Siberian Mathematical Journal, Vol. 45, *No.* 1, *pp.* 1–6, 2004
Original Russian Text Copyright © 2004 *Aydin H., Gültekyn I., and Mulazzani M.*

TORUS KNOTS AND DUNWOODY MANIFOLDS

H. Aydin, I. Gültekyn, and M. Mulazzani

UDC 515.16

Abstract: We obtain an explicit representation as Dunwoody manifolds of all cyclic branched coverings of torus knots of type $(p, mp \pm 1)$ with $p > 1$ and $m > 0$.

Keywords: torus knot, Heegaard splitting, Dunwoody manifold

1. Introduction

Many interesting examples of cyclic branched coverings of knots in $\mathbf{S}^3$ admitting cyclic presentations for their fundamental groups have been found recently (see [1–7]). In order to investigate these relations, M. J. Dunwoody introduced in [2] a class of 3-manifolds, depending on six integer parameters, with cyclically presented fundamental groups. It has been shown in [8] that all these manifolds turn out to be strongly-cyclic branched coverings of $(1,1)$-knots in lens spaces (possibly $\mathbf{S}^3$). Moreover, the explicit Dunwoody representation for all cyclic branched coverings of 2-bridge knots has been obtained.

In this paper we give a similar result for a wide class of torus knots which are, together with 2-bridge knots, the most important examples of $(1,1)$-knots in $\mathbf{S}^3$. The Dunwoody parameters are obtained for all cyclic branched coverings of torus knots of type $(p, mp \pm 1)$, with $p > 1$ and $m > 0$, thus including all torus knots with bridge number ≤ 4. These manifolds have been considered from another point of view in [9] and [1].

We refer to [10, 11] for details on knot theory and cyclic branched coverings of knots and to [12] for details on cyclic presentation of groups.

2. (1,1)-Knots and Dunwoody Manifolds

A knot K in a 3-manifold N^3 is called a $(1,1)$-*knot* if there exists a Heegaard splitting of genus one

$$(N^3, K) = (T, A) \cup_\varphi (T', A'),$$

where T and T' are solid tori, $A \subset T$ and $A' \subset T'$ are properly embedded trivial arcs, and $\varphi : (\partial T', \partial A') \to (\partial T, \partial A)$ is an attaching homeomorphism (see Fig. 1). Obviously, N^3 turns out to be a lens space $L(p, q)$ (including $\mathbf{S}^3 = L(1, 0)$).

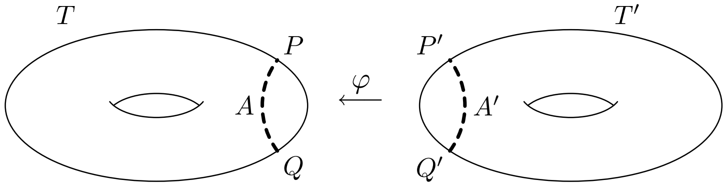

Fig. 1. $(1,1)$-decomposition.

It is well known that the family of $(1,1)$-knots contains all torus knots and all 2-bridge knots in $\mathbf{S}^3$. Several topological properties of $(1,1)$-knots have recently been investigated (see references in [13]).

An algebraic representation of $(1,1)$-knots has been developed in [9] and [13], where it is shown that there is a natural surjective map

$$\psi \in PMCG_2(\partial T) \mapsto K_\psi \in \mathscr{K}_{1,1}$$

from the pure mapping class group of the twice punctured torus $PMCG_2(\partial T)$ to the class $\mathscr{K}_{1,1}$ of all $(1,1)$-knots.

Erzurum (Turkey); Bologna (Italy). Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 45, No. 1, pp. 3–10, January–February, 2004. Original article submitted February 28, 2003.

The fundamental group of the exterior of a (1, 1)-knot K_ψ can be explicitly obtained using its representation ψ. Let $\alpha, \beta, \gamma \subset \partial T$ be the loops depicted in Fig. 2. They represent a set of free generators for $\pi_1(\partial T - \partial A, *)$, while α and γ freely generate $\pi_1(T - A, *)$. A straightforward application of Seifert–Van Kampen theorem gives:

Lemma 1 [9]. *The fundamental group of the exterior of a* (1, 1)*-knot* $K_\psi \subset L(p, q)$ *admits the presentation*

$$\pi_1(L(p, q) - K_\psi, *) = \langle\, \alpha, \gamma \,|\, r(\alpha, \gamma) \,\rangle,$$

where $r(\alpha, \gamma)$ *is the homotopy class of* $i(\psi(\beta))$ *with* $i : (\partial T, \partial A) \to (T, A)$ *the inclusion map.*

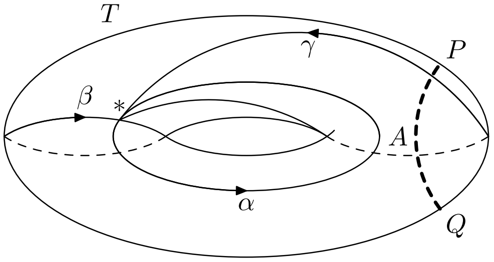

Fig. 2

As a consequence, all (1, 1)-knots in $\mathbf{S}^3$ are prime [14], and can therefore be classified, up to mirror image, by their fundamental groups (see [10, p. 76]). Many results concerning the connections between cyclically presented groups and strongly-cyclic branched coverings of (1, 1)-knots (see the definition in [9]) have been obtained. It has been proved in [15] that every n-fold strongly-cyclic branched covering of a (1, 1)-knot admits a Heegaard diagram of genus n which encodes a cyclic presentation for the fundamental group. This result has been improved in [9], where a constructive algorithm which explicitly gives the cyclic presentations is obtained.

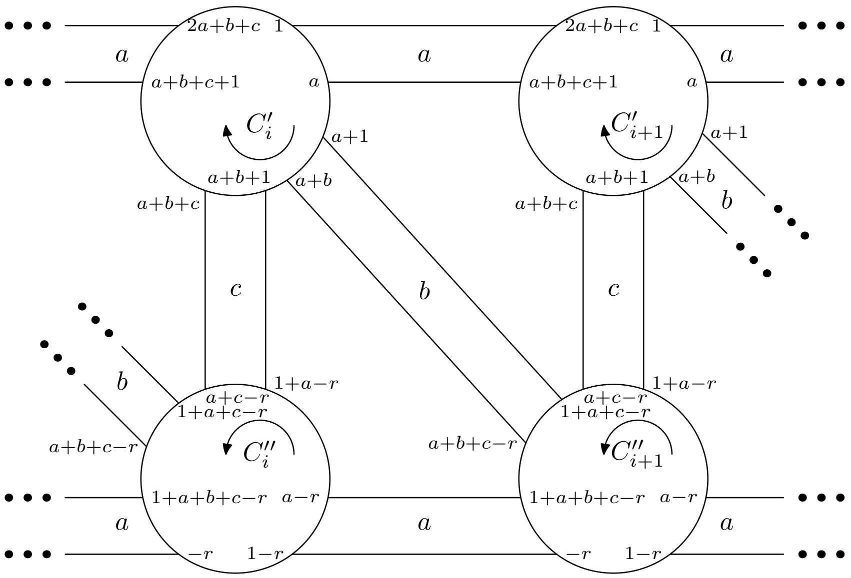

Fig. 3. A Dunwoody diagram.

The family of Dunwoody manifolds has been introduced in [2] by a class of trivalent regular planar graphs (called *Dunwoody diagrams*) with cyclic symmetry, depending on six integers a, b, c, n, r, s, such that $n > 0$, $a, b, c \geq 0$, and $a + b + c > 0$. For certain values of the parameters, called *admissible*, the Dunwoody diagrams $D(a,b,c,n,r,s)$ turn out to be Heegaard diagrams, so defining a wide class of closed, orientable 3-manifolds $M(a,b,c,n,r,s)$ with cyclically presented fundamental groups.

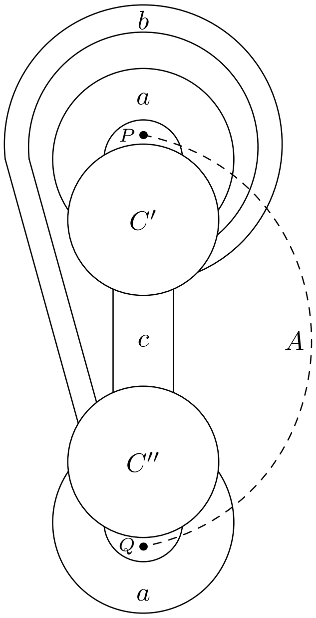

Fig. 4. $D(a,b,c,1,r,0)$.

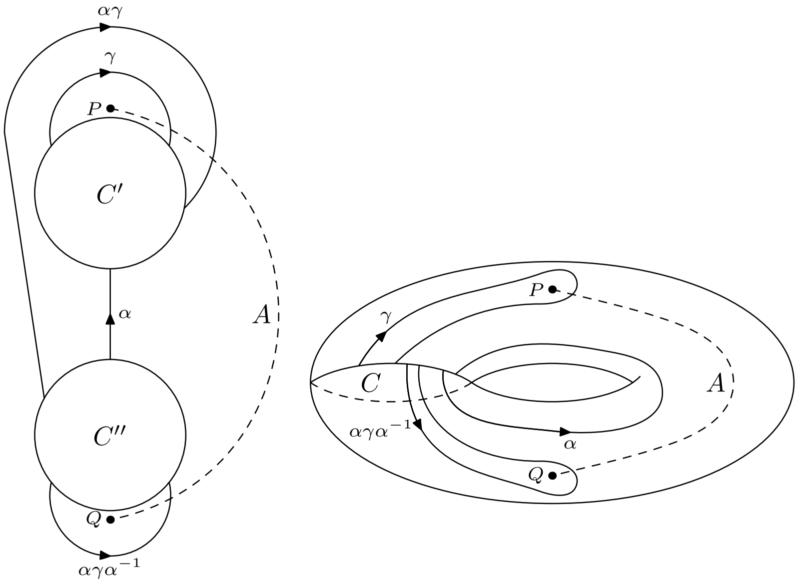

Fig. 5

More precisely, an admissible Dunwoody diagram $D(a,b,c,n,r,s)$ is an open Heegaard diagram of genus n (see Fig. 3) which contains n upper cycles $C'_1,\ldots,C'_n$ and n lower cycles $C''_1,\ldots,C''_n$, each having $d=2a+b+c$ vertices. For every $i=1,\ldots,n$, the cycle C'_i (resp. C''_i) is connected to the cycle C'_{i+1} (resp. C''_{i+1}) by a parallel arcs, to the cycle C''_i by c parallel arcs, and to the cycle C''_{i+1} by b parallel arcs. We denote by $\mathscr{B}$ the set of all these arcs. The cycle C'_i is glued to the cycle C''_{i-s} (subscripts mod n) so that equally labeled vertices are identified together. Observe that the parameters r and s can be considered mod d and n respectively. Since the identification rule and the diagram are invariant

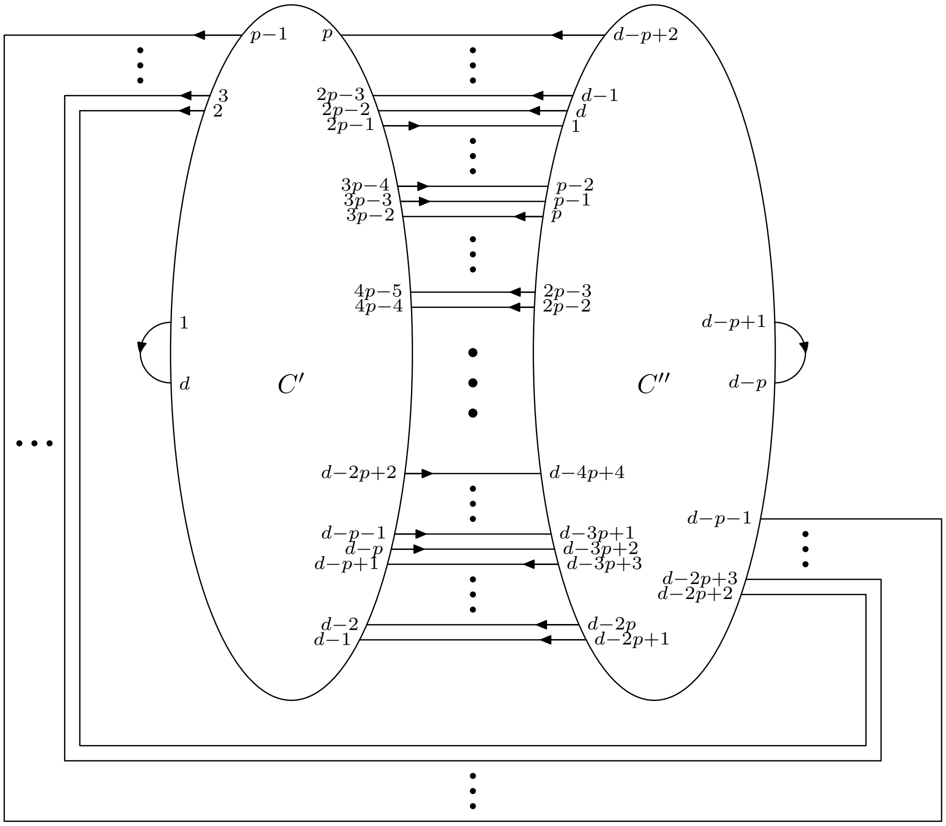

Fig. 6. $D(1, p-2, 2mp-2m-p+1, 1, p, 0)$.

with respect to a cyclic action of order n, the Dunwoody manifolds admit a cyclic symmetry of order n. Obviously, the Dunwoody manifold $M(a, b, c, 1, r, 0)$ is homeomorphic to a lens space (possibly $\mathbf{S}^3$), since it admits a Heegaard splitting of genus one.

A characterization of all Dunwoody manifolds as strongly-cyclic branched coverings of $(1, 1)$-knots is given by the following:

Proposition 2 [8]. *The Dunwoody manifold $M(a, b, c, n, r, s)$ is the n-fold strongly-cyclic covering of the lens space $M(a, b, c, 1, r, 0)$ (possibly $\mathbf{S}^3$), branched over a $(1, 1)$-knot $K(a, b, c, r)$ only depending on the integers a, b, c, r.*

The converse of this result is also true, as proved in [16]. So the class of the Dunwoody manifolds coincides with the class of the strongly-cyclic branched coverings of $(1, 1)$-knots.

The $(1, 1)$-knots $K(a, b, c, r)$, occurring in Proposition 2, admit a natural $(1, 1)$-decomposition $(T, A) \cup_{\varphi} (T', A')$ depicted in Fig. 4, where the arcs of $\mathscr{B}$ constitute the curve $\varphi(\beta') = \psi(\beta)$, with ψ the element of $PMCG_2(\partial T)$ corresponding to φ. The fundamental group of the exterior of $K(a, b, c, r)$ can be directly read in the Dunwoody diagram of $D(a, b, c, 1, r, 0)$. The relation $r(\alpha, \gamma)$ of Lemma 1 is obtained by walking along the arcs of $\mathscr{B}$, following a fixed orientation: associate to each arc a word in α and γ representing its homotopy class in the fundamental group of $T - A$ (see Fig. 5), where a properly embedded disk with boundary C is considered squeezed to the base point $*$.

3. Main Results

An interesting problem is to find the Dunwoody parameters of the cyclic branched coverings of important classes of $(1, 1)$-knots, in particular when the knot lies in $\mathbf{S}^3$. This type of result has been obtained in [8, Theorem 8] for all 2-bridge knots.

Now we obtain a similar result for the torus knots $\mathbf{t}(p, mp \pm 1)$, with $m > 0$ and $p > 1$.

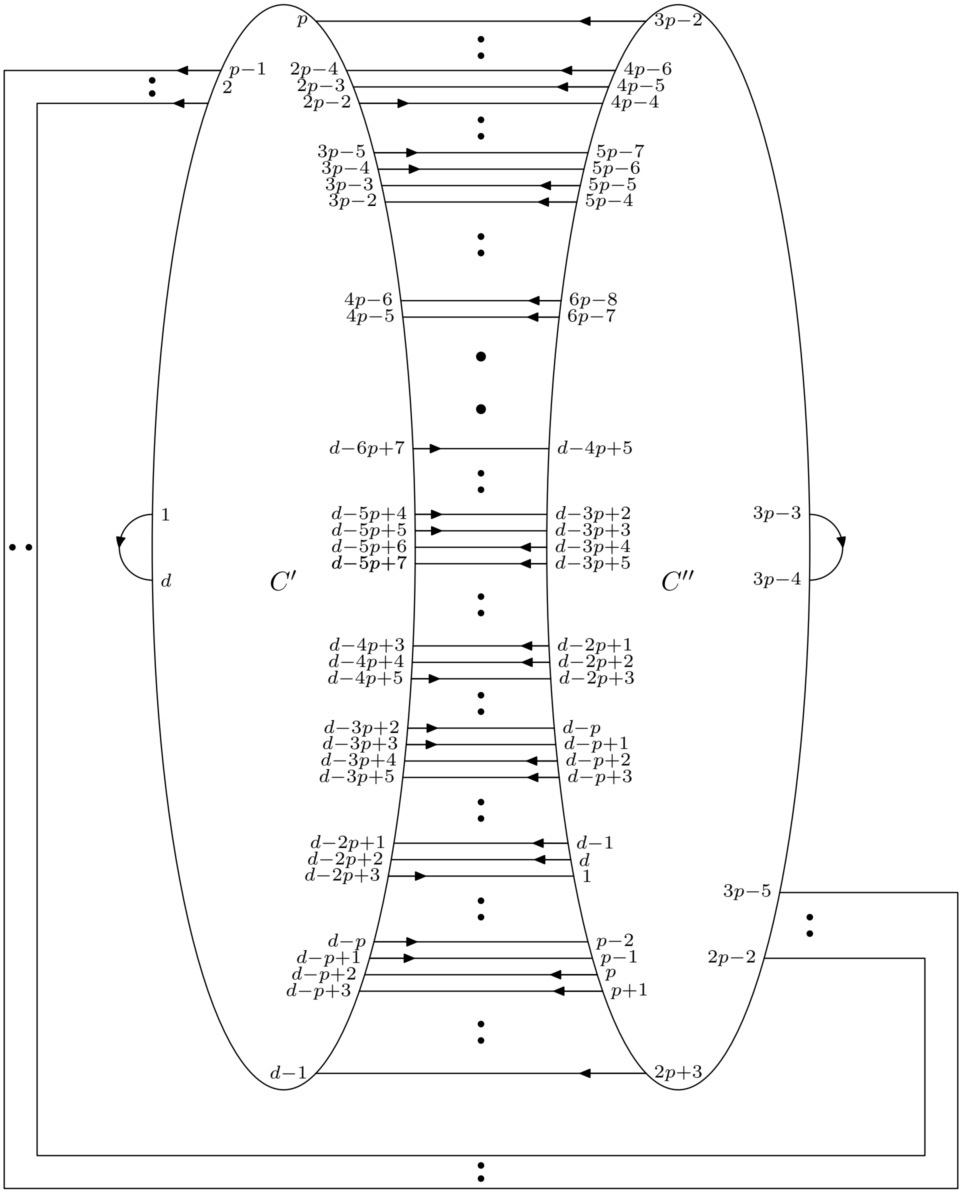

Fig. 7. $D(1, p-2, 2mp-2m-p-1, 1, -3p+4, 0)$.

Proposition 3. 1. *$K(1, p-2, 2mp-2m-p+1, p)$ is the torus knot $\mathbf{t}(p, mp+1)$ for all $m > 0$ and $p > 1$;*

2. *$K(1, p-2, 2mp-2m-p-1, -3p+4)$ is the torus knot $\mathbf{t}(p, mp-1)$ for all $m > 1$ and $p > 1$.*

Proof. 1. Let $K = K(1, p-2, 2mp-2m-p+1, p)$. Fig. 6 depicts the Dunwoody diagram $D = D(1, p-2, 2mp-2m-p+1, 1, p, 0)$. The number of vertices of each cycle is $d = 2m(p-1)+1$. Starting from the vertex of C'' with label d, we pass along all arcs of $\mathscr{B}$ with the described orientation, obtaining the word $w = \alpha^{m(p-1)}\alpha\gamma^{-1}\alpha^{-1}(\alpha^{-(m-1)}\gamma^{-1}\alpha^{-1})^{p-2}\alpha^{-(m-1)}\gamma^{-1}$. Since the conditions of [8, Corollary 4] are satisfied, D is therefore admissible. Moreover, the fundamental group of $M(1, p-2, 2mp-2m-p+1, 1, p, 0)$ is $\langle \alpha, \gamma \mid w, \gamma \rangle$, which is trivial. So $M(1, p-2, (p-1)(2m-1), 1, p, 0) \cong \mathbf{S}^3$. Moreover, $\pi_1(\mathbf{S}^3 - K) = \langle \alpha, \gamma \mid w \rangle$. Since $w = \alpha^{m(p-1)+1}(\gamma^{-1}\alpha^{-m})^{p-1}\gamma^{-1} = \alpha^{-m}\alpha^{mp+1}(\gamma^{-1}\alpha^{-m})^{p}\alpha^{m}$, we have $\pi_1(\mathbf{S}^3 - K) \cong \langle \alpha, \gamma \mid w' \rangle$, with $w' = \alpha^{mp+1}(\gamma^{-1}\alpha^{-m})^{p}$. Obviously, this group is isomorphic to the group $\langle x, y \mid x^{mp+1}y^{-p} \rangle$, which is the group of the torus knot $\mathbf{t}(p, mp+1)$. As a consequence K is precisely $\mathbf{t}(p, mp+1)$.

2. The proof is analogous to the previous one. Let $K = K(1, p-2, 2mp-2m-p-1, -3p+4)$. Fig. 7 depicts the Dunwoody diagram $D = D(1, p-2, 2mp-2m-p-1, 1, -3p+4, 0)$. The number of vertices of each cycle is $d = 2m(p-1)-1$ (note that the labeling is considered mod d). Starting from the vertex of C'' with label d, we pass along all arcs of $\mathscr{B}$, obtaining the word $w = \alpha^{m-1}(\alpha^m)^{p-3}\alpha^{m-1}\alpha\gamma^{-1}\alpha^{-1}\alpha^{-(m-1)}(\gamma^{-1}\alpha^{-1}\alpha^{-(m-1)})^{p-2}\gamma^{-1}$. So D is admissible. Moreover, $M(1, p-2, 2mp-2m-p-1, 1, -3p+4, 0)$ is homeomorphic to $\mathbf{S}^3$, since its fundamental group is trivial. Since $w = \alpha^{m(p-1)-1}(\gamma^{-1}\alpha^{-m})^{p-1}\gamma^{-1} = \alpha^{mp-1}(\alpha^{-m}\gamma^{-1})^p$, the group $\pi_1(\mathbf{S}^3 - K) \cong \langle \alpha, \gamma \mid w\rangle$ is isomorphic to the group $\langle x, y \mid x^{mp-1}y^{-p}\rangle$. Therefore, K is the torus knot $\mathbf{t}(p, mp-1)$.

Corollary 4. 1. *For all $m > 0$ and $p > 1$, the n-fold cyclic branched covering of the torus knot $\mathbf{t}(p, mp+1)$ is the Dunwoody manifold $M(1, p-2, 2mp-2m-p+1, n, p, p)$.*

2. *For all $m > 1$ and $p > 1$, the n-fold cyclic branched covering of the torus knot $\mathbf{t}(p, mp-1)$ is the Dunwoody manifold $M(1, p-2, 2mp-2m-p-1, n, -3p+4, -p)$.*

PROOF. From Proposition 2, we only need to find the sixth parameter s of the 6-tuples. The relation $q_\sigma + sp_\sigma \equiv 0 \mod n$ must be satisfied for all n, where p_σ is the number of arcs of $\mathscr{B}$ oriented from C' to C'' minus the number of those oriented from C'' to C' and q_σ is the number of arcs of $\mathscr{B}$ oriented from right to left minus the number of those oriented from left to right in the Dunwoody diagram (see [8, p. 385]). In other words, $-p_\sigma$ is the total exponent of α in w and $-q_\sigma$ is the total exponent of γ in w. In the first case $q_\sigma = p$ and $p_\sigma = -1$; therefore, $s = p$. In the second case $q_\sigma = p$ and $p_\sigma = 1$; therefore, $s = -p$.

An extension of the previous results to the whole class of torus knots by using the same technique seems to be very complicated, although some partial results were obtained. A possible alternative method could be the induction on the number of steps of the Euclidean division algorithm needed to find the greatest common divisor of the parameters of the torus knot. In this approach, our results could provide a base of induction.

Acknowledgements. This research was partially conducted during the third author's visit to the Department of Mathematics of Atatürk University, Erzurum (Turkey) which was supported by TOKTEN/UNISTAR funds and fulfilled under the auspices of the G.N.S.A.G.A. of I.N.d.A.M. (Italy) and the funds of Bologna University for selected research topics.

References

1. *Cavicchioli A., Hegenbarth F., and Kim A. C.*, "A geometric study of Sieradski groups," Algebra Colloq., **5**, No. 2, 203–217 (1998).
2. *Dunwoody M. J.*, "Cyclic presentations and 3-manifolds," in: Proc. Inter. Conf., Groups-Korea 94, Walter de Gruyter, Berlin and New York, 1995, pp. 47–55.
3. *Helling H., Kim A. C., and Mennicke J. L.*, "A geometric study of Fibonacci groups," J. Lie Theory, **8**, 1–23 (1998).
4. *Kim A. C.*, "On the Fibonacci group and related topics," Contemp. Math., **184**, 231–235 (1995).
5. *Kim A. C., Kim Y., and Vesnin A.*, "On a class of cyclically presented groups," in: Proc. Inter. Conf., Groups-Korea 98, Walter de Gruyter, Berlin; New York, 2000, pp. 211–220.
6. *Maclachlan C. and Reid A. W.*, "Generalised Fibonacci manifolds," Transform. Groups, **2**, 165–182 (1997).
7. *Vesnin A. and Kim A. C.*, "The fractional Fibonacci groups and manifolds," Siberian Math. J., **39**, 655–664 (1998).
8. *Grasselli L. and Mulazzani M.*, "Genus one 1-bridge knots and Dunwoody manifolds," Forum Math., **13**, 379–397 (2001).
9. *Cattabriga A. and Mulazzani M.*, "Strongly-cyclic branched coverings of (1,1)-knots and cyclic presentations of groups," Math. Proc. Cambridge Philos. Soc., **135**, 137–146 (2003).
10. *Kawauchi A.*, A Survey of Knot Theory, Birkhäuser-Verlag, Basel (1996).
11. *Rolfsen D.*, Knots and Links, Publish or Perish, Berkeley (1976).
12. *Johnson D. L.*, Topics in the Theory of Group Presentations, Cambridge Univ. Press, Cambridge (1980). (London Math. Soc. Lecture Note Ser.; **42**.)
13. *Cattabriga A. and Mulazzani M.* "$(1,1)$-knots via the mapping class group of the twice punctured torus," Adv. Geom. (2004) (to appear).
14. *Norwood F. H.*, "Every two-generator knot is prime," Math. Soc., **86**, 143–147 (1982).
15. *Mulazzani M.*, "Cyclic presentations of groups and cyclic branched coverings of $(1,1)$-knots," Bull. Korean Math. Soc., **40**, 101–108 (2003).
16. *Cattabriga A. and Mulazzani M.*, All Strongly-Cyclic Branched Coverings of $(1,1)$-Knots Are Dunwoody Manifolds [Preprint] (2003).