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Isomorphisms and homeomorphisms of a class of graphs and spaces

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Summary. We solve the isomorphism problem for the whole class of Lins–Mandel gems (*g*raphs *e*ncoded *m*anifold*s*). We also present certain homeomorphisms of branched cyclic coverings of two-bridge hyperbolic links. As a consequence, we prove that, in a wide subset of interesting cases, the isomorphism conditions for Lins–Mandel gems are equivalent to the homeomorphism conditions for the encoded 3-manifolds.

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1. Introduction

Lins–Mandel spaces have been introduced in [19] as a direct combinatorial generalization of lens spaces, by a four parameter family of 4-coloured graphs. The encoded spaces $S(n,p,q,m)$ are closed orientable 3-manifolds, possibly with isolated singular points. This class of spaces has been extensively studied by several researchers (see [2], [3], [4], [5], [6], [7], [8], [10], [13], [16], [19], [22], [23]) and appears to be fairly rich, since it contains several interesting 3-manifolds (see [18]), such as the Poincaré homology sphere $S(5,3,2,1) \cong S(3,5,4,1)$ [17], the Seifert–Weber hyperbolic dodecahedron space $S(5,8,3,2) \cong S(5,8,3,3)$ [25], the euclidean Hantzsche–Wendt manifold $S(3,5,2,1)$ [29], the hyperbolic Fomenko–Matveev–Weeks manifold $S(3,7,4,1)$ [15], which is the hyperbolic 3-manifold with the smallest known volume, and also an infinite family of Brieskorn manifolds $M(n,p,2) \cong S(n,p,1,n-1)$ [21]. The necessary and sufficient conditions on the four parameters of a Lins–Mandel coloured graph to encode a 3-manifold (i.e. to be a 3-gem) have been obtained in [22]. Moreover, [23] shows that every Lins–

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Mandel manifold is a cyclic covering of $\mathbf{S}^3$, branched over a two-bridge knot or link.

In this paper we find the necessary and sufficient conditions for the isomorphism of Lins–Mandel gems. Actually, this problem has been studied in [8] for a particular subfamily of graphs, called crystallizations, but its results appear to be incorrect (see Remark 2).

We also present certain homeomorphisms of branched cyclic coverings of two-bridge hyperbolic links, obtained from results of Zimmermann [28] and Sakuma [24]. As a consequence, we prove that, in several interesting cases, the isomorphism conditions for Lins–Mandel gems are equivalent to the homeomorphism conditions for the encoded 3-manifolds.

2. Isomorphisms of Lins–Mandel gems

Regarding the theory of PL-manifolds represented by edge-coloured graphs, we refer to [12] and [18]. We recall that, in this theory, a $(d+1)$-coloured graph encodes a d-dimensional pseudo-manifold. When the encoded space is a manifold, the graph is called a *d-gem*. Notice that every manifold admits representation by gems.

The family of Lins–Mandel 4-coloured graphs

$$\mathcal{G} = \{G(n,p,q,m) \mid n,p \in \mathbb{Z}^+,\, q \in \mathbb{Z}_{2p},\, m \in \mathbb{Z}_n\}$$ [1]

has been defined in [19] by the rules: the set of vertices of $G = G(n,p,q,m)$ is

$$V(G) = \mathbb{Z}_n \times \mathbb{Z}_{2p}$$

and the coloured edges are obtained by the four fixed-point-free involutions $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$ on $V(G)$

$$\begin{aligned}
\varepsilon_0(i,j) &= (i + m\mu(j-q), 1-j+2q),\\
\varepsilon_1(i,j) &= (i, j-(-1)^j),\\
\varepsilon_2(i,j) &= (i, j+(-1)^j),\\
\varepsilon_3(i,j) &= (i+\mu(j), 1-j),
\end{aligned}$$

where $\mu : \mathbb{Z}_{2p} \to \{-1,+1\}$ is the function

$$\mu(j) = \begin{cases} +1, & \text{if } 1 \le j \le p,\\ -1, & \text{otherwise.}\end{cases}$$

For each $k \in \{0,1,2,3\}$, the vertices $v, w \in V(G)$ are joined by a k-edge if and only if $\varepsilon_k(v) = w$.

Roughly speaking, the graph $G(n,p,q,m)$ is constructed by taking n copies C_i of a bicoloured cycle of length $2p$ involving colours 1 and 2, for $i \in \mathbb{Z}_n$, so that

[1] By definition, the third and the fourth coordinate of $G(n,p,q,m)$ will be always considered mod $2p$ and mod n respectively.

$V(C_i) = \{(i,j) \mid j \in \mathbb{Z}_{2p}\}$. The cycle C_i is joined to the cycles $C_{i\pm1}$ by 3-edges, and to the cycles $C_{i\pm m}$ by 0-edges.

Each $G(n,p,q,m) \in \mathcal{G}$ represents a 3-dimensional (possibly singular) manifold $S(n,p,q,m)$ and

$$\mathcal{S} = \{S(n,p,q,m) \mid n,p \in \mathbb{Z}^+,\ q \in \mathbb{Z}_{2p},\ m \in \mathbb{Z}_n\}$$

will be called the family of Lins–Mandel spaces. Since every $G(n,p,q,m) \in \mathcal{G}$ is bipartite, every $S(n,p,q,m) \in \mathcal{S}$ is an orientable (singular) 3-manifold. The spaces $S(n,p,q,m)$ and $S(n,kp,kq,m)$ are homeomorphic [23], therefore we shall assume $\gcd(p,q) = 1$ in the following, without loss of generality.

In most cases $S(n,p,q,m)$ is a genuine manifold (i.e., without singular points).

Lemma 1 ([22])**.** *The graph $G(n,p,q,m)$ is a 3-gem and therefore $S(n,p,q,m)$ is a 3-manifold if and only if either* (i) *p is even or* (ii) *p is odd and $m = 0, (-1)^q$.*

Figures 1, 2 and 3 show three examples of graphs of the family, each representing an interesting manifold. Observe that in the figures the (missing) 0-edges connect vertices labelled with the same letter.

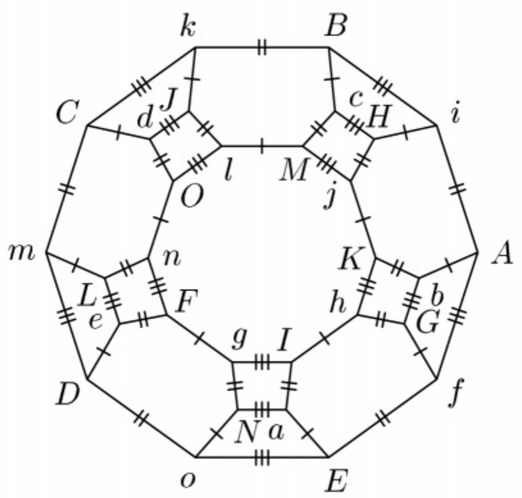

Figure 1. $G(5,3,2,1)$, representing the Poincaré homology sphere.

Remark 1. The family of Lins–Mandel spaces has been introduced as a combinatorial generalization of lens spaces. In fact, for each $p, q > 0$ such that $\gcd(p,q) = 1$,

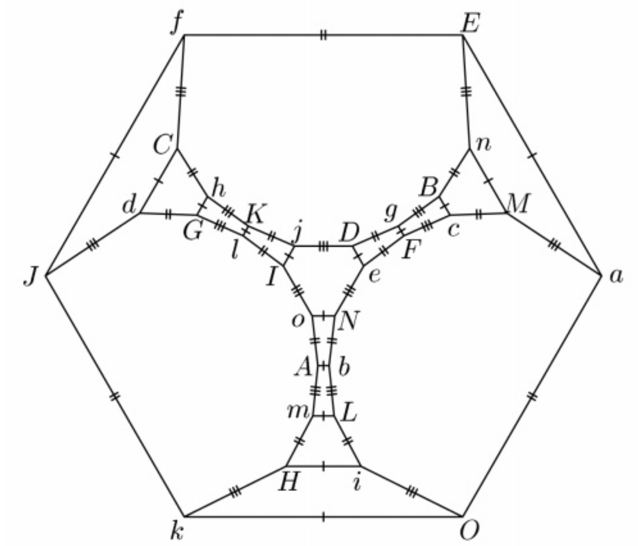

Figure 2. $G(3,5,2,1)$, representing the Hantzsche–Wendt manifold.

the spaces $S(2,p,q,1)$ and $S(p,2,1,q)$ are both homeomorphic to the lens space $L(p,q)$. Moreover, $S(n,p,q,0)$ and $S(n,1,1,-1)$ are homeomorphic to $\mathbf{S}^3$, for every $n,p>0$ and $q \in \mathbb{Z}_{2p}$.

Each graph $G(n,p,q,m)$ is defined by two different 4-tuples of parameters, as stated by

Lemma 2 ([19])**.** *The graphs $G(n,p,q,m)$ and $G(n,p,q+p,-m)$ are equal.*

Proof. Let $\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3$ (resp. $\varepsilon'_0,\varepsilon'_1,\varepsilon'_2,\varepsilon'_3$) be the involutions defining $G(n,p,q,m)$ (resp. $G(n,p,q+p,-m)$). Obviously, $\varepsilon'_1=\varepsilon_1$, $\varepsilon'_2=\varepsilon_2$ and $\varepsilon'_3=\varepsilon_3$. Moreover, we have $\varepsilon'_0(i,j)=(i-m\mu(j-q-p),1-j+2q-2p)=(i+m\mu(j-q),1-j+2q)=\varepsilon_0(i,j)$.[2] □

Now we list the residues[3] of $G(n,p,q,m)$ (see [22]). When p is even, the graph $G(n,p,q,m)$ contains

- n $\{1,2\}$-residues of length $2p$,
- n $\{0,3\}$-residues of length $2p$,
- 2 $\{2,3\}$-residues of length $2n$ and $n(p-2)/2$ $\{2,3\}$-residues of length 4,

[2] Observe that $\mu(j+p)=\mu(j-p)=\mu(1-j)=-\mu(j)$, for every $j\in\mathbb{Z}_{2p}$.
[3] If $k',k''\in\{0,1,2,3\}$, a $\{k',k''\}$-residue is a bicoloured cycle involving colours k' and k''.

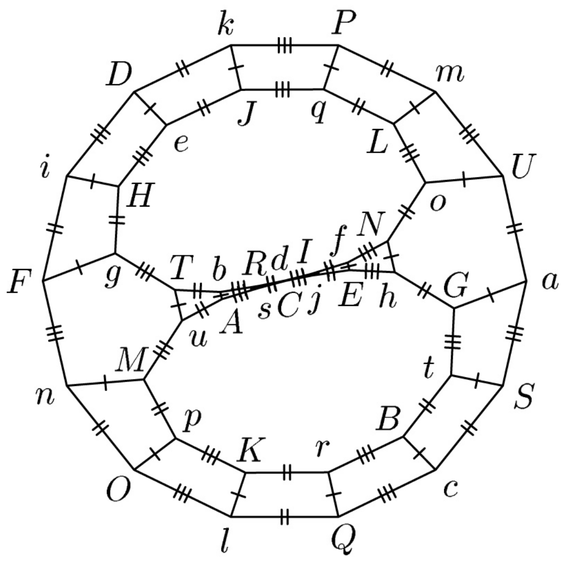

Figure 3. $G(3,7,4,1)$, representing the Fomenko–Matveev–Weeks manifold.

- $2\gcd(n,m)$ $\{0,1\}$-residues of length $2n/\gcd(n,m)$ and $n(p-2)/2$ $\{0,1\}$-residues of length 4,
- $np/2$ $\{1,3\}$-residues of length 4,
- $np/2$ $\{0,2\}$-residues of length 4.

On the other hand, when p is odd, the graph $G(n,p,q,m)$ contains
- n $\{1,2\}$-residues of length $2p$,
- $\gcd(n,m-(-1)^q)$ $\{0,3\}$-residues of length $2pn/\gcd(n,m-(-1)^q)$,
- 1 $\{2,3\}$-residue of length $2n$ and $n(p-1)/2$ $\{2,3\}$-residues of length 4,
- $\gcd(n,m)$ $\{0,1\}$-residues of length $2n/\gcd(n,m)$ and $n(p-1)/2$ $\{0,1\}$-residues of length 4,
- 1 $\{1,3\}$-residue of length $2n$ and $n(p-1)/2$ $\{1,3\}$-residues of length 4,
- $\gcd(n,m)$ $\{0,2\}$-residues of length $2n/\gcd(n,m)$ and $n(p-1)/2$ $\{0,2\}$-residues of length 4.

An isomorphism between two Lins–Mandel graphs G and G' is uniquely defined by a pair (f,ϕ), where $f: V(G) \to V(G')$ is a bijection and ϕ is a permutation of the colour set $X = \{0,1,2,3\}$, such that $f\varepsilon_k = \varepsilon'_{\phi(k)}f$, for each $k \in \{0,1,2,3\}$. Actually, since G and G' are connected, only ϕ and the image $f(v)$ of a chosen vertex v of G are required. Of course, isomorphic graphs encode homeomorphic spaces.

The next three lemmas give the main isomorphisms of Lins–Mandel graphs. The proofs of these lemmas are rather technical and have been included in the Appendix.

Lemma 3 ([19])**.** *$G(n,p,q,m) \cong G(n,p,-q,m)$.*

Lemma 4. a) *If p is even, then $G(n,p,q,m) \cong G(n,p,q^{-1},m)$.*
b') *If p and q are odd, then $G(n,p,q,-1) \cong G(n,p,q^{-1},-1)$.*
b") *If p is odd and q is even, then $G(n,p,q,1) \cong G(n,p,(q+p)^{-1}+p,1)$.*

Lemma 5 ([8])**.** *If $(n,m)=1$, then $G(n,p,q,m) \cong G(n,p,q,m^{-1})$.*

From Lemmas 2, 3, 4 and 5 we get:

Corollary 6. a') *If p is even,* $\gcd(n,m) \neq 1$ *and*

$$\textit{either} \quad \begin{cases} q' = \pm q^{\pm 1} \\ m' = m \end{cases} \quad \textit{or} \quad \begin{cases} q' = \pm q^{\pm 1} + p \\ m' = -m \end{cases},$$

then $G(n,p,q',m')$ is isomorphic to $G(n,p,q,m)$.
a") *If p is even,* $\gcd(n,m) = 1$ *and*

$$\textit{either} \quad \begin{cases} q' = \pm q^{\pm 1} \\ m' = m^{\pm 1} \end{cases} \quad \textit{or} \quad \begin{cases} q' = \pm q^{\pm 1} + p \\ m' = -m^{\pm 1} \end{cases},$$

then $G(n,p,q',m')$ is isomorphic to $G(n,p,q,m)$.
b) *If p is odd and $q' \equiv \pm q^{\pm 1} \bmod p$, then $G(n,p,q',(-1)^{q'})$ is isomorphic to $G(n,p,q,(-1)^{q})$.*

Proof. Statements a') and a") follow from Lemmas 2, 3, 4 and 5. As concerns part b), from the same lemmas we get:
b') if q is odd and $q' = \pm q^{\pm 1}, \pm q^{\pm 1} + p$, then $G(n,p,q',(-1)^{q'}) \cong G(n,p,q,(-1)^{q})$;
b") if q is even and $q' = \pm(q+p)^{\pm 1}, \pm(q+p)^{\pm 1} + p$, then $G(n,p,q',(-1)^{q'}) \cong G(n,p,q,(-1)^{q})$.
It is easy to check that b') + b") is equivalent to b). □

Since for either $n \leq 2$ or $p \leq 2$ or $m = 0$, the corresponding space is trivial ($\mathbf{S}^3$ or a lens space), we are mainly interested in the cases $n, p \geq 3$ and $m \neq 0$.

Lemma 7 ([8])**.** *Let $n, p \geq 3$. If $G(n',p',q',m')$ is isomorphic to $G(n,p,q,m)$, then $n' = n$ and $p' = p$.*

Proof. As explained above, the $\{1,2\}$-residues of the graph $G(n,p,q,m)$ are exactly n and they are all of length $2p$. When $n, p \geq 3$, the same property holds for the $\{1,2\}$-residues (and possibly for the $\{0,3\}$-residues) of $G(n',p',q',m')$ if and only if $n' = n$ and $p' = p$, while it does not hold for the other types of residues. This proves the statement. □

The previous result does not hold when either $n \leq 2$ or $p \leq 2$. For example, $G(2,p,q,1)$ and $G(p,2,1,q)$ are isomorphic.

Corollary 6 can be reversed when $n, p \geq 3$; the next theorem completely describes the isomorphisms of these graphs.

Theorem 8. *Assume $n, p \geq 3$.*

a') *If p is even and* $\gcd(n,m) \neq 1$*, then $G(n',p',q',m') \cong G(n,p,q,m)$ if and only if*

$$n' = n, \quad p' = p$$

and

$$\textit{either} \quad \begin{cases} q' = \pm q^{\pm 1} \\ m' = m \end{cases} \quad \textit{or} \quad \begin{cases} q' = \pm q^{\pm 1} + p \\ m' = -m \end{cases}.$$

a") *If p is even and* $\gcd(n,m) = 1$*, then $G(n',p',q',m') \cong G(n,p,q,m)$ if and only if*

$$n' = n, \quad p' = p$$

and

$$\textit{either} \quad \begin{cases} q' = \pm q^{\pm 1} \\ m' = m^{\pm 1} \end{cases} \quad \textit{or} \quad \begin{cases} q' = \pm q^{\pm 1} + p \\ m' = -m^{\pm 1} \end{cases}.$$

b) *If p is odd, then $G(n',p',q',(-1)^{q'}) \cong G(n,p,q,(-1)^q)$ if and only if*

$$n' = n, \quad p' = p \quad \textit{and} \quad q' \equiv \pm q^{\pm 1} \textit{ mod } p.$$

Proof. The "if" part follows from Corollary 6. The "only if" part will be proved in the Appendix. □

Observe that, for the second and the third parameter, the isomorphism conditions of part b) of Theorem 8 are the same as the homeomorphism conditions for lens spaces. This is not true for part a), since, in this case, the situation is complicated by the presence of the additional parameter m.

Remark 2. Proposition 4.1 of [8] states that, when $\gcd(n,m) = 1$, the graphs $G(n,p,q,m)$ and $G(n,p,q',m')$ are isomorphic if and only if $q' \equiv \pm q^{\pm 1} \bmod p$ and $m' = \pm m^{\pm 1}$. This result is incorrect when p is even, since, for example, $H_1\big(S(3,4,1,1)\big) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6$ and $H_1\big(S(3,4,5,1)\big) = H_1\big(S(3,4,1,2)\big) \cong \mathbb{Z}_3$ (see Appendix of [19]).

Cases where p is even are particularly interesting because the graph always represents a manifold without any restriction on m. From Theorem 8 we get:

Corollary 9. *Let n, p, q be fixed, with $n, p \geq 3$ and p even. Then $G(n,p,q,m') \cong G(n,p,q,m)$ if and only if*

(i) *$m' = m$, when* $\gcd(n,m) \neq 1$ *and $q^2 \neq p \pm 1$;*
(ii) *$m' = \pm m$, when* $\gcd(n,m) \neq 1$ *and $q^2 = p \pm 1$;*
(iii) *$m' = m^{\pm 1}$, when* $\gcd(n,m) = 1$ *and $q^2 \neq p \pm 1$;*
(iv) *$m' = \pm m^{\pm 1}$, when* $\gcd(n,m) = 1$ *and $q^2 = p \pm 1$.*

Proof. Since $p \geq 2$, $\gcd(p,q) = 1$ and q is odd, the condition $q = \pm q + p$ cannot be satisfied and the condition $q = \pm q^{-1} + p$ is equivalent to $q^2 = \pm 1 + p$. This proves the statement. □

3. Connections with branched cyclic coverings of two-bridge knots and links

A b-fold branched cyclic covering of an oriented μ-component link $L = \bigcup_{i=1}^{\mu} L_i \subset \mathbf{S}^3$ is completely determined (up to equivalence) by assigning to each component L_i an integer $k_i \in \mathbb{Z}_b - \{0\}$, such that the set $\{k_1, \ldots, k_\mu\}$ generates the group $\mathbb{Z}_b$. The monodromy associated to the covering sends each meridian of L_i to the permutation $(1\,2\,\cdots\,b)^{k_i} \in \Sigma_b$. By multiplying each k_i by the same invertible element k of $\mathbb{Z}_b$, we get an equivalent covering.

Following [20] we will call a branched cyclic covering

a) *strictly-cyclic*, if $k_i = k_j$, for every $i, j \in \{1, \ldots, \mu\}$,

b) *almost-strictly-cyclic*, if $k_i = \pm k_j$, for every $i, j \in \{1, \ldots, \mu\}$,

c) *meridian-cyclic*, if $\gcd(b, k_i) = 1$, for every $i \in \{1, \ldots, \mu\}$,

d) *singly-cyclic*, if $\gcd(b, k_i) = 1$, for some $i \in \{1, \ldots, \mu\}$,

e) *monodromy-cyclic*, if it is cyclic.

The following implications are straightforward:

$$\text{a)} \;\Rightarrow\; \text{b)} \;\Rightarrow\; \text{c)} \;\Rightarrow\; \text{d)} \;\Rightarrow\; \text{e)}\ .$$

Moreover, the five definitions are equivalent when L is a knot.

If L has two components and the branched covering is singly-cyclic, we can always suppose that $k_1 = 1$, up to equivalence and renumbering of the components of L. Therefore, the covering is completely determined by an integer $k = k_2 \in \mathbb{Z}_b - \{0\}$.

Branched cyclic coverings of two-bridge knots and links are of great interest, since a double branched covering of a two-bridge knot or link is homeomorphic to a lens space (for notations and properties of two-bridge knots and links we refer to [1] and [26]). Let us denote by $\mathbf{b}(\alpha, \beta)$ the two-bridge knot or link of type (α, β). It is well known that $\mathbf{b}(\alpha, \beta)$ is a knot when α is odd and a two-component link when α is even. Moreover, $\mathbf{b}(\alpha, \beta)$ is hyperbolic if and only if it is not toroidal (that is, $\beta \not\equiv \pm 1 \bmod \alpha$). We denote by $M_{b,k}(\alpha, \beta)$ the b-fold singly-cyclic branched covering of the link $\mathbf{b}(\alpha, \beta)$ defined by k. Observe that the branched covering $M_{b,k}(\alpha, \beta)$ is strictly-cyclic if $k = 1$, almost-strictly-cyclic if $k = \pm 1$ and meridian-cyclic if $\gcd(b, k) = 1$.

From results of Zimmermann [28] and Sakuma [24] it is possible to obtain the homeomorphism conditions for these manifolds, when the covering is meridian-cyclic and the branching set is a hyperbolic link.

Theorem 10. *Let b, α, β be fixed, with α even and $\beta \not\equiv \pm 1$ mod α. For* $\gcd(b,k) = 1$*, the manifolds $M_{b,k'}(\alpha,\beta)$ and $M_{b,k}(\alpha,\beta)$ are homeomorphic if and only if*

(i) $k' = k^{\pm 1}$, *when* $\beta^2 \neq \alpha \pm 1$;

(ii) $k' = \pm k^{\pm 1}$, *when* $\beta^2 = \alpha \pm 1$.

Proof. Apply Theorem 1 of [28] (including the note (a) of page 293) and Theorem 4.1 of [24] (see tables of page 184). □

Notice that, for the particular case of the Whitehead link $\mathbf{b}(8,3)$, the previous result is contained in [9] and [28].

The graph $G(n,p,q,m)$ has a rotational cyclic symmetry of order n, which sends each cycle C_i onto C_{i+1} (see details in Lemma 18). As a direct consequence, the space $S(n,p,q,m)$ also admits a cyclic symmetry. The next lemma states this important property.

Lemma 11 ([23])**.** a) *If p is even and $m \neq 0$, then $S(n,p,q,m)$ is homeomorphic to the singly-cyclic branched covering $M_{n,-m}(p,q)$ of the two-bridge link $\mathbf{b}(p,q)$.*

b) *If p is odd, then $S(n,p,q,(-1)^q)$ is the n-fold branched cyclic covering of the two-bridge knot $\mathbf{b}(p,q)$.*

As a consequence of the previous result, the geometric structure of $S(n,p,q,m)$, when the branching set $\mathbf{b}(p,q)$ is a hyperbolic knot or link, can be obtained from Thurston [27] and Dunbar [11]. Moreover, when the branching set is toroidal and $m = (-1)^q$, then $S(n,p,q,m)$ turns out to be the Brieskorn manifold $M(n,p,2)$ (see [3] and [10]). Thus, we have the following result for the geometric structure of the manifold $S(n,p,q,m)$.

Proposition 12. *Let $S(n,p,q,m)$ be a manifold.*

1) *If* $\gcd(n,m) = 1$ *and $q \not\equiv \pm 1$ mod p, then $S(n,p,q,m)$ is hyperbolic for* (i) *$p = 5$, $n \geq 4$ and* (ii) *$p \neq 5$, $n \geq 3$. Moreover, $S(3,5,3,-1) \cong S(3,5,2,1)$ is euclidean and $S(2,p,q,1)$ is spherical for all p, q.*

2) *If $q \equiv \pm 1 \bmod p$, then $S(n,p,q,(-1)^q)$ is spherical for $n^{-1} + p^{-1} > 1/2$, a Nil-manifold for $n^{-1} + p^{-1} = 1/2$ and a $\widetilde{SL}(2,\mathbb{R})$-manifold for $n^{-1} + p^{-1} < 1/2$.*

Proof. 1) See Theorem 3.1 and Remark 3.3 of [14]. 2) See [21]. □

4. Homeomorphisms of Lins–Mandel manifolds

Since isomorphic graphs encode homeomorphic spaces, from Corollary 6 we get the following homeomorphisms of Lins–Mandel manifolds

Proposition 13. a') *If p is even,* $\gcd(n,m) \neq 1$ *and*

$$\text{either} \quad \begin{cases} q' = \pm q^{\pm 1} \\ m' = m \end{cases} \quad \text{or} \quad \begin{cases} q' = \pm q^{\pm 1} + p \\ m' = -m \end{cases},$$

then $S(n,p,q',m')$ is homeomorphic to $S(n,p,q,m)$.

a") *If p is even,* $\gcd(n,m) = 1$ *and*

$$\text{either} \quad \begin{cases} q' = \pm q^{\pm 1} \\ m' = m^{\pm 1} \end{cases} \quad \text{or} \quad \begin{cases} q' = \pm q^{\pm 1} + p \\ m' = -m^{\pm 1} \end{cases},$$

then $S(n,p,q',m')$ is homeomorphic to $S(n,p,q,m)$.

b) *If p is odd and $q' \equiv \pm q^{\pm 1}$ mod p, then $S(n,p,q',(-1)^{q'})$ is homeomorphic to $S(n,p,q,(-1)^q)$.*

This proposition cannot be reversed, even when $n,p \geq 3$. For example, [18] shows that $S(5,3,2,1)$ and $S(3,5,4,1)$ are both homeomorphic to the Poincaré homology sphere.

Theorem 10 and Lemma 11 give us the possibility of partially translating the combinatorial results of Corollary 9 regarding graphs to the topological results for manifolds. In fact, we have

Theorem 14. *Let n,p,q be fixed, with $n,p \geq 3$, p even and $q \not\equiv \pm 1$ mod p. For* $\gcd(n,m) = 1$*, the manifolds $S(n,p,q,m')$ and $S(n,p,q,m)$ are homeomorphic if and only if*

(i) $m' = m^{\pm 1}$, *when $q^2 \neq p \pm 1$;*

(ii) $m' = \pm m^{\pm 1}$, *when $q^2 = p \pm 1$.*

Observe that Theorem 14 is only a partial analogue of Corollary 9. So it is natural to state the

Conjecture. *Let n,p,q be fixed, with $n,p \geq 3$, p even and $q \not\equiv \pm 1$ mod p. For $(n,m) \neq 1$, the manifolds $S(n,p,q,m')$ and $S(n,p,q,m)$ are homeomorphic if and only if*

(i) $m' = m$, *when $q^2 \neq p \pm 1$;*

(ii) $m' = \pm m$, *when $q^2 = p \pm 1$.*

5. Appendix

In this section we give the proofs of Lemmas 3, 4 and 5 and complete the proof of Theorem 8.

Proof of Lemma 3 ([19]). Let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$ (resp. $\varepsilon'_0, \varepsilon'_1, \varepsilon'_2, \varepsilon'_3$) be the involutions defining $G = G(n,p,q,m)$ (resp. $G' = G(n,p,-q,m)$). Moreover, let $\phi_1 \in \Sigma_X$ and

$f_1 : \mathbb{Z}_n \times \mathbb{Z}_{2p} \to \mathbb{Z}_n \times \mathbb{Z}_{2p}$ be the maps

$$\phi_1 = 1, \qquad f_1(i,j) = (-i, 1-j).$$

Since $f_1^2 = 1$, the map f_1 is a bijection. The pair (f_1, ϕ_1) is an isomorphism between the graphs $G(n,p,q,m)$ and $G(n,p,-q,m)$. In fact, we get:

- $f_1\varepsilon_0(i,j) = (-i - m\mu(j-q), j-2q)$;
- $f_1\varepsilon_1(i,j) = (-i, 1-j+(-1)^j)$;
- $f_1\varepsilon_2(i,j) = (-i, 1-j-(-1)^j)$;
- $f_1\varepsilon_3(i,j) = (-i-\mu(j), j)$;
- $\varepsilon'_0 f_1(i,j) = (-i + m\mu(1-j+q), j-2q) = (-i - m\mu(j-q), j-2q)$;
- $\varepsilon'_1 f_1(i,j) = (-i, 1-j-(-1)^{1-j}) = (-i, 1-j+(-1)^j)$;
- $\varepsilon'_2 f_1(i,j) = (-i, 1-j+(-1)^{1-j}) = (-i, 1-j-(-1)^j)$;
- $\varepsilon'_3 f_1(i,j) = (-i+\mu(1-j), j) = (-i-\mu(j), j)$. □

In order to prove Lemma 4 we need some technical results.

Lemma 15 ([22])**.** *Let ε_0, ε_1, ε_2, ε_3 be the involutions defining the graph $G(n,p,q,m)$.*

a) *If p is even, then:* (i) $\varepsilon_3\varepsilon_1 = \varepsilon_1\varepsilon_3$, (ii) $\varepsilon_0\varepsilon_2 = \varepsilon_2\varepsilon_0$, (iii) $\varepsilon_3\varepsilon_2(i,j) = \varepsilon_2\varepsilon_3(i,j)$ *for every i and for every $j \neq 0, 1, p, p+1$,* (iv) $\varepsilon_0\varepsilon_1(i,j) = \varepsilon_1\varepsilon_0(i,j)$ *for every i and for every $j \neq q, q+1, q+p, q+p+1$.*

b') *If p and q are odd, then:* (i) $\varepsilon_3\varepsilon_1(i,j) = \varepsilon_1\varepsilon_3(i,j)$ *for every i and for every $j \neq p, p+1$,* (ii) $\varepsilon_0\varepsilon_2(i,j) = \varepsilon_2\varepsilon_0(i,j)$ *for every i and for every $j \neq q+p, q+p+1$,* (iii) $\varepsilon_3\varepsilon_2(i,j) = \varepsilon_2\varepsilon_3(i,j)$ *for every i and for every $j \neq 0, 1$,* (iv) $\varepsilon_0\varepsilon_1(i,j) = \varepsilon_1\varepsilon_0(i,j)$ *for every i and for every $j \neq q, q+1$.*

b") *If p is odd and q is even, then:* (i) $\varepsilon_3\varepsilon_1(i,j) = \varepsilon_1\varepsilon_3(i,j)$ *for every i and for every $j \neq p, p+1$,* (ii) $\varepsilon_0\varepsilon_2(i,j) = \varepsilon_2\varepsilon_0(i,j)$ *for every i and for every $j \neq q, q+1$,* (iii) $\varepsilon_3\varepsilon_2(i,j) = \varepsilon_2\varepsilon_3(i,j)$ *for every i and for every $j \neq 0, 1$, (iv) $\varepsilon_0\varepsilon_1(i,j) = \varepsilon_1\varepsilon_0(i,j)$ for every i and for every $j \neq q+p, q+p+1$.*

Lemma 16 ([22])**.** *If $G(n,p,q,m)$ has n $\{0,3\}$-residues, then the $2p$ vertices of each $\{0,3\}$-residue have distinct second coordinates.*

Lemma 17. *Let (f,ϕ) be an isomorphism between $G = G(n,p,q,m)$ and $G' = G(n',p',q',m')$ and let $\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3$ (resp. $\varepsilon'_0,\varepsilon'_1,\varepsilon'_2,\varepsilon'_3$) be the involutions defining G (resp. defining G'). If $\varepsilon'_{\phi(k)}f(i,j) = f\varepsilon_k(i,j)$ and $(i',j') = \varepsilon_k(i,j)$, then $\varepsilon'_{\phi(k)}f(i',j') = f\varepsilon_k(i',j')$.*

Proof. We have $f\varepsilon_k(i',j') = f\varepsilon_k\varepsilon_k(i,j) = f(i,j) = \varepsilon'_{\phi(k)}\varepsilon'_{\phi(k)}f(i,j) = \varepsilon'_{\phi(k)}f\varepsilon_k(i,j) = \varepsilon'_{\phi(k)}f(i',j')$. □

Define $\pi'' : \mathbb{Z}_n \times \mathbb{Z}_{2p} \to \mathbb{Z}_{2p}$ as the projection $\pi''(i,j) = j$.

Proof of Lemma 4. a) Let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$ (resp. $\varepsilon'_0, \varepsilon'_1, \varepsilon'_2, \varepsilon'_3$) be the involutions defining $G = G(n, p, q, m)$ (resp. $G' = G(n, p, q^{-1}, m)$). Moreover, let $\phi_2 \in \Sigma_X$ and $f_2 : \mathbb{Z}_n \times \mathbb{Z}_{2p} \to \mathbb{Z}_n \times \mathbb{Z}_{2p}$ be the maps

$$\phi_2 = (0\,1)(2\,3), \qquad f_2(i,j) = \begin{cases} (\varepsilon'_0\varepsilon'_3)^{j/2}(-i,0), & \text{if } j \text{ is even} \\ \varepsilon'_3(\varepsilon'_0\varepsilon'_3)^{(j-1)/2}(-i,0), & \text{if } j \text{ is odd} \end{cases}.$$

Since f_2 sends the $\{1,2\}$-residues of G injectively onto the $\{0,3\}$-residues of G', then it is a bijection. We claim that (f_2, ϕ_2) is an isomorphism between $G(n,p,q,m)$ and $G(n,p,q^{-1},m)$. From the definition of f_2, we immediately get $\varepsilon'_3 f_2 = f_2\varepsilon_2$ and $\varepsilon'_0 f_2 = f_2\varepsilon_1$. A direct computation shows that $f_2(i,j) = (-i + h'_j, jq^{-1})$ if j is even and $f_2(i,j) = (-i + h''_j, 1-(j-1)q^{-1})$ if j is odd, where h'_j and h''_j only depend on j. Since $\pi'' f_2(i,0) = 0$, $\pi'' f_2(i,1) = 1$, $\pi'' f_2(i,p) = p$, $\pi'' f_2(i,p+1) = p+1$, $\pi'' f_2(i,q) = q^{-1}$, $\pi'' f_2(i,q+1) = q^{-1}+1$, $\pi'' f_2(i,q+p) = q^{-1}+p$ and $\pi'' f_2(i,q+p+1) = q^{-1}+p+1$, for every $i \in \mathbb{Z}_n$, from Lemma 16 we get: $\pi'' f_2(i,j) \in \{0,1,p,p+1\}$ if and only if $j \in \{0,1,p,p+1\}$ and $\pi'' f_2(i,j) \in \{q^{-1}, q^{-1}+1, q^{-1}+p, q^{-1}+p+1\}$ if and only if $j \in \{q, q+1, q+p, q+p+1\}$. Now we show, by induction, that (i') $\varepsilon'_2 f_2(i,j) = f_2\varepsilon_3(i,j)$, for every $(i,j) \in \mathbb{Z}_n \times \mathbb{Z}_{2p}$. By Lemma 17 we only need to prove (i') for $(i,1), (i,2), ..., (i,p)$. First of all, we have $f_2\varepsilon_3(i,1) = f_2(i+1,0) = (-i-1,0)$ and $\varepsilon'_2 f_2(i,1) = \varepsilon'_2(-i-1,1) = (-i-1,0)$. Therefore, (i') holds for $(i,1)$, for all i. Let us suppose that (i') holds for $(i,1), (i,2), \ldots, (i,k)$, with $1 \le k \le p-1$. From Lemma 15 we get, if k is odd, $f_2\varepsilon_3(i,k+1) = f_2\varepsilon_3\varepsilon_1(i,k) = f_2\varepsilon_1\varepsilon_3(i,k) = \varepsilon'_0 f_2\varepsilon_3(i,k) = \varepsilon'_0\varepsilon'_2 f_2(i,k) = \varepsilon'_2\varepsilon'_0 f_2(i,k) = \varepsilon'_2 f_2\varepsilon_1(i,k) = \varepsilon'_2 f_2(i,k+1)$ and, if k is even, $f_2\varepsilon_3(i,k+1) = f_2\varepsilon_3\varepsilon_2(i,k) = f_2\varepsilon_2\varepsilon_3(i,k) = \varepsilon'_3 f_2\varepsilon_3(i,k) = \varepsilon'_3\varepsilon'_2 f_2(i,k) = \varepsilon'_2\varepsilon'_3 f_2(i,k) = \varepsilon'_2 f_2\varepsilon_2(i,k) = \varepsilon'_2 f_2(i,k+1)$. Therefore, (i') holds for $(i,k+1)$, for all i. Then we show, by induction, that (i") $\varepsilon'_1 f_2(i,j) = f_2\varepsilon_0(i,j)$, for every $(i,j) \in \mathbb{Z}_n \times \mathbb{Z}_{2p}$. By Lemma 17 we only need to prove (i") for $(i,q+1), (i,q+2), \ldots, (i,q+p)$. First of all, we have $f_2\varepsilon_0(i,q+1) = f_2(i+m,q) = (-i-m+h''_q, 1-(q-1)q^{-1}) = (-i-m+h''_q, q^{-1})$ and $\varepsilon'_1 f_2(i,q+1) = \varepsilon'_1 f_2\varepsilon_1(i,q) = \varepsilon'_1\varepsilon'_0 f_2(i,q) = \varepsilon'_1\varepsilon'_0(-i+h''_q, q^{-1}) = \varepsilon'_1(-i-m+h''_q, 1+q^{-1}) = (-i-m+h''_q, q^{-1})$. Therefore, (i") holds for $(i,q+1)$, for all i. Let us suppose that (i") holds for $(i,q+1), (i,q+2), \ldots, (i,k)$ with $q+1 \le k \le q+p-1$. From Lemma 15 we get, if k is odd, $f_2\varepsilon_0(i,k+1) = f_2\varepsilon_0\varepsilon_1(i,k) = f_2\varepsilon_1\varepsilon_0(i,k) = \varepsilon'_0 f_2\varepsilon_0(i,k) = \varepsilon'_0\varepsilon'_1 f_2(i,k) = \varepsilon'_1\varepsilon'_0 f_2(i,k) = \varepsilon'_1 f_2\varepsilon_1(i,k) = \varepsilon'_1 f_2(i,k+1)$ and, if k is even, $f_2\varepsilon_0(i,k+1) = f_2\varepsilon_0\varepsilon_2(i,k) = f_2\varepsilon_2\varepsilon_0(i,k) = \varepsilon'_3 f_2\varepsilon_0(i,k) = \varepsilon'_3\varepsilon'_1 f_2(i,k) = \varepsilon'_1\varepsilon'_3 f_2(i,k) = \varepsilon'_1 f_2\varepsilon_2(i,k) = \varepsilon'_1 f_2(i,k+1)$. Therefore (i') holds for $(i,k+1)$, for all i.

b') Let ε_0, ε_1, ε_2, ε_3 (resp. ε'_0, ε'_1, ε'_2, ε'_3) be the involutions defining $G = G(n,p,q,-1)$ (resp. $G' = G(n,p,q^{-1},-1)$). Moreover, let $\phi_2 \in \Sigma_X$ and $f_2 : \mathbb{Z}_n \times \mathbb{Z}_{2p} \to \mathbb{Z}_n \times \mathbb{Z}_{2p}$ be the maps

$$\phi_2 = (0\,2)(1\,3), \qquad f_2(i,j) = \begin{cases} (\varepsilon'_3\varepsilon'_0)^{j/2}(-i, q^{-1}+1), & \text{if } j \text{ is even} \\ \varepsilon'_0(\varepsilon'_3\varepsilon'_0)^{(j-1)/2}(-i, q^{-1}+1), & \text{if } j \text{ is odd} \end{cases}.$$

Since f_2 sends the $\{1,2\}$-residues of G injectively onto the $\{0,3\}$-residues of

G', it is a bijection. We claim that (f_2, ϕ_2) is an isomorphism between $G(n,p,q,-1)$ and $G(n,p,q^{-1},-1)$. From the definition of f_2, we immediately get $\varepsilon'_3 f_2 = f_2\varepsilon_1$ and $\varepsilon'_0 f_2 = f_2\varepsilon_2$. A direct computation shows that $f_2(i,j) = (-i+l'_j, 1-(j-1)q^{-1})$, if j is even and $f_2(i,j) = (-i+l''_j, jq^{-1})$, if j is odd, where l'_j and l''_j only depend on j. Since $\pi'' f_2(i,0) = q^{-1}+1$, $\pi'' f_2(i,1) = q^{-1}$, $\pi'' f_2(i,p) = p$, $\pi'' f_2(i,p+1) = p+1$, $\pi'' f_2(i,q) = 1$, $\pi'' f_2(i,q+1) = 0$, $\pi'' f_2(i,q+p) = q^{-1}+p$ and $\pi'' f_2(i,q+p+1) = q^{-1}+p+1$, for every $i \in \mathbb{Z}_n$, from Lemma 16 we get: $\pi'' f_2(i,j) \in \{0,1\}$ if and only if $j \in \{q,q+1\}$, $\pi'' f_2(i,j) \in \{p,p+1\}$ if and only if $j \in \{p,p+1\}$, $\pi'' f_2(i,j) \in \{q^{-1},q^{-1}+1\}$ if and only if $j \in \{0,1\}$ and $\pi'' f_2(i,j) \in \{q^{-1}+p, q^{-1}+p+1\}$ if and only if $j \in \{q+p,q+p+1\}$. Now we show, by induction, that (i') $\varepsilon'_1 f_2(i,j) = f_2\varepsilon_3(i,j)$, for every $(i,j) \in \mathbb{Z}_n \times \mathbb{Z}_{2p}$. By Lemma 17 we only need to prove (i') for $(i,1),(i,2)\ldots,(i,p)$. First of all, we have $f_2\varepsilon_3(i,1) = f_2(i+1,0) = (-i-1, q^{-1}+1)$ and $\varepsilon'_1 f_2(i,1) = \varepsilon'_1(-i-1,q^{-1}) = (-i-1,q^{-1}+1)$. Therefore, (i') holds for $(i,1)$, for all i. Let us suppose that (i') holds in $(i,1),(i,2),...,(i,k)$ with $1 \le k \le p-1$. From Lemma 15 we get, if k is odd, $f_2\varepsilon_3(i,k+1) = f_2\varepsilon_3\varepsilon_1(i,k) = f_2\varepsilon_1\varepsilon_3(i,k) = \varepsilon'_3 f_2\varepsilon_3(i,k) = \varepsilon'_3\varepsilon'_1 f_2(i,k) = \varepsilon'_1\varepsilon'_3 f_2(i,k) = \varepsilon'_1 f_2\varepsilon_1(i,k) = \varepsilon'_1 f_2(i,k+1)$ and, if k is even, $f_2\varepsilon_3(i,k+1) = f_2\varepsilon_3\varepsilon_2(i,k) = f_2\varepsilon_2\varepsilon_3(i,k) = \varepsilon'_0 f_2\varepsilon_3(i,k) = \varepsilon'_0\varepsilon'_1 f_2(i,k) = \varepsilon'_1\varepsilon'_0 f_2(i,k) = \varepsilon'_1 f_2\varepsilon_2(i,k) = \varepsilon'_1 f_2(i,k+1)$. Therefore, (i') holds for $(i,k+1)$, for all i. Then we show, by induction, that (i") $\varepsilon'_2 f_2(i,j) = f_2\varepsilon_0(i,j)$, for every $(i,j) \in \mathbb{Z}_n \times \mathbb{Z}_{2p}$. By Lemma 17 we only need to prove (i") for $(i,q+1),(i,q+2),\ldots,(i,q+p)$. First of all, we have $f_2\varepsilon_0(i,q+1) = f_2(i-1,q) = (-i+1+l''_q,1)$ and $\varepsilon'_2 f_2(i,q+1) = \varepsilon' 2 f_2\varepsilon_1(i,q) = \varepsilon'_2\varepsilon'_3 f_2(i,q) = \varepsilon'_2\varepsilon'_3(-i+l''_q,1) = \varepsilon'_2(-i+1+l''_q,0) = (-i+1+l''_q,1)$. Therefore, (i") holds for $(i,q+1)$, for all i. Let us suppose that (i") holds for $(i,q+1),(i,q+2),\ldots,(i,k)$ with $q+1 \le k \le q+p-1$. From Lemma 15 we get, if k is odd, $f_2\varepsilon_0(i,k+1) = f_2\varepsilon_0\varepsilon_1(i,k) = f_2\varepsilon_1\varepsilon_0(i,k) = \varepsilon'_3 f_2\varepsilon_0(i,k) = \varepsilon'_3\varepsilon'_2 f_2(i,k) = \varepsilon'_2\varepsilon'_3 f_2(i,k) = \varepsilon'_2 f_2\varepsilon_1(i,k) = \varepsilon'_2 f_2(i,k+1)$ and, if k is even, $f_2\varepsilon_0(i,k+1) = f_2\varepsilon_0\varepsilon_2(i,k) = f_2\varepsilon_2\varepsilon_0(i,k) = \varepsilon'_0 f_2\varepsilon_0(i,k) = \varepsilon'_0\varepsilon'_2 f_2(i,k) = \varepsilon'_2\varepsilon'_0 f_2(i,k) = \varepsilon'_2 f_2\varepsilon_2(i,k) = \varepsilon'_2 f_2(i,k+1)$. Therefore, (ii') holds for $(i,k+1)$, for all i.

b") Follows directly from point b') and Lemma 2. □

Proof of Lemma 5 ([8]). Let $\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3$ (resp. $\varepsilon'_0,\varepsilon'_1,\varepsilon'_2,\varepsilon'_3$) be the involutions defining $G = G(n,p,q,m)$ (resp. $G' = G(n,p,q,m^{-1})$). Moreover, let $\phi_3 \in \Sigma_X$ and $f_3 : \mathbb{Z}_n \times \mathbb{Z}_{2p} \to \mathbb{Z}_n \times \mathbb{Z}_{2p}$ be the maps

$$\phi_3 = \begin{cases} (0\,3)(1\,2), & \text{if } q \text{ is odd} \\ (0\,3), & \text{if } q \text{ is even} \end{cases}, \qquad f_3(i,j) = (-m^{-1}i, 1+q-j).$$

It is easy to check that the map $g : \mathbb{Z}_n \times \mathbb{Z}_{2p} \to \mathbb{Z}_n \times \mathbb{Z}_{2p}$, defined by $g(i,j) = (-mi, 1+q-j)$, is the inverse map of f_3; therefore, f_3 is a bijection. The pair (f_3,ϕ_3) is an isomorphism between G and G'. In fact we get:

- $f_3\varepsilon_0(i,j) = (-m^{-1}i - \mu(j-q), j-q)$;
- $f_3\varepsilon_1(i,j) = (-m^{-1}i, 1+q-j+(-1)^j)$;

– $f_3\varepsilon_2(i,j) = (-m^{-1}i, 1+q-j-(-1)^j)$;
– $f_3\varepsilon_3(i,j) = (-m^{-1}i - m^{-1}\mu(j), q+j)$;
– $\varepsilon'_0 f_3(i,j) = (-m^{-1}i + m^{-1}\mu(1-j), q+j) = (-m^{-1}i - m^{-1}\mu(j), q+j)$;
– $\varepsilon'_1 f_3(i,j) = (-m^{-1}i, 1+q-j-(-1)^{1+q-j})$
$= \begin{cases} (-m^{-1}i, 1+q-j+(-1)^j), & \text{if } q \text{ is even} \\ (-m^{-1}i, 1+q-j-(-1)^j), & \text{if } q \text{ is odd} \end{cases}$;
– $\varepsilon'_2 f_3(i,j) = (-m^{-1}i, 1+q-j+(-1)^{1+q-j})$
$= \begin{cases} (-m^{-1}i, 1+q-j-(-1)^j), & \text{if } q \text{ is even} \\ (-m^{-1}i, 1+q-j+(-1)^j), & \text{if } q \text{ is odd} \end{cases}$;
– $\varepsilon'_3 f_3(i,j) = (-m^{-1}i + \mu(1+q-j), j-q) = (-m^{-1}i - \mu(j-q), j-q)$. □

In order to prove Theorem 8 we need some preparatory results.

Lemma 18. *Let $\sigma \in \Sigma_X$ and $r, s : \mathbb{Z}_n \times \mathbb{Z}_{2p} \to \mathbb{Z}_n \times \mathbb{Z}_{2p}$ be the maps*

$$\sigma = \begin{cases} 1, & \text{if } p \text{ is even} \\ (1\,2), & \text{if } p \text{ is odd} \end{cases}, \quad r(i,j) = (i+1, j), \quad s(i,j) = (-i, p+j).$$

Then $(r, 1)$ and (s, σ) are automorphisms of $G(n,p,q,m)$.

Proof. Since $r^n = 1 = s^2$, both r and s are bijections. Then we have
– $\varepsilon_0 r(i,j) = (i+1+m\mu(j-q), 1-j+2q) = r\varepsilon_0(i,j)$,
– $\varepsilon_1 r(i,j) = (i+1, j-(-1)^j) = r\varepsilon_1(i,j)$,
– $\varepsilon_2 r(i,j) = (i+1, j+(-1)^j) = r\varepsilon_2(i,j)$,
– $\varepsilon_3 r(i,j) = (i+1+\mu(j), 1-j) = r\varepsilon_3(i,j)$.
– $\varepsilon_0 s(i,j) = (-i-m\mu(j-q), 1+p-j+2q)$,
– $\varepsilon_1 s(i,j) = (-i, p+j-(-1)^{p+j}) = \begin{cases} (-i, p+j-(-1)^j), & \text{if } p \text{ is even} \\ (-i, p+j+(-1)^j), & \text{if } p \text{ is odd} \end{cases}$;
– $\varepsilon_2 s(i,j) = (-i, p+j+(-1)^{p+j}) = \begin{cases} (-i, p+j+(-1)^j), & \text{if } p \text{ is even} \\ (-i, p+j-(-1)^j), & \text{if } p \text{ is odd} \end{cases}$;
– $\varepsilon_3 s(i,j) = (-i-\mu(j), 1-j+p)$,
– $s\varepsilon_0(i,j) = (-i-m\mu(j-q), 1+p-j+2q)$,
– $s\varepsilon_1(i,j) = (-i, j+p-(-1)^j$,
– $s\varepsilon_2(i,j) = (-i, j+p+(-1)^j$,
– $s\varepsilon_3(i,j) = (-i-\mu(j), 1-j+p)$. □

Lemma 19. *Let (f,ϕ) be an isomorphism between the graphs $G(n,p,q,m)$ and $G(n,p,q',m')$, with $n,p \geq 3$.*

a') *If p is even and* $\gcd(n,m) \neq 1$, *then $\phi \in \{1, (0\,1)(2\,3)\}$ and $\pi'' f(0,0) \in \{0, 1, p, p+1\}$.*

a") *If p is even and* $\gcd(n,m) = 1$, *then $\phi \in \{1, (0\,1)(2\,3), (0\,3)(1\,2), (0\,2)(1\,3)\}$ and* (i) *$\pi'' f(0,0) \in \{0, 1, p, p+1\}$, if $\phi \in \{1, (0\,1)(2\,3)\}$,* (ii) *$\pi'' f(0,0) \in \{q', q'+1, q'+p, q'+p+1\}$, if $\phi \in \{(0\,3)(1\,2), (0\,2)(1\,3)\}$.*

b) *If p is odd, then $\phi \in \{1, (1\,2), (0\,3), (0\,3)(1\,2), (0\,1)(2\,3), (0\,2)(1\,3), (0\,1\,3\,2), (0\,2\,3\,1)\}$ and $\pi'' f(0,0) \in \{0, 1, q', q'+1, p, p+1, q'+p, q'+p+1\}$.*

Proof. First of all, observe that the vertex $(0,0)$ of $G = G(n,p,q,m)$ lies in a $\{2,3\}$-residue of length $2n$. All the $\{1,2\}$-residues (resp. the $\{1,2\}$-residues) of G are mapped by any isomorphism either to the $\{1,2\}$-residues or to the $\{0,3\}$-residues of $G' = G(n,p,q',m')$.

a') Each of the two $\{2,3\}$-residues of length $2n$ is mapped to a $\{2,3\}$-residue (where each vertex has second coordinate $j \in \{0,1,p,p+1\}$) and each of the two $\{0,1\}$-residues of length $2n/\gcd(n,m)$ is mapped to a $\{0,1\}$-residue.

a") Each of the two $\{2,3\}$-residues (resp. the two $\{0,1\}$-residues) of length $2n$ is mapped either to a $\{2,3\}$-residue (where each vertex has second coordinates $j \in \{0,1,p,p+1\}$) or to a $\{0,1\}$-residue (where each vertex has second coordinates $j \in \{q', q'+1, q'+p, q'+p+1\}$).

b) The $\{2,3\}$-residue, the $\{1,3\}$-residue, the $\{0,1\}$-residue and the $\{0,2\}$-residue of length $2n$ are mapped to either the $\{2,3\}$-residue or the $\{1,3\}$-residue or the $\{0,1\}$-residue or the $\{0,2\}$-residue. Since each vertex of these four residues has second coordinate $j \in \{0,1,q',q'+1,p,p+1,q'+p,q'+p+1\}$), the statement holds. □

Proof of Theorem 8. (*"only if" part*) From Lemma 7 we get $n' = n$ and $p' = p$. Let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$ (resp. $\varepsilon'_0, \varepsilon'_1, \varepsilon'_2, \varepsilon'_3$) be the involutions defining either $G = G(n,p,q,m)$ or $G = G(n,p,q,(-1)^q)$ (resp. either $G' = G(n,p,q',m')$ or $G' = G(n,p,q',(-1)^{q'})$) and let f_1, f_2, f_3 be the bijections defined in Lemma 3, 4 and 5 respectively. Moreover, observe that the pairs $(r,1)$ and (s,σ) defined in Lemma 18 are automorphisms of both G and G'.

a') If (f,ϕ) is an isomorphism between G and G', then, by Lemma 19, $\phi \in \{1,(0\,1)(2\,3)\}$ and, up to the action of r and s, we can suppose $f(0,0) = (0,j)$ with $j \in \{0,1\}$. Then we have the following four cases:

(i) If $f(0,0) = (0,0)$ and $\phi = 1$, then $f = 1$ and (f,ϕ) is an automorphism of $G(n,p,q,m)$.

(ii) If $f(0,0) = (0,0)$ and $\phi = (0\,1)(2\,3)$, then $f = f_2$ and (f,ϕ) is an isomorphism between $G(n,p,q,m)$ and $G(n,p,q^{-1},m)$.

(iii) If $f(0,0) = (0,1)$ and $\phi = 1$, then $f = f_1$ and (f,ϕ) is an isomorphism between $G(n,p,q,m)$ and $G(n,p,-q,m)$.

(iv) If $f(0,0) = (0,1)$ and $\phi = (0\,1)(2\,3)$, then $f = f_1 f_2$ and (f,ϕ) is an isomorphism between $G(n,p,q,m)$ and $G(n,p,-q^{-1},m)$.

a") If (f,ϕ) is an isomorphism between G and G', then, by Lemma 19, $\phi \in \{1,(0\,3)(1\,2),(0\,1)(2\,3),(0\,2)(1\,3)\}$ and, up to the action of r and s, we can suppose $f(0,0) = (0,j)$, with $j \in \{0,1\}$ if $\phi \in \{1,(0\,1)(2\,3)\}$ and with $j \in \{q', q'+1\}$ if $\phi \in \{(0\,3)(1\,2),(0\,2)(1\,3)\}$. Then we have the four cases of the previous point and the following four cases: (i') $f(0,0) = (0,1+q')$ and $\phi = (0\,3)(1\,2)$; (ii') $f(0,0) = (0,1+q')$ and $\phi = (0\,2)(1\,3)$; (iii') $f(0,0) = (0,q')$ and $\phi = (0\,3)(1\,2)$ and (iv') $f(0,0) = (0,q')$ and $\phi = (0\,2)(1\,3)$. In (i'), if j is even (resp. if j is odd) we get

$$\begin{aligned} f(i,j) &= f(\varepsilon_1\varepsilon_2)^{j/2}(\varepsilon_3\varepsilon_2)^i(0,0) = (\varepsilon'_2\varepsilon'_1)^{j/2}(\varepsilon'_0\varepsilon'_1)^i f(0,0) \\ &= (\varepsilon'_2\varepsilon'_1)^{j/2}(-m'i, 1+q') = (-m'i, 1+q'-j) \end{aligned}$$

(resp. $f(i,j) = f\varepsilon_2(\varepsilon_1\varepsilon_2)^{(j-1)/2}(\varepsilon_3\varepsilon_2)^i(0,0) = \varepsilon_1'(\varepsilon_2'\varepsilon_1')^{(j-1)/2}(\varepsilon_0'\varepsilon_1')^i f(0,0) = \varepsilon_1'(\varepsilon_2'\varepsilon_1')^{(j-1)/2}(-m'i, 1+q') = (-m'i, 1+q'-j)$). Since $f\varepsilon_0(i,j) = (-m'i - m'm\mu(j-q), j+q'-2q)$ and $\varepsilon_3' f(i,j) = (-m'i - \mu(j-q'), j-q')$, we get $q' = q$, $m' = m^{-1}$, and therefore $f = f_3$. Case (ii') is a composition of cases (i') and (ii), case (iii') is a composition of cases (i') and (iii), case (iv') is a composition of cases (i') and (iv). Therefore, we have:

(i') If $f(0,0) = (0, 1+q)$ and $\phi = (0\,3)(1\,2)$, then $f = f_3$ and (f, ϕ) is an isomorphism between $G(n,p,q,m)$ and $G(n,p,q,m^{-1})$.

(ii') If $f(0,0) = (0, 1+q)$ and $\phi = (0\,2)(1\,3)$, then $f = f_3 f_2$ and (f, ϕ) is an isomorphism between $G(n,p,q,m)$ and $G(n,p,q^{-1},m^{-1})$.

(iii') If $f(0,0) = (0, q)$ and $\phi = (0\,3)(1\,2)$, then f is the map $f_3 f_1$ and (f, ϕ) is an isomorphism between $G(n,p,q,m)$ and $G(n,p,-q,m^{-1})$.

(iv') If $f(0,0) = (0, q)$ and $\phi = (0\,2)(1\,3)$, then f is the map $f_3 f_1 f_2$ and (f, ϕ) is an isomorphism between $G(n,p,q,m)$ and $G(n,p,-q^{-1},m^{-1})$.

b) The pair $(f_3, (1\,2))$ is an automorphism of G'. Therefore, if (f, ϕ) is an isomorphism between G and G', then, up to the action of r, s and f_3, we can suppose that $f(0,0) = (0, j)$, with $j \in \{0, 1\}$. By this assumption, f sends the $\{2,3\}$-residue of G onto the $\{2,3\}$-residue of G' and therefore either $\phi = 1$ or $\phi = (0\,1)(2\,3)$. We have four cases:

(i") If $f(0,0) = (0,0)$ and $\phi = 1$, then $f = 1$ and (f, ϕ) is an automorphism of $G(n,p,q,(-1)^q)$.

(ii") If $f(0,0) = (0,0)$ and $\phi = (0\,1)(2\,3)$, then $f = f_2$ and (f, ϕ) is an isomorphism between $G(n,p,q,-1)$ and $G(n,p,q^{-1},-1)$ (resp. between $G(n,p,q,1)$ and $G(n,p,(q+p)^{-1}+p,1)$) if q is odd (resp. even).

(iii") If $f(0,0) = (0,1)$ and $\phi = 1$, then $f = f_1$ and (f, ϕ) is an isomorphism between $G(n,p,q,(-1)^q)$ and $G(n,p,-q,(-1)^q)$.

(iv") If $f(0,0) = (0,1)$ and $\phi = (0\,1)(2\,3)$, then $f = f_1 f_2$ and (f, ϕ) is an isomorphism between $G(n,p,q,-1)$ and $G(n,p,-q^{-1},-1)$ (resp. between $G(n,p,q,1)$ and $G(n,p,-(q+p)^{-1}+p,1)$) if q is odd (resp. even).

Therefore, if $G(n,p,q',(-1)^{q'})$ is isomorphic to $G(n,p,q,(-1)^q)$, then $(*)$ $q' = \pm q^{\pm 1}$ when q is odd and $(**)$ $q' = \pm(q+p)^{\pm 1} + p$ when q is even. Since $(*)$ + $(**)$ is equivalent to $q' \equiv \pm q^{\pm 1} \bmod p$, our proof is completed. □

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