

ALL STRONGLY-CYCLIC BRANCHED COVERINGS OF $(1, 1)$ -KNOTS ARE DUNWOODY MANIFOLDS

ALESSIA CATTABRIGA AND MICHELE MULAZZANI

ABSTRACT

It is shown that every strongly-cyclic branched covering of a $(1, 1)$ -knot is a Dunwoody manifold. This result, together with the converse statement previously obtained by Grasselli and Mulazzani, proves that the class of Dunwoody manifolds coincides with the class of strongly-cyclic branched coverings of $(1, 1)$ -knots. As a consequence, a parametrization of $(1, 1)$ -knots by 4-tuples of integers is obtained. Moreover, using a representation of $(1, 1)$ -knots by the mapping class group of the twice-punctured torus, an algorithm is provided which gives the parametrization of all torus knots in $\mathbf{S}^3$.

1. Introduction

In order to investigate the relations between cyclic branched coverings of knots in $\mathbf{S}^3$ and manifolds admitting cyclically presented fundamental groups, Dunwoody introduced in [6] a class of 3-manifolds depending on six integer parameters. As proved in [7], all these manifolds turn out to be strongly-cyclic coverings of lens spaces (possibly $\mathbf{S}^3$), branched over $(1, 1)$ -knots. Moreover, it has been shown in [10] that every n -fold strongly-cyclic branched covering of a $(1, 1)$ -knot admits a genus n Heegaard diagram encoding a cyclic presentation for the fundamental group. This result has been improved in [3], obtaining a constructive algorithm which, starting from a representation of $(1, 1)$ -knots through the elements of the mapping class group of the twice-punctured torus, explicitly gives the cyclic presentations.

In this paper we prove that all strongly-cyclic branched coverings of $(1, 1)$ -knots are actually Dunwoody manifolds. As a consequence, the class of Dunwoody manifolds coincides with the class of strongly-cyclic branched coverings of $(1, 1)$ -knots.

We also obtain, as a further consequence, a parametrization of all $(1, 1)$ -knots (with the exception of the ‘core’ knot $\{P\} \times \mathbf{S}^1 \subset \mathbf{S}^2 \times \mathbf{S}^1$, which admits no strongly-cyclic branched coverings) by means of four of the six Dunwoody parameters. Moreover, we give an algorithm that allows us to find the parametrization of all torus knots in $\mathbf{S}^3$.

We refer to [9, 2] for details on knot theory and cyclic branched coverings of knots, and to [8] for details on cyclic presentations of groups.

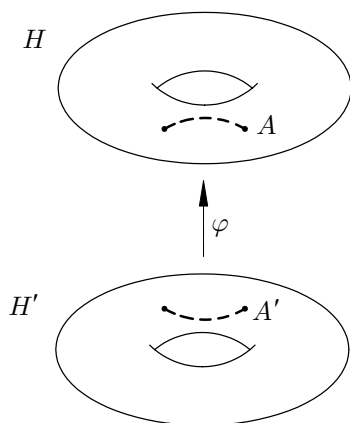
2. Strongly-cyclic branched coverings of $(1, 1)$ -knots and Dunwoody manifolds

An n -fold cyclic covering of a 3-manifold N^3 branched over a knot $K \subset N^3$ is called *strongly-cyclic* if the branching index of K is n (that is, the fibre of each

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FIGURE 1. A $(1, 1)$ -decomposition.

point of K contains a single point), so the homology class of a meridian loop m around K is mapped by the associated monodromy $\omega: H_1(N^3 - K) \rightarrow \mathbb{Z}_n$ to a generator of $\mathbb{Z}_n$ (up to equivalence we can always suppose that $\omega[m] = 1$).

Observe that a cyclic branched covering of a knot K in $\mathbf{S}^3$ is always strongly-cyclic and uniquely determined, up to equivalence, since $H_1(\mathbf{S}^3 - K) \cong \mathbb{Z}$. Obviously, this property is no longer true for a knot in a more general 3-manifold. Also, if p is a prime number, any p -fold cyclic branched covering of a knot K is automatically strongly-cyclic.

In this paper we deal with strongly-cyclic branched coverings of $(1, 1)$ -knots, which are knots in lens spaces (possibly in $\mathbf{S}^3$).

A knot K in a 3-manifold N^3 is called a $(1, 1)$ -knot if there exists a Heegaard splitting of genus one

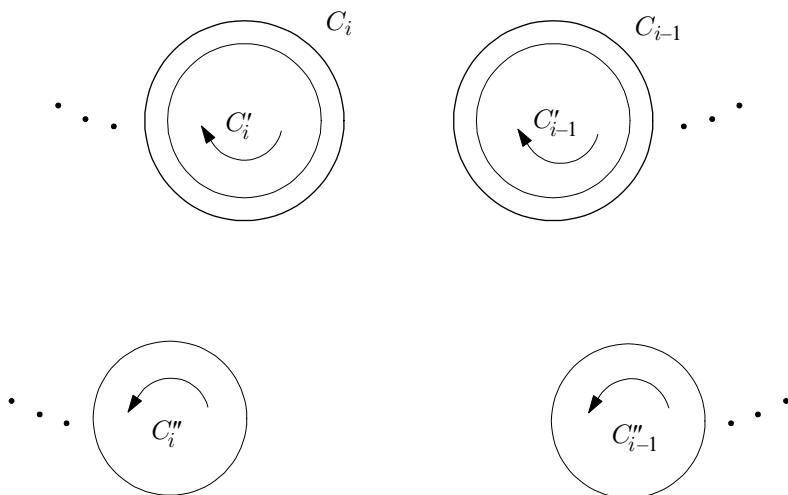
$$(N^3, K) = (H, A) \bigcup_{\varphi} (H', A'),$$

where H and H' are solid tori, $A \subset H$ and $A' \subset H'$ are properly embedded trivial arcs, and $\varphi: (\partial H', \partial A') \rightarrow (\partial H, \partial A)$ is an attaching homeomorphism (see Figure 1). Obviously, N^3 turns out to be a lens space $L(p, q)$ (including $\mathbf{S}^3 = L(1, 0)$).

It is well known that the family of $(1, 1)$ -knots contains all torus knots and all two-bridge knots in $\mathbf{S}^3$. Several topological properties of $(1, 1)$ -knots have recently been investigated (see references in [4]).

PROPOSITION 1. *A $(1, 1)$ -knot $K \subset L(p, q)$ with $(1, 1)$ -decomposition $(L(p, q), K) = (H, A) \bigcup_{\varphi} (H', A')$ is completely determined, up to equivalence, by $\varphi(\beta')$, where β' is the boundary of a meridian disk $D' \subset H'$ which does not intersect A' . Moreover, if $(L(p, q), \bar{K}) = (H, A) \bigcup_{\bar{\varphi}} (H', A')$ is a decomposition of a $(1, 1)$ -knot $\bar{K}$ such that $\bar{\varphi}(\beta')$ is isotopic to $\varphi(\beta')$ in $\partial H - \partial A$, then $\bar{K}$ is equivalent to K .*

Proof. The first statement follows from the fact that two properly embedded trivial arcs in a ball B , with the same endpoints, are isotopic rel ∂B . The second statement is straightforward. $\square$

FIGURE 3. The diagram $D(0,0,0,n,r,s)$.

$M(a,b,c,r,n,s)$ admits a cyclic symmetry of order n . Of course, $M(a,b,c,1,r,0)$ is homeomorphic to a lens space or to $\mathbf{S}^3$, since it admits a genus one Heegaard splitting. Moreover, the trivial case $M(0,0,0,n,0,s)$ is homeomorphic to the connected sum of n copies of $\mathbf{S}^2 \times \mathbf{S}^1$, for all n and s .

A characterization of all Dunwoody manifolds as strongly-cyclic branched coverings of $(1,1)$ -knots is given by the following result.

PROPOSITION 2 [7]. *The Dunwoody manifold $M(a,b,c,n,r,s)$ is the n -fold strongly-cyclic covering of the lens space $M(a,b,c,1,r,0)$ (possibly $\mathbf{S}^3$), branched over a $(1,1)$ -knot only depending on the integers a, b, c, r .*

An interesting example of a Dunwoody manifold is $M(1,1,1,3,2,1)$, which is homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$. It is well known that this manifold cannot be a cyclic branched covering of any knot in $\mathbf{S}^3$, but turns out to be a 3-fold cyclic covering of $\mathbf{S}^2 \times \mathbf{S}^1 \cong M(1,1,1,1,2,0)$, branched over a $(1,1)$ -knot, which will be referred to as $K(1,1,1,2)$.

In the next section we prove the converse of Proposition 2. As a consequence, the class of Dunwoody manifolds coincides with the class of strongly-cyclic branched coverings of $(1,1)$ -knots.

3. Main result

Now we establish the main result of this paper.

THEOREM 3. *Every strongly-cyclic branched covering of a $(1,1)$ -knot is a Dunwoody manifold.*

Proof. Let $K \subset L(p,q)$ be a $(1,1)$ -knot and let $(L(p,q), K) = (H, A) \bigcup_{\varphi} (H', A')$ be a $(1,1)$ -decomposition of K . Let β (respectively β') be a meridian of ∂H (respectively $\partial H'$) that bounds a disk in H (respectively H') not intersecting A

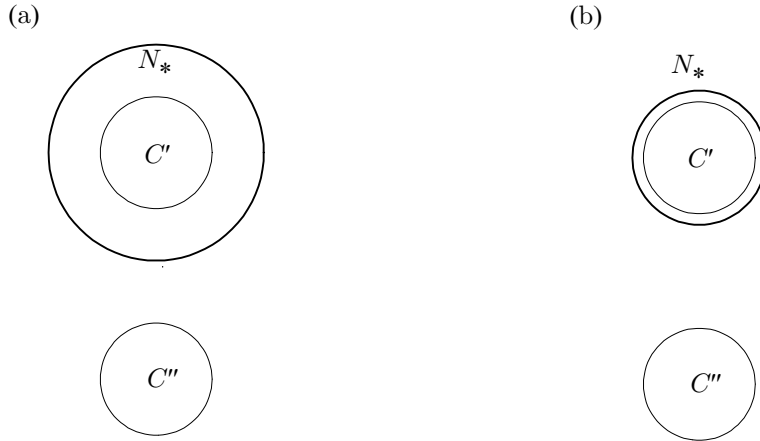


FIGURE 4.

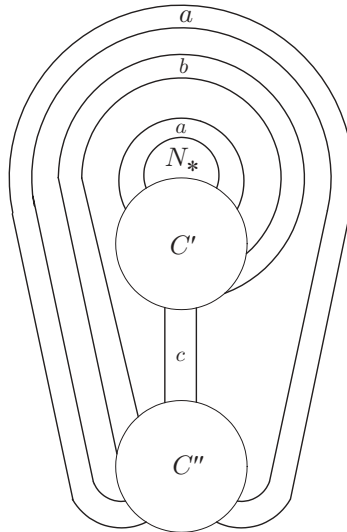


FIGURE 5.

(respectively A'). The system of curves $(\beta, \varphi(\beta'))$ on $T = \partial H$ defines a genus one Heegaard diagram of $L(p, q)$, which does not intersect $\partial A = \{N, S\}$. Let H_φ be the open Heegaard diagram on $\mathbb{R}^2$ obtained by cutting T along β , and considering S as the point at the infinity of $\mathbf{S}^2 = \mathbb{R}^2 \cup \{S\}$. The diagram consists of two canonical circles C' and C'' , corresponding to β , and a closed curve or a set of arcs with endpoints on the canonical circles, which corresponds to $\varphi(\beta')$ and will be denoted by $\mathcal{E}$. Suppose that one of the following holds.

- (1) H_φ is the diagram depicted in Figure 4(a).
- (2) H_φ is the diagram depicted in Figure 4(b).
- (3) There exist integers a, b, c , with $a, b, c \geq 0$ and $a + b + c > 0$, such that H_φ is the diagram depicted in Figure 5.

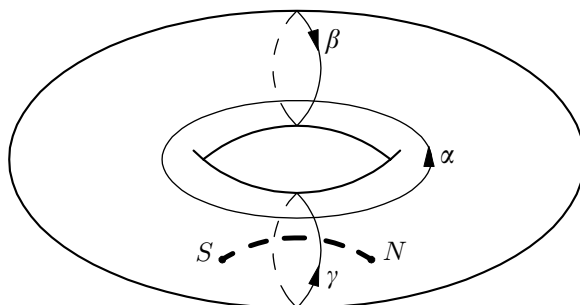


FIGURE 6.

In the first case, K is the core knot $\{P\} \times \mathbf{S}^1 \subset \mathbf{S}^2 \times \mathbf{S}^1$, where P is a point of $\mathbf{S}^2$. Therefore, from [3, Corollary 2], we have $H_1(\mathbf{S}^2 \times \mathbf{S}^1 - K) = \langle \alpha, \gamma | \gamma \rangle \cong \mathbb{Z}$, where α and γ are the curves on T depicted in Figure 6. Thus, by [3, Theorem 4], there exists no strongly-cyclic branched covering of K .

In the second case, K is the trivial knot in $\mathbf{S}^2 \times \mathbf{S}^1$. Therefore, by [3, Corollary 2], we have $H_1(\mathbf{S}^2 \times \mathbf{S}^1 - K) = \langle \alpha, \gamma | \emptyset \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus, by [3, Theorem 4], there exist exactly n n -fold strongly-cyclic branched coverings of K , depending on the choice of $\omega(\alpha) \in \mathbb{Z}_n$, where $\omega: H_1(\mathbf{S}^2 \times \mathbf{S}^1 - K) \rightarrow \mathbb{Z}_n$ is the monodromy map of the covering such that $\omega(\gamma) = 1$. If we denote by $C_{n,s}(K)$ the n -fold strongly-cyclic branched covering of K such that $\omega(\alpha) = s$, we have $C_{n,s}(K) = M(0, 0, 0, n, 0, s)$. Actually, as previously observed, $C_{n,s}(K)$ is homeomorphic to the connected sum of n copies of $\mathbf{S}^2 \times \mathbf{S}^1$, for all n, s .

Let us consider the third case. If $f: M \rightarrow L(p, q)$ is an n -fold strongly-cyclic branched covering of K , then the $(1, 1)$ -decomposition of K lifts to a genus n Heegaard splitting for M (see [10]). Since $\omega(\gamma) = 1$, up to equivalence, then the lifting of H_φ is the Dunwoody diagram $D(a, b, c, n, r, s)$, where $s = \omega(\alpha)$. In other words, M is the Dunwoody manifold $M(a, b, c, n, r, s)$.

By Proposition 1, to prove the theorem it is enough to show that H_φ is equivalent, up to Singer moves fixing N , to one of the three diagrams discussed above.

Denote by D' and D'' the disks of $\mathbb{R}^2$ bounded by C' and C'' , respectively. Moreover, let $\mathcal{A}'$ (respectively $\mathcal{A}''$) be the set of arcs of $\mathcal{E}$ with both the endpoints on C' (respectively C''), and denote by $\mathcal{B}$ the remaining arcs of $\mathcal{E}$. Of course, $|\mathcal{A}'| = |\mathcal{A}''|$. An arc $e \in \mathcal{A}'$ (respectively $\mathcal{A}''$) is called *trivial* if the closed curve $e \cup e'$, where e' is one of the two arcs of C' (respectively C'') with the same endpoints of e , bounds a disk containing neither N nor D'' (respectively D'). As illustrated in Figure 7, each trivial arc can be removed by a Singer move of type IB (see [11]). Thus, up to equivalence, we can suppose that H_φ contains no trivial arcs. Observe that this assumption implies that $e \cup e'$ bounds a disk in $\mathbb{R}^2$ containing the point N , for every $e \in \mathcal{A}' \cup \mathcal{A}''$. In fact, if there exists a non-trivial arc e of $\mathcal{A}'$ (respectively $\mathcal{A}''$) such that $e \cup e'$ bounds a disk D in $\mathbb{R}^2$ not containing N , then D contains D'' (respectively D') and therefore there exists a trivial arc in $\mathcal{A}''$ (respectively $\mathcal{A}'$).

In order to simplify the proof, let us consider the planar graph Γ obtained from H_φ by collapsing the disks D' and D'' to their centres, which we still indicate by C' and C'' , respectively. Of course, the arcs of $\mathcal{A}'$ and $\mathcal{A}''$ become loops in Γ bounding disks all containing N .

We say that two elements of $\mathcal{E}$ are *parallel* if they are isotopic rel $\{C', C'', N\}$. It is easy to see that any two elements of $\mathcal{A}'$ (respectively $\mathcal{A}''$) are parallel. In fact, if the

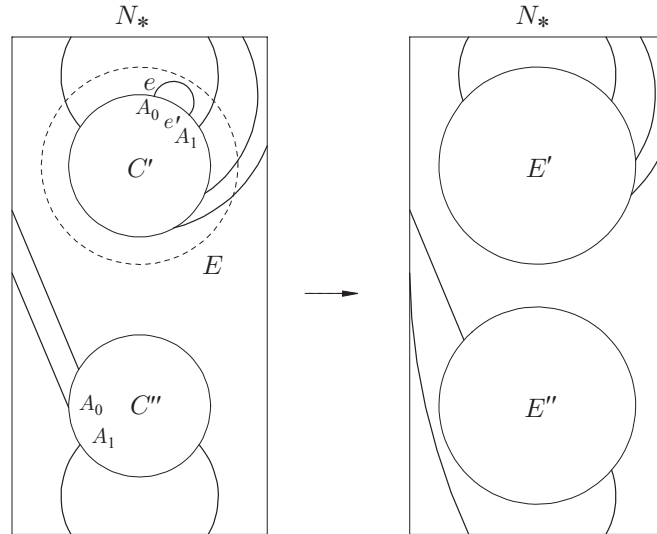


FIGURE 7. Singer move of type IB.

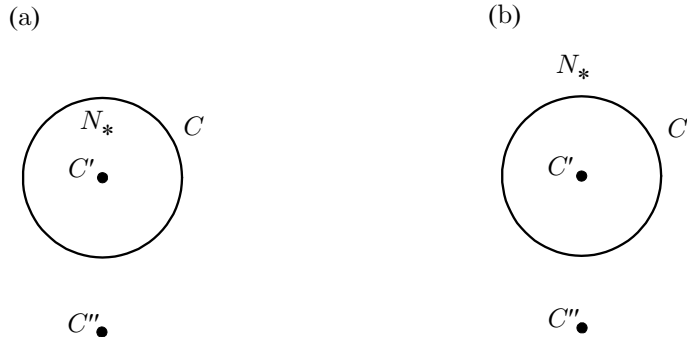


FIGURE 8.

disk bounded by a loop of $\mathcal{A}'$ (respectively $\mathcal{A}''$) contains C'' (respectively C'), then all the disks bounded by the loops of $\mathcal{A}'$ (respectively $\mathcal{A}''$) contain C'' (respectively C'). Otherwise, each loop of $\mathcal{A}''$ (respectively $\mathcal{A}'$) bounds a disk not containing N . As regards the elements of $\mathcal{B}$, we note that two different arcs $g, g' \in \mathcal{B}$ are parallel if and only if the closed curve $g \cup g'$ bounds a disk $D_{g,g'}$ not containing N . It is not difficult to see that there are at most two isotopy classes. For, if $g, g', g'' \in \mathcal{B}$ are different arcs such that g is not parallel to either g' or g'' , then $N \in D_{g,g'}$ and $N \in D_{g,g''}$. Moreover, either $D_{g',g''} = (D_{g,g'} - D_{g,g''}) \cup g''$ or $D_{g',g''} = (D_{g,g''} - D_{g,g'}) \cup g'$. In both cases $N \notin D_{g',g''}$ and therefore g' is parallel to g'' .

If $\mathcal{A}' = \mathcal{A}'' = \mathcal{B} = \emptyset$, $\mathcal{E}$ consists of a closed curve C . Thus, up to isotopy in $\mathbb{R}^2 - N$, we can suppose that C is a standard circle. There are two possibilities, depending on whether the point N is contained inside or outside C . However, in both cases, since C is a curve of a Heegaard diagram, C' is inside C if and only if C'' is outside C . Hence, up to a possible exchange between C' and C'' , the two possibilities are those depicted in Figure 8, which are the same as in Figure 4.

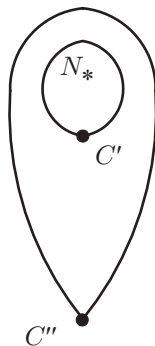


FIGURE 9.

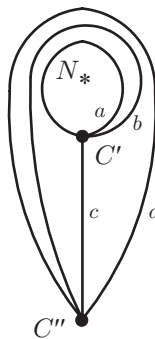


FIGURE 10.

If $\mathcal{A}' \cup \mathcal{A}'' \cup \mathcal{B} \neq \emptyset$, we can consider the graph Γ' obtained from Γ by taking only one element for each isotopy class of arcs. Γ' is a graph embedded in $\mathbb{R}^2 - N$ with two vertices, a loop in each vertex if $\mathcal{A}' \neq \emptyset$, and one or two edges linking the vertices if $\mathcal{B} \neq \emptyset$. If $\mathcal{A}' \neq \emptyset$, one of the two loops is contained in the disk bounded by the other, since both of the disks bounded by the loops contain N . Up to isotopy in $\mathbb{R}^2 - N$ and to a possible exchange between C' and C'' , they are as in Figure 9. The other edges of Γ' , if any, must be contained in the annulus bounded by the two loops, so, up to an isotopy of $\mathbb{R}^2 - N$, which can be chosen as the identity outside C''' , they are as in Figure 10. Of course, the same configuration of these edges holds when $\mathcal{A}' = \emptyset$.

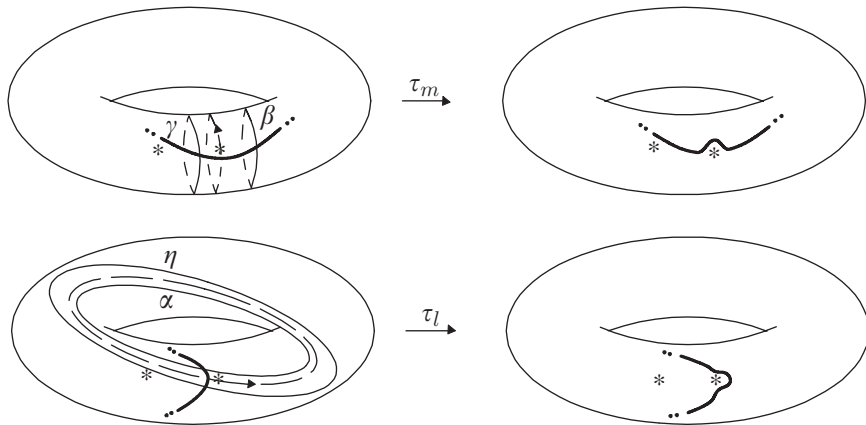
Thus H_φ is the diagram depicted in Figure 5, where a, b, c are the cardinalities of the isotopy classes. $\square$

By Theorem 3 and Proposition 2 we have the following.

COROLLARY 4. *The class of Dunwoody manifolds coincides with the class of strongly-cyclic branched coverings of $(1, 1)$ -knots.*

4. Parametrization of $(1, 1)$ -knots

As a consequence of the proof of Theorem 3, any $(1, 1)$ -knot K , with the sole exception of the core knot $\{P\} \times \mathbf{S}^1 \subset \mathbf{S}^2 \times \mathbf{S}^1$ (which admits no strongly-cyclic

FIGURE 11. Action of τ_m and τ_l .

branched coverings), has a $(1, 1)$ -decomposition which can be represented by an admissible Dunwoody diagram $D(a, b, c, 1, r, 0)$, for suitable integers $a, b, c \geq 0$ and r . In this case, we set $K = K(a, b, c, r)$, and the Dunwoody manifold $M(a, b, c, n, r, s)$ is an n -fold strongly-cyclic branched covering of the lens space $M(a, b, c, 1, r, 0)$ (possibly homeomorphic to $\mathbf{S}^3$), branched over the $(1, 1)$ -knot $K(a, b, c, r)$.

EXAMPLES 5. By [7, Theorem 8], the two-bridge knot with Schubert parametrization $(2a + 1, 2r)$ is the $(1, 1)$ -knot $K(a, 0, 1, r)$. The trivial knot in $\mathbf{S}^2 \times \mathbf{S}^1$ is $K(0, 0, 0, 0)$ and the trivial knot in $L(p, q)$ (including $L(1, 0) \cong \mathbf{S}^3$) is $K(0, 0, p, q)$.

Note that a different parametrization of $(1, 1)$ -knots, which involves four parameters for the knot and two additional parameters for the ambient space, can be found in [5].

Now we describe an algorithm that gives the parametrization $K(a, b, c, r)$ of all torus knots in $\mathbf{S}^3$.

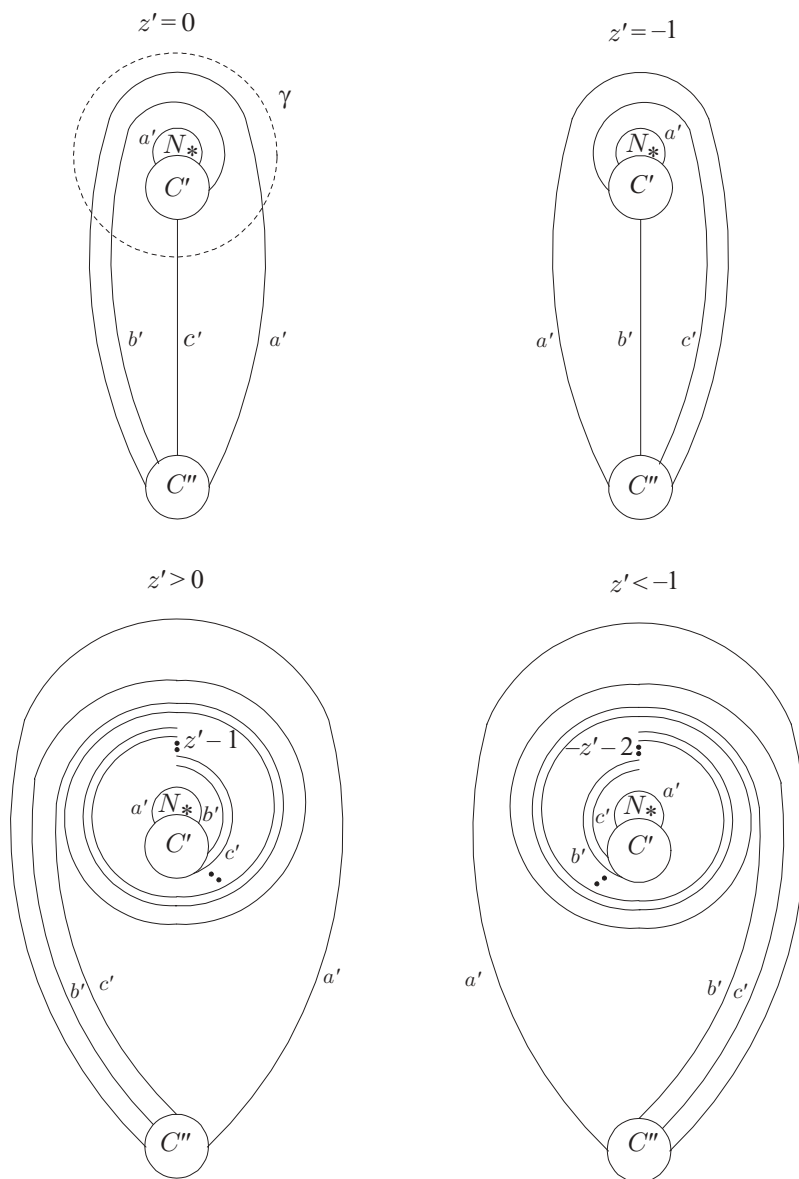
Given a closed simple curve $\delta \subset \partial H$, denote by $t_\delta \in \text{PMCG}_2(\partial H)$ the right-hand Dehn twist along δ . Moreover, let $\tau_m = t_\beta t_\gamma^{-1}$ and $\tau_l = t_\eta t_\alpha^{-1}$, where $\beta, \gamma, \alpha, \eta$ are the curves depicted in Figure 11. The effect of τ_m and τ_l is to slide one puncture, for example N , along the broken curves depicted in Figure 11, that is, along a meridian and a longitude of the torus, respectively.

As shown in [4], for every $1 < k < h$, the torus knot $\mathbf{t}(k, h) \subset \mathbf{S}^3$ is the $(1, 1)$ -knot K_ψ with

$$\psi = \prod_{j=0}^{h-1} (\tau_l^{-1} \tau_m^{\varepsilon_{h-j}}) t_\beta t_\alpha t_\beta, \quad (1)$$

where $\varepsilon_{h-j} = \lfloor (j+1)k/h \rfloor - \lfloor (j+2)k/h \rfloor$ ($\lfloor x \rfloor$ denotes the integral part of x). Since $k < h$, we have $\varepsilon_{h-j} \in \{-1, 0\}$, for all j .

In order to find the parameters a, b, c, r for $\mathbf{t}(k, h)$, it is enough to illustrate how the Heegaard diagram $D(0, 0, 0, 1, 0, 0)$ is modified by the initial application of $t_\beta t_\alpha t_\beta$ and by the successive applications of the elements τ_l^{-1} and $\tau_l^{-1} \tau_m^{-1}$ composing ψ , according to (1). In this way we construct a Heegaard diagram $D(a, b, c, 1, r, 0)$ representing $\mathbf{t}(k, h)$.

FIGURE 12. The Heegaard diagram $D_{z'}(a', b', c', 1, r', 0)$.

Actually, during the process, the Heegaard diagrams involved at each step are diagrams which can be obtained by performing a certain number $z' \in \mathbb{Z}$ of Dehn twists along the curve γ to a standard Dunwoody diagram $D(a', b', c', 1, r', 0)$ (see Figure 12). We will call this diagram $D_{z'}(a', b', c', 1, r', 0)$. These types of diagrams are depicted in Figure 12, where an arc labelled k denotes k parallel arcs. Obviously, $D_0(a', b', c', 1, r', 0) = D(a', b', c', 1, r', 0)$.

Observe that, at the end of the process, we can reduce z' to zero, since $K_{t_\gamma \psi}$ and K_ψ are equivalent knots.

PROPOSITION 6. Let $\mathbf{t}(k, h) \subset \mathbf{S}^3$ be a torus knot and ψ be its representation described in (1). Then $\mathbf{t}(k, h) = K(a, b, c, r)$, where $(a, b, c, r) = (a_h, b_h, c_h, r_h)$ is the final step of the following algorithm, applied for $i = h - j = 1, \dots, h$.

- (1) $(a_0, b_0, c_0, r_0) = (0, 0, 1, 0)$ and $z_0 = 0$.
- (2) For $i = 1, \dots, h$,

$$\begin{cases} a_i = a_{i-1} + v \\ b_i = r_{i-1} - 2w - ud \\ c_i = d - b_i \\ r_i = a_{i-1} + v + w \\ z_i = u - \varepsilon_i \end{cases}$$

where

$$w = \begin{cases} a_{i-1} + b_{i-1} + c_{i-1} & \text{if } z_{i-1} < -1 - \varepsilon_i \\ a_{i-1} + c_{i-1} & \text{if } z_{i-1} = -1 - \varepsilon_i \\ a_{i-1} & \text{if } z_{i-1} > -1 - \varepsilon_i, \end{cases}$$

$$v = \begin{cases} -(b_{i-1} + c_{i-1})(z_{i-1} + 1 + \varepsilon_i) - b_{i-1} & \text{if } z_{i-1} < -1 - \varepsilon_i \\ 0 & \text{if } z_{i-1} = -1 - \varepsilon_i \\ (b_{i-1} + c_{i-1})(z_{i-1} + 1 + \varepsilon_i) - c_{i-1} & \text{if } z_{i-1} > -1 - \varepsilon_i, \end{cases}$$

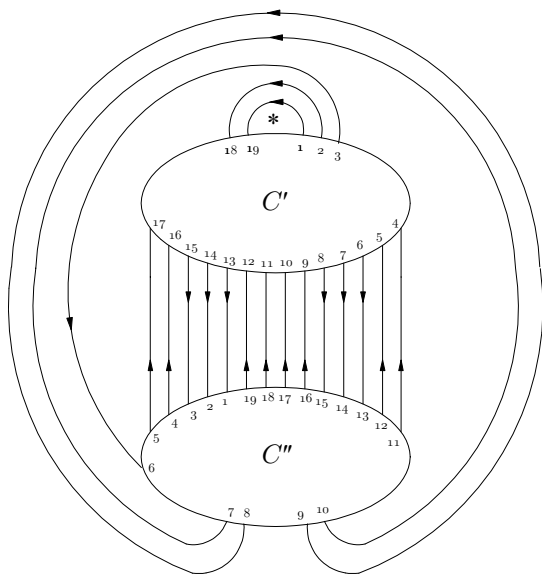
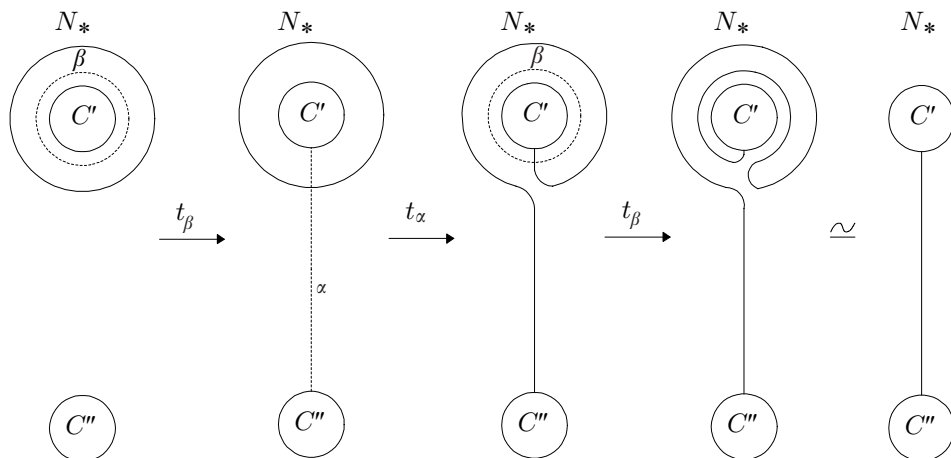
and $u = \lfloor (r_{i-1} - 2w)/d \rfloor$, with $d = 2a_{i-1} + b_{i-1} + c_{i-1}$.

The proof of Proposition 6 will be given at the end of this section. Now we give some examples and applications.

REMARK 7. Given an admissible Dunwoody diagram $D(a, b, c, 1, r, 0)$, with $a + b + c > 0$, we fix an orientation on the arcs of $\mathcal{E}$ that induces an orientation on the corresponding curve of the Heegaard diagram in such a way that the vertex on C' labelled 1 is the first endpoint of the corresponding edge. Let $p_{a,b,c,r}$ be the number of arcs of $\mathcal{B}$ oriented from C' to C'' minus the number of arcs oriented from C'' to C' , and let $q_{a,b,c,r}$ be the number of arcs of $\mathcal{E}$ oriented from right to left minus the number of arcs oriented from left to right (see [7, p. 385]). If $K(a, b, c, r)$ is a $(1, 1)$ -knot in $\mathbf{S}^3$, then the n -fold cyclic branched covering of $K(a, b, c, r)$ is the Dunwoody manifold $M(a, b, c, r, n, s)$, where $s = -p_{a,b,c,r}q_{a,b,c,r}$. In fact, by Proposition 2, there exists a unique $s \pmod{n}$ such that $M(a, b, c, n, r, s)$ is the n -fold cyclic covering of $M(a, b, c, 1, r, 0) \cong \mathbf{S}^3$, branched over $K(a, b, c, r)$. Moreover, by [7], s must satisfy the condition $q_{a,b,c,r} + sp_{a,b,c,r} \equiv 0 \pmod{n}$ and we have $p_{a,b,c,r} = \pm 1$.

EXAMPLE 8. Let us consider $\mathbf{t}(5, 8)$. By (1), a representation of $\mathbf{t}(5, 8)$ is given by $\psi = \tau_l^{-1} \tau_m^{-1} \tau_l^{-1} (\tau_l^{-1} (\tau_m^{-1} \tau_l^{-1})^2)^2 t_\beta t_\alpha t_\beta$. Then, by Proposition 6, we have $\mathbf{t}(5, 8) = K(2, 1, 14, 11)$. Moreover, from the diagram $D(2, 1, 14, 1, 11, 0)$ depicted in Figure 13, we get $p_{2,1,14,11} = -1$ and $q_{2,1,14,11} = 5$. Thus, by Remark 7, the n -fold cyclic branched covering of $\mathbf{t}(5, 8)$ is the Dunwoody manifold $M(2, 1, 14, n, 11, 5)$, for all $n > 1$.

As an application, we explicitly determine the parametrization of $\mathbf{t}(k, ck + 1)$ as well as the Dunwoody representation of its cyclic branched coverings.

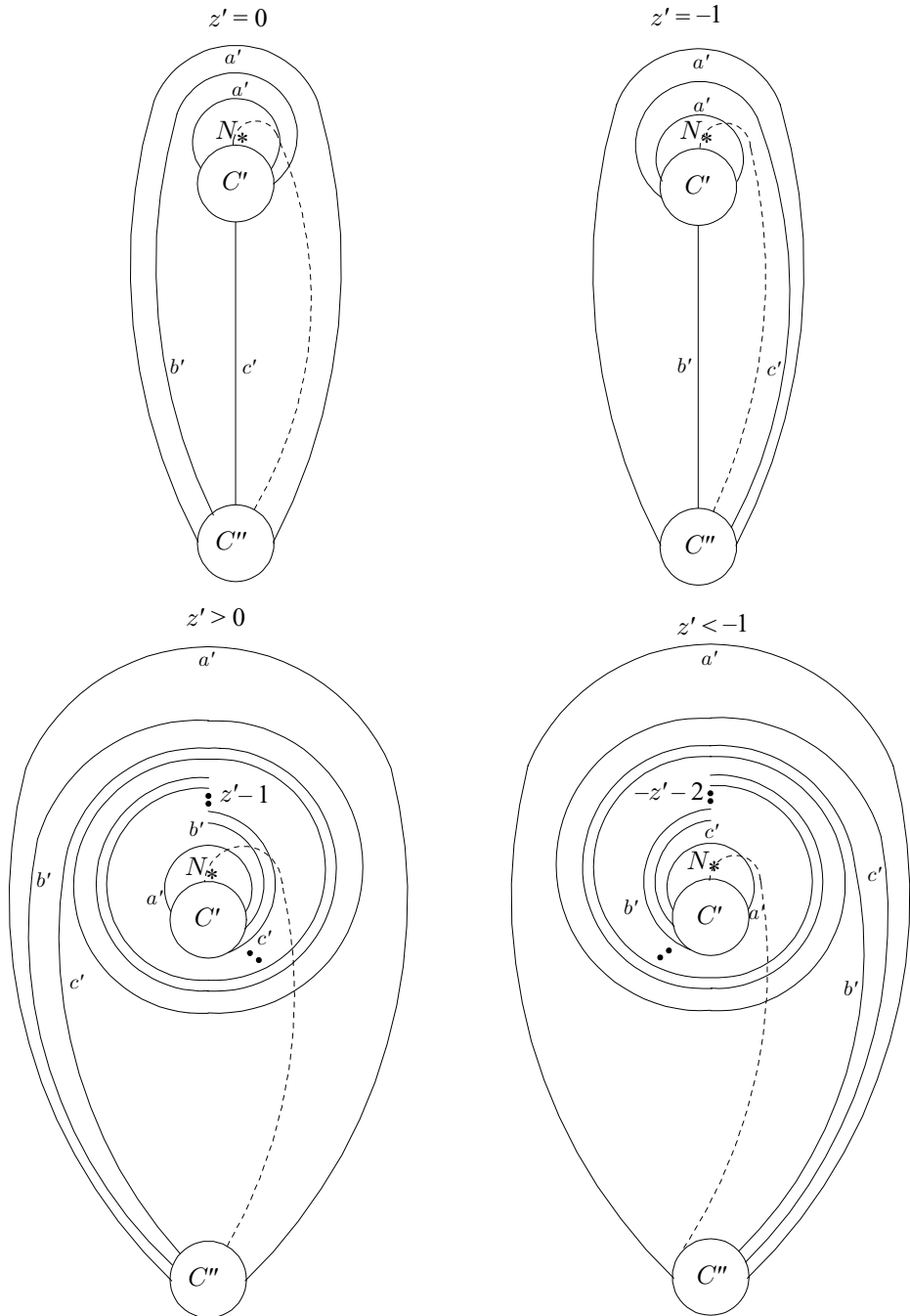

 FIGURE 13. $D(2, 1, 14, 1, 11, 0)$.

 FIGURE 14. Action of $t_\beta t_\alpha t_\beta$ on $D(0, 0, 0, 1, 0, 0)$.

COROLLARY 9. For every $c > 0$ and $k > 1$, the torus knot $\mathbf{t}(k, ck + 1)$ is $K(1, k - 2, 2kc - 2c - k + 1, k)$. Moreover, the n -fold cyclic branched covering of $\mathbf{t}(k, ck + 1)$ is the Dunwoody manifold $M(1, k - 2, 2kc - 2c - k + 1, n, k, k)$, for all $n > 1$.

Proof. By (1), $\mathbf{t}(k, ck + 1)$ is represented by $\psi = (\tau_l^{-c} \tau_m^{-1})^k \tau_l^{-1} t_\beta t_\alpha t_\beta$. Applying Proposition 6 and Remark 7 we get the statement. $\square$

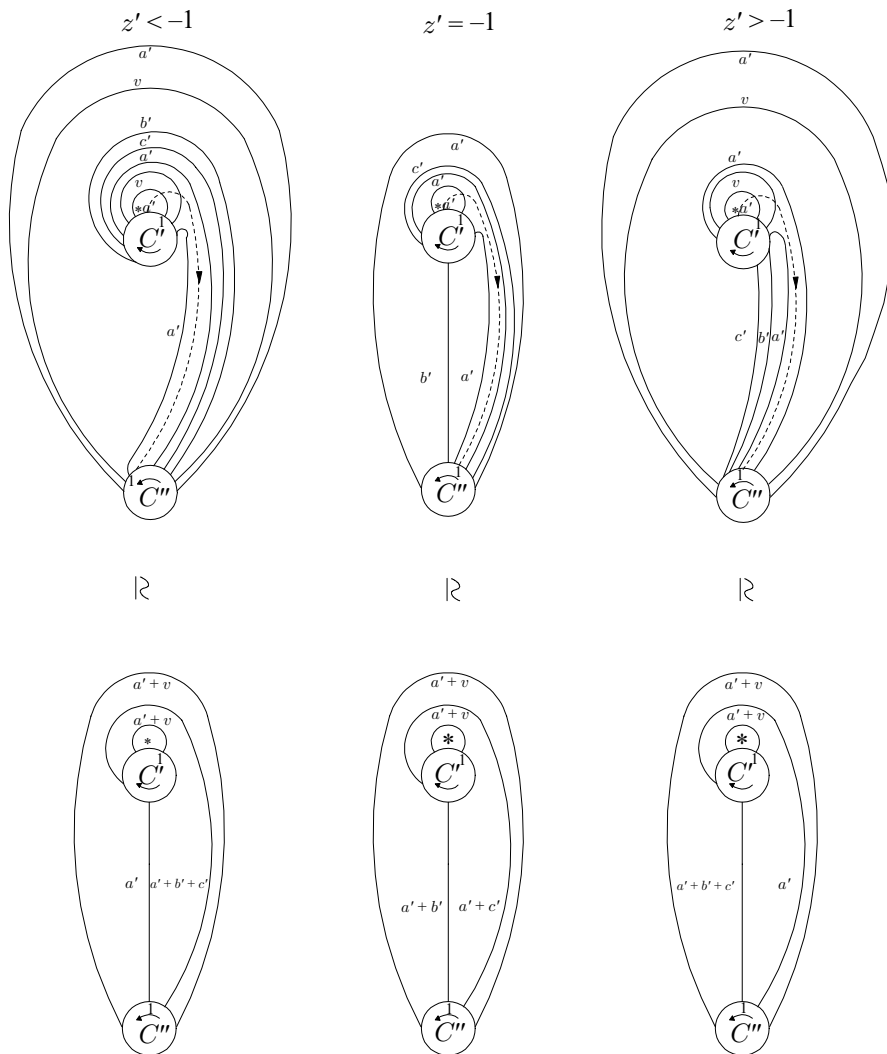
Observe that Corollary 9 agrees with the result obtained in [1] with different techniques.

Proof of Proposition 6. As shown in Figure 14, the application of $t_\beta t_\alpha t_\beta$ to $D(0, 0, 0, 1, 0, 0)$ gives the diagram $D(0, 0, 1, 1, 0, 0)$.

FIGURE 15. The parameter w .

In order to simplify the notations in the figures, we set $(a_{i-1}, b_{i-1}, c_{i-1}, r_{i-1}) = (a', b', c', r')$ and $z_{i-1} = z'$. To obtain the parameters a, b, c and r , we consider the application of $\tau_l^{-1} \tau_m^{\varepsilon_i}$ to $D_{z'}(a', b', c', 1, r', 0)$.

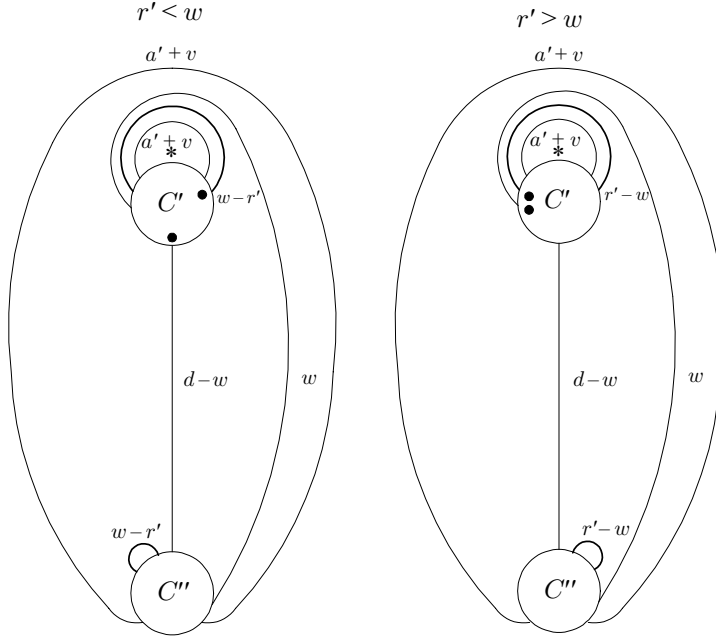
Let us first consider the case $\varepsilon_i = 0$. We recall that the effect of τ_l^{-1} is to slide N along the longitude of the torus, illustrated by the broken line in Figure 11, in the


 FIGURE 16. Action of τ_l^{-1} on $D_{z'}(a', b', c', 1, r', 0)$ for $r' = w$.

opposite direction to the arrow. This curve will always be represented on a Heegaard diagram by a dashed arc connecting an internal point of the arc on C' , with endpoints labelled d and 1 (according to the orientation), with the corresponding point on C'' . The number of intersections of the longitude with the arcs of a given diagram depends on r' . Let w be the value of r' such that the number of these intersections is minimal. Then, as illustrated in Figure 15, we have

$$w = \begin{cases} a' + b' + c' & \text{if } z' < -1 \\ a' + c' & \text{if } z' = -1 \\ a' & \text{if } z' > -1. \end{cases}$$

In this figure, and in the following ones, an arc labelled f denotes f parallel arcs, and we take the convention that a label of a vertex is the label corresponding to the endpoint of the first of the f parallel arcs.

FIGURE 17. Action of τ_l^{-1} on $D_{z'}(a', b', c', 1, r', 0)$ for $r' < w$ and $r' > w$.

First of all, we consider the case $r' = w$. In this case the longitude has $a' + v$ intersections, and the action of τ_l^{-1} is illustrated in Figure 16. We obtain $(a_i, b_i, c_i, r_i) = (a' + v, d - w, w, a' + v + w)$ and $z_i = -1$, which is the same result of the statement when $r' = w \neq 0$ (in this case $-d \leq r' - 2w = -w < 0$ and so $u = -1$). If $r' = w = 0$, we have $u = 0$, and therefore the statement gives $(a_i, b_i, c_i, r_i) = (a' + v, 0, d, a' + v)$ and $z_i = 0$; but it is easy to check that $D_0(a' + v, 0, d, 1, a' + v, 0) = D_{-1}(a' + v, d, 0, 1, a' + v, 0)$.

When $r' > w$ or $r' < w$, the result of the application of τ_l^{-1} is depicted in Figure 17. In both cases, the further $|r' - w|$ intersections determine $|r' - w|$ trivial arcs on C'' . The j th of these arcs has endpoints on C'' labelled $a' + v + d + j$ and $a' + v + d + 2(r' - w) - j + 1$ if $w < r'$, and labelled $a' + v + j$ and $a' + v + 2(w - r') - j + 1$ if $w > r'$. Each time we eliminate a trivial arc e , we glue together the two arcs whose endpoints on C' have the same label as the endpoints of e on C'' . In Figure 17, the black points indicate which arcs are glued together. After the elimination of all the trivial arcs, we obtain, as above, $a_i = a' + v$ and $r_i = a' + v + w$, while the value of the other three parameters depends on the quotient of the division of $|r' - 2w|$ by d . Suppose that $r' > w$; then we have two cases.

(1) If $r' - w < w$, we obtain $b_i = d - w + r' - w = d + r' - 2w$, $c_i = w - (r' - w) = 2w - r'$ and $z_i = -1$.

(2) If $r' - w \geq w$, after the elimination of the first w trivial arcs, we obtain the diagram depicted in Figure 18. During the elimination of the remaining $r' - 2w$ arcs, each time we eliminate d arcs the parameter z' increases by one. Therefore, if u is the integer defined by $u = \lfloor (r' - 2w)/d \rfloor$, we have $b_i = r' - 2w - ud$, $c_i = (u + 1)d - (r' - 2w)$ and $z_i = u$.

Analysing the case $r' < w$ in an analogous way, we complete the case $\varepsilon_i = 0$.

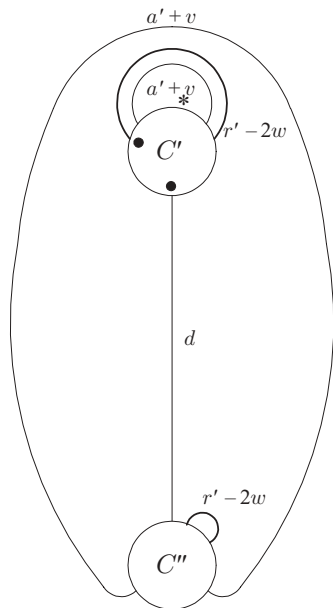


FIGURE 18. Action of τ_l^{-1} in the case $r' - w \geq w$.

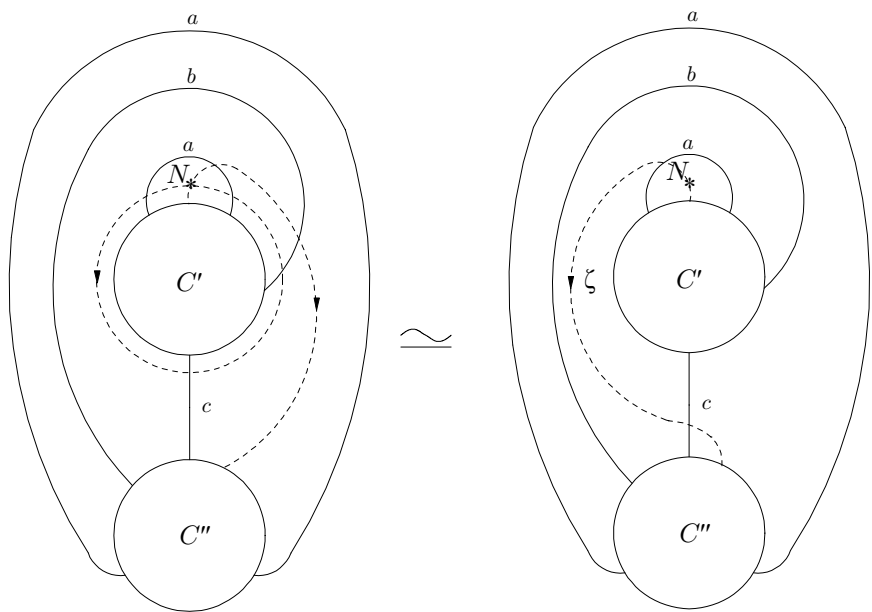


FIGURE 19. Action of $\tau_l^{-1} \tau_m^{-1}$.

In the case $\varepsilon_i = -1$ we examine the action of $\tau_l^{-1} \tau_m^{-1}$. This can be done in a similar way as before, since, as depicted in Figure 19, the action of $\tau_l^{-1} \tau_m^{-1}$ is equivalent to an action that moves N along the longitude ζ . $\square$

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Alessia Cattabriga
 Department of Mathematics
 University of Bologna
 Italy

cattabri@dm.unibo.it

Michele Mulazzani
 Department of Mathematics and CIRM
 University of Bologna
 Italy

mulazza@dm.unibo.it