mu(j + (-1)^j - t), 1 - j - (-1)^j + 2t),$$

$$(\alpha\varepsilon)^2(i, j) = (i + c\mu(j + (-1)^j - t) - c\mu(j - t), j).$$

Let us consider the conditions:

$$1 - j - (-1)^j + 2t \not\equiv j \pmod{2l}, \tag{5}$$

$$\mu(j - t + (-1)^j) = \mu(j - t). \tag{6}$$

They are verified in the following cases:

- (a) for $j \neq t, t+1, l+t, l+t+1$ if l, t are even;
- (b) for every $j \in \mathbb{Z}_{2l}$ if l is even, t is odd;
- (c) for $j \neq t, t+1$ if l is odd, t is even;
- (d) for $j \neq l+t, l+t+1$ if l, t are odd.

Now, if t even, we have $\varepsilon(i, t) = (i, t+1)$, $\alpha(i, t+1) = (i+c, t)$ and therefore $(\alpha\varepsilon)^{b'}(i, j) = (i, j)$. If $l+t$ is even, $\varepsilon(i, l+t) = (i, l+t+1)$, $\alpha(i, l+t+1) = (i-c, l+t)$ and hence $(\alpha\varepsilon)^{b'}(i, l+t) = (i, l+t)$. In case (a) we have $2d$ ε, α -cycles of length $2b'$, while in

cases (c) and (d) there are d ε, α -cycles of length $2b'$. All the other ε, α -cycles are of length 4. Finally,

$$\begin{aligned} g_{\varepsilon, \alpha} &= 2d + (2bl - 4b)/4 = 2d + b(l-2)/2 \text{ if } l, t \text{ are even;} \\ g_{\varepsilon, \alpha} &= 2bl/4 = bl/2 \text{ if } l \text{ is even, } t \text{ is odd;} \\ g_{\varepsilon, \alpha} &= d + (2bl - 2b)/4 = d + b(l-1)/2 \text{ if } l \text{ is odd. } \quad \square \end{aligned}$$

Lemma 2.5. For every $S(b, l, t, c)$:

$$g_{\pi, \alpha} = \begin{cases} 2(b, c) + b(l-2)/2 & \text{if } t \text{ is odd, } l \text{ is even,} \\ bl/2 & \text{if } t, l \text{ are even,} \\ (b, c) + b(l-1)/2 & \text{if } l \text{ is odd.} \end{cases}$$

Proof. Set $d = (b, c)$, $b = db'$ and $c = dc'$. Now we have:

$$\begin{aligned} \alpha\pi(i, j) &= (i + c\mu(j - (-1)^j - t), 1 - j + (-1)^j + 2t), \\ (\alpha\pi)^2(i, j) &= (i + c\mu(j - (-1)^j - t) - c\mu(j - t), j). \end{aligned}$$

Let us consider the conditions:

$$1 - j + (-1)^j + 2t \not\equiv j \pmod{2l}, \quad (7)$$

$$\mu(j - t - (-1)^j) = \mu(j - t). \quad (8)$$

They are verified in the following cases:

- (a) for $j \neq t$, $t+1$, $l+t$, $l+t+1$ if l even and t odd;
- (b) for every $j \in \mathbb{Z}_{2l}$ if l, t even;
- (c) for $j \neq t$, $t+1$ if l, t odd;
- (d) for $j \neq l+t$, $l+t+1$ if l odd and t even.

Now, if t is odd, we have $\pi(i, t) = (i, t+1)$, $\alpha(i, t+1) = (i+c, t)$ and therefore $(\alpha\pi)^{b'}(i, j) = (i, j)$. If $l+t$ is odd, $\pi(i, l+t) = (i, l+t+1)$, $\alpha(i, l+t+1) = (i-c, l+t)$ and hence $(\alpha\pi)^{b'}(i, l+t) = (i, l+t)$. In case (a) we have $2d$ π, α -cycles of length $2b'$, while in cases (c) and (d) there are d π, α -cycles of length $2b'$. All the other π, α -cycles are of length 4. Finally,

$$\begin{aligned} g_{\pi, \alpha} &= 2d + (2bl - 4b)/4 = 2d + b(l-2)/2 \text{ if } l \text{ even, } t \text{ odd;} \\ g_{\pi, \alpha} &= 2bl/4 = bl/2 \text{ if } l, t \text{ are even;} \\ g_{\pi, \alpha} &= d + (2bl - 2b)/4 = d + b(l-1)/2 \text{ if } l \text{ is odd. } \quad \square \end{aligned}$$

Corollary 2.2. For every $S(b, l, t, c)$: $g_{\varepsilon, \alpha} + g_{\pi, \alpha} = 2(b, c) + b(l-1)$.

Lemma 2.6. Set

$$t' = \frac{t}{(l, t)}, \quad l' = \frac{l}{(l, t)}, \quad b' = \frac{b}{(b, c)}, \quad c = \frac{c}{(b, c)}$$

and $\tilde{n} = \min \{ h \mid h \geq 1, h(1 - (-1)^{l'} c) \equiv 0 \pmod{b} \}$; then it is always: $(b, c) \mid \tilde{n}$ and $\tilde{n} \mid b$.

Proof. If $d=(b,c)$ then for a suitable $s \in \mathbb{Z}$ we have: $(1 \pm c)\tilde{n} = sb \Rightarrow \tilde{n} \pm dc'\tilde{n} = sdb' \Rightarrow d | \tilde{n}$. Moreover, since $(\tilde{n}, s) = 1$, we easily obtain $\tilde{n} | b$.

Lemma 2.7. Let l', \tilde{n} be as in the previous lemma; then for every $S(b, l, t, c)$:

$$g_{v,\alpha} = \begin{cases} b(l, t) & \text{if } l' \text{ is even,} \\ \frac{b}{\tilde{n}}(l, t) & \text{if } l' \text{ is odd.} \end{cases}$$

Proof. First let us set $d=(l, t)$ and $t=dt'$. Then we obtain: $(\alpha v)^n(i, j) = (i + \sum_{k=0}^{n-1} [\mu(j+2kt) - c\mu(t+j+2kt)], j+2nt)$. Let us define $N = \{n \in \mathbb{Z}^+ \mid (\alpha v)^n(i, j) = (i, j)\}$ and $m = \min N$. If $n \in N$ we have: $2l | 2nt \Rightarrow l' | nt' \Rightarrow l' | n$; set $n = n'l'$.

Now if l' is even (and hence t' is odd) we obtain for each $j \in \mathbb{Z}_{2l}$: $\mu(j+2(k+(l'/2))t) = \mu(j+2kt+l't) = \mu(j+2kt+lt') = -\mu(j+2kt)$. In this case $\sum_{k=0}^{l'-1} \mu(j+2kt) = \sum_{k=0}^{l'-1} \mu(t+j+2kt) = 0$ and $j+2l't = j+2lt' \equiv j \pmod{2l}$; therefore, $l' \in N$ and $m = l'$.

Whereas if l' is odd we have $\forall s \geq 0$: $\mu(j+2(k+sl')t) = \mu(j+2kt+2sl't) = \mu(j+2kt+2slt') = \mu(j+2kt)$; hence $\sum_{k=0}^{n'l'-1} \mu(j+2kt) = n' \sum_{k=0}^{l'-1} \mu(j+2kt)$ and $\sum_{k=0}^{n'l'-1} \mu(t+j+2kt) = n' \sum_{k=0}^{l'-1} \mu(t+j+2kt)$. In the same way of [5, p. 223] it is not difficult to show that $\sum_{k=0}^{l'-1} \mu(j+2kt) = \pm 1$ and $\sum_{k=0}^{l'-1} \mu(t+j+2kt) = (-1)^{l'} \sum_{k=0}^{l'-1} \mu(j+2kt)$. Thus $(\alpha v)^{n'l'}(i, j) = (i \pm n'(1 - (-1)^{l'}c), j)$ and therefore $m = \tilde{n}l'$.

If l' is even each v, α -cycle has length $2l'$ and $g_{v,\alpha} = 2bl/2l' = bd$. If l' is odd each v, α -cycle has length $2\tilde{n}l'$ and $g_{v,\alpha} = 2bl/2\tilde{n}l' = bd/\tilde{n}$. \square

Proposition 2.1. For every $S(b, l, t, c) \in S$:

$$g = \begin{cases} 2 + 2bl + 2(b, c) + b(l, t) - b & \text{if } l' \text{ is even,} \\ 2 + 2bl + 2(b, c) + \frac{b}{\tilde{n}}(l, t) - b & \text{if } l' \text{ is odd,} \end{cases}$$

where $l' = l/(l, t)$.

Proof. By summing the results of Lemmas 2.1, 2.7 and Corollaries 2.1, 2.2. \square

3. Computation of the 3-residues

Lemma 3.1. For every $S(b, l, t, c)$: $r_{\hat{2}} = 1$.

Proof. We can take $(i, 0)$ as representatives for the ε, π -cycles $C_i (i \in \mathbb{Z}_b)$. Since for each $i \in \mathbb{Z}_b$ it is $(\varepsilon v)^i(i, 0) = (0, 0)$ the subgraph $S_{\hat{2}}$ is connected. \square

Lemma 3.2. For every $S(b, l, t, c)$: $r_{\hat{3}} = (b, c)$.

Proof. Set $d=(b,c)$, $b=db'$, $c=dc'$. Since $\alpha(i,t)=(i-c,1+t)$ and $\alpha(i,t+1)=(i+c,t)$ the cycle C_i is joined with both cycles $C_{i\pm c}$ by α -edges. Therefore, C_i is in the same \hat{v} -residue of the cycles C_{i+zc} for every $z\in\mathbb{Z}$. We have: $i+z'c\equiv i+zc\pmod b \Leftrightarrow (z'-z)dc'\equiv 0\pmod b \Leftrightarrow (z'-z)c'\equiv 0\pmod{b'} \Leftrightarrow z'\equiv z\pmod{b'}$. Hence the set $\{[i+zc]_b \mid z\in\mathbb{Z}\}$ has cardinality b' and there are $b/b'=(b,c)$ \hat{v} -residues. \square

Lemma 3.3. Let $l'=l/(l,t)$ be an even number. Then the vertices (i',j) , (i'',j) are in the same $\{v,\alpha\}$ -residue iff $i'=i''$.

Proof. Suppose $(\alpha v)^n(i',j)=(i'',j)$, then we obtain: $j+2nt\equiv j\pmod{2l} \Rightarrow nt\equiv 0\pmod l \Rightarrow l' \mid nt' \Rightarrow l' \mid n$. Since from Lemma 2.7 the v,α -cycles have length $2l'$ then $i'=i''$. \square

Lemma 3.4. Let \tilde{n},l' be as in Lemma 2.6, then for every $S(b,l,t,c)$:

$$r_{\tilde{\pi}} = \begin{cases} 1 + \frac{b(l,t)-1}{\tilde{n}} & \text{if } l \text{ is odd,} \\ 1 + b \frac{(l,t)-1}{2} & \text{if } l \text{ is even, } t \text{ is odd,} \\ 2 + \frac{b(l,t)-2}{\tilde{n}} & \text{if } l,t \text{ are even, } l' \text{ is odd,} \\ 1 + (b,c) + b \frac{(l,t)-2}{2} & \text{if } l,t,l' \text{ are even.} \end{cases}$$

Proof. We set $d=(l,t)$, $t=dt'$ and call ϕ',ϕ'' the projections over the first and the second coordinate of $V(S)$.

(a) Let l' be an odd number. Then by Lemma 2.3 there is an ε, v -cycle C of length $2b$ containing all the vertices $(i,0)$ and $(i,1)$ ($i\in\mathbb{Z}_b$), and $b(l-1)/2$ ε, v -cycles $\mathcal{C}_{i,k}$ ($i\in\mathbb{Z}_b$, $1\leq k\leq(l-1)/2$) of length 4 with vertices $(i,2k),(i,2k+1),(i+1,-2k),(i+1,-2k+1)$. It is clear that the conditions for a cycle $\mathcal{C}_{i,k}$ to lie in the $\hat{\pi}$ -residue of C do not depend on i . Since

$$\begin{aligned} \phi''\alpha(i,2(k+st)) &= -2(k+(s-1)t)+1, & \phi''\alpha(i,2(k+st)+1) &= -2(k+(s-1)t), \\ \phi''\alpha(i,-2(k+st)) &= 2(k+(s+1)t)+1, & \phi''\alpha(i,-2(k+st)+1) &= 2(k+(s+1)t), \end{aligned}$$

it is easy to see that a cycle $\mathcal{C}_{i,k}$ is in the same $\hat{\pi}$ -residue of C iff $\exists z\in\mathbb{Z}$ such that $k\equiv zt\pmod l$. We have $\#\{k \mid k\equiv zt\pmod l, z\in\mathbb{Z} \text{ and } 1\leq k\leq(l-1)/2\}=(l'-1)/2$ and so there are $b(l'-1)/2$ cycles of length 4 in the component of C . Let A be the subset of vertices of that component; obviously it is $\#A=2b+2b(l'-1)=2bl'$. Then if t is even (resp. if t is odd) the vertices $(i,t),(i,t+1)\in A$ (resp. $(i,l+t),(i,l+t+1)\in A$) for each $i\in\mathbb{Z}_b$. In fact they lie in $\mathcal{C}_{i,t/2}$ (resp. in $\mathcal{C}_{i-1,(l-t)/2}$) and we have

$$\begin{aligned} \frac{l'+1}{2}t &= \frac{l'dt'+t}{2} = l\frac{t'}{2} + \frac{t}{2} \equiv \frac{t}{2} \pmod l \\ \left(\text{resp. } \frac{l'-1}{2}t &= \frac{lt'-1}{2} = l\frac{t'-1}{2} + \frac{l-t}{2} \equiv \frac{l-t}{2} \pmod l \right). \end{aligned}$$

Hence $\forall (i, j) \notin A$ it is $(\alpha\varepsilon)^2(i, j) = (i, j)$ (see proof of Lemma 2.4) and $(v\varepsilon)^2(i, j) = (i, j)$. Therefore, the $\{v, \alpha\}$ -residues Ω_s which do not lie in the component of C are coupled through ε -edges. Now we can easily compute the number of $\hat{\pi}$ -residues:

$$r_{\hat{\pi}} = 1 + \frac{2bl - \#A}{2 \#V(\Omega_s)} = 1 + \frac{2bl - 2bl'}{4\tilde{n}l'} = 1 + \frac{b(l, t) - 1}{\tilde{n} \cdot 2}.$$

(b) Now let l be even. Then there are 2 ε, v -cycles C', C'' of length $2b$ (containing, respectively, the subsets of vertices $\{(i, 0), (i, 1)\}_{i \in \mathbb{Z}_b}$ and $\{(i, l), (i, l+1)\}_{i \in \mathbb{Z}_b}$) and $b((l-2)/2)$ ε, v -cycles $\mathcal{C}_{i,k}$ of length 4 ($i \in \mathbb{Z}_b, 1 \leq k \leq (l-2)/2$) with vertices $(i, 2k), (i, 2k+1), (i+1, -2k), (i+1, -2k+1)$. As above we can see that a cycle $\mathcal{C}_{i,k}$ is in the same $\hat{\pi}$ -residue of C' (resp. C'') iff $\exists z \in \mathbb{Z}$ such that $k \equiv zt \pmod{l}$ (resp. $k \equiv l/2 + zt \pmod{l}$). Note that if l' is even (and therefore t' is odd) the components of C' and C'' are the same because $l'/2t = l'/2t' = l/2t' \equiv l/2 \pmod{l}$. Let B be the subset of vertices of that component; since $\#\{k \mid k \equiv zt \pmod{l}, 1 \leq k \leq (l-2)/2\} = (l'/2)/2$ we have $\#B = 4b + 2b(l'-2) = 2bl'$. If l' is odd the two components are distinct and, since $\#\{k \mid k \equiv zt \pmod{l}, 1 \leq k \leq (l-2)/2\} = (l'-1)/2$, each of them contains $2b + 2b(l'-1) = 2bl'$ vertices. We call B' (resp. B'') the set of vertices in the C' (resp. C'') component.

(b') If t is odd (and therefore l' even) we always have $(\alpha\varepsilon)^2(i, j) = (i, j)$ and then, as in (a), we obtain

$$r_{\hat{\pi}} = 1 + \frac{2bl - 2bl'}{4l'} = 1 + b \frac{(l, t) - 1}{2}.$$

(b'') if t is even and l' is odd the vertices $(i, t), (i, t+1), (i, l+t), (l+t+1) \in B' \cup B''$ for each $i \in \mathbb{Z}_b$. In fact the first two vertices lie in $\mathcal{C}_{i, t/2}$, the others lie in $\mathcal{C}_{i-1, (l-t)/2}$ and if t' is even (resp. if t' is odd) then we have

$$\frac{l'+1}{2}t = l\frac{t'}{2} + \frac{t}{2} \equiv \frac{t}{2} \pmod{l} \left(\text{resp. } \frac{l'-1}{2}t = l\frac{t'-1}{2} + \frac{l-t}{2} \equiv \frac{l-t}{2} \pmod{l} \right)$$

and

$$\left(\frac{l}{2} - \frac{l'+1}{2}t \right) \equiv \frac{l-t}{2} \pmod{l} \left(\text{resp. } \left(\frac{l}{2} - \frac{l'-1}{2}t \right) \equiv \frac{t}{2} \pmod{l} \right).$$

Therefore, it is $(\alpha\varepsilon)^2(i, j) = (i, j)$ and $(v\varepsilon)^2(i, j) = (i, j)$ for every $(i, j) \notin B' \cup B''$ and as above we obtain

$$r_{\hat{\pi}} = 2 + \frac{2bl - 4bl'}{4\tilde{n}l'} = 2 + \frac{b(l, t) - 2}{\tilde{n} \cdot 2}.$$

(b''') if t, l' are even then we have $(\alpha\varepsilon)^2(i, j) = (i, j)$ when $j \neq t, t+1, l+t, l+t+1$. Let $D = \{(i, j) \in V(S) \mid j = t, t+1, l+t, l+t+1\}$. By Lemma 3.3 the vertices $(i, t) (i \in \mathbb{Z}_b)$ are in b distinct v, α -cycles D_i . Moreover, since $\alpha(i, t) = (i-c, t+1)$, $\alpha(i, l+t) = (i+c, l+t+1)$ and $\phi''(\alpha v)^{l'/2}(i, t) = t + tl' = t + t'l \equiv t + l \pmod{2l}$, the set D is contained in the above mentioned v, α -cycles. Now it is easy to prove that $D_{i'}, D_{i''}$ are in the same $\hat{\pi}$ -residue iff

$i' \equiv i'' \pmod{(b, c)}$. Furthermore the other v, α -cycles are coupled by the ε -edges. Hence we obtain

$$r_{\varepsilon} = 1 + (b, c) + \frac{2bl - 2bl' - 2bl'}{4l'} = 1 + (b, c) + b \frac{(l, t) - 2}{2}. \quad \square$$

Lemma 3.5. *Let \tilde{n}, l' be as in Lemma 2.6, then for every $S(b, l, t, c)$:*

$$r_{\varepsilon} = \begin{cases} 1 + \frac{b}{\tilde{n}} \frac{(l, t) - 1}{2} & \text{if } l \text{ is odd,} \\ (b, c) + b \frac{(l, t) - 1}{2} & \text{if } l \text{ is even, } t \text{ is odd,} \\ \frac{b}{\tilde{n}} \frac{(l, t)}{2} & \text{if } l, t \text{ are even, } l' \text{ is odd,} \\ b \frac{(l, t)}{2} & \text{if } l, t, l' \text{ are even.} \end{cases}$$

Proof. (a) First suppose l odd. We define a bijection τ on $V(S)$ with $\tau(i, j) = (-i, j + l)$. Since $\varepsilon\tau = \tau\varepsilon$, $\pi\tau = \tau\varepsilon$, $v\tau = \tau v$ and $\alpha\tau = \tau\alpha$ we obtain an automorphism of graph S that exchanges the colours ε and π . Then the statement follows immediately from Lemma 3.4.

(b) If l is even there are $b(l/2)$ π, v -cycles $\mathcal{C}_{i, k}$ of length 4 ($i \in \mathbb{Z}_b$, $1 \leq k \leq l/2$) with vertices $(i, 2k), (i, 2k - 1), (i + 1, -2k + 2), (i + 1, -2k + 1)$.

(b'–b'') if t is even we always have $(\alpha\pi)^2(i, j) = (i, j)$ and then we obtain

$$r_{\varepsilon} = \frac{2bl}{4l'} = b \frac{(l, t)}{2} \quad \text{if } l' \text{ is even,}$$

$$r_{\varepsilon} = \frac{2bl}{4\tilde{n}l'} = \frac{b}{\tilde{n}} \frac{(l, t)}{2} \quad \text{if } l' \text{ is odd.}$$

(b''') if t is odd then we have $(\alpha\pi)^2(i, j) = (i, j)$ whenever $j \neq t, t + 1, l + t, l + t + 1$. As in part (b''') of the proof of the previous lemma we obtain:
 $r_{\varepsilon} = (b, c) + (2bl - 2bl')/4l' = (b, c) + b((l, t) - 1)/2$. \square

Proposition 3.1. *For every $S(b, l, t, c) \in \mathcal{S}$:*

$$r = \begin{cases} 2 + 2(b, c) + b(l, t) - b & \text{if } l' \text{ is even,} \\ 3 + (b, c) + \frac{b}{\tilde{n}}(l, t) - \frac{b}{\tilde{n}} & \text{if } l' \text{ is odd,} \end{cases}$$

where $l' = l/(l, t)$.

Proof. By summing the results of Lemmas 3.1, 3.2, 3.4 and 3.5. \square

4. Main results

Now it is easy to find necessary and sufficient conditions for $S(b, l, t, c)$ to represent a 3-manifold. In fact the following relation must be verified:

$$2bl + r = g. \quad (9)$$

Theorem 4.1.

- (a) Let $l' = l/(l, t)$ be even, then every $S(b, l, t, c)$ represents a 3-manifold.
 (b) Let $l' = l/(l, t)$ be odd, then each $S(b, l, t, c)$ represents a 3-manifold iff either $c = b$ or $c = (-1)^{l'}$ (where $t' = t/(l, t)$).

Proof.

- (a) We always have: $2bl + r - g = 0$.
 (b) In this case $2bl + r - g = 1 + b - (b, c) - b/\tilde{n}$, and therefore S represents a 3-manifold iff:

$$\frac{b}{\tilde{n}} - 1 = b - (b, c). \quad (10)$$

If $b = c$ then, by Lemma 2.6, $\tilde{n} = (b, c)$ and (10) holds. If $c = (-1)^{l'}$ then $\tilde{n} = 1 = (b, c)$ and (10) still holds. On the contrary, let us suppose that $b > c \neq (-1)^{l'}$, then $\tilde{n} > 1$ and, moreover, $(b, c) < b$. We obtain the following inequalities:

- (A) if $\tilde{n} < b$ then $0 < b/\tilde{n} - 1 < b - \tilde{n} = b - n(b, c) \leq b - (b, c)$;
 (B) if $\tilde{n} = b$ then $0 = b/\tilde{n} - 1 < b - (b, c)$.

Therefore in both cases the condition (10) does not hold. \square

Remarks. It is interesting to compare the results of Propositions 4.1 and 4.2 with the conditions (i)–(iii) mentioned in the introduction. Conditions (i) and (ii) are not necessary for the graph $S(b, l, t, c)$ to represent a 3-manifold; whereas condition (iii) (replacing l with l') is necessary if $c \neq b$.

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