

# On the Vertex Group of $n$ -Manifolds

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## Abstract

In this paper<sup>1 2</sup> we obtain the decomposition of the vertex group of  $n$ -manifolds, extending the one given by Kauffman and Lins for dimension 3 and solving the related conjecture. The result is obtained in the more general category of gems: the vertex group of a gem  $\Gamma$ , representing an  $n$ -manifold  $M$ , is the free product of  $n$  copies of the fundamental group of  $M$  and a free group  $F$  of rank  $N - n$ , where  $N$  is the number of  $n$ -residues of  $\Gamma$ . In particular, for crystallizations  $F \cong \mathbf{Z}$  and consequently the vertex group is an invariant of  $M$ .

## 1 Introduction and Notation

Throughout this paper, manifolds are closed and graphs are finite. For the basic concepts of graph theory we refer to [BM76]; for groups given by generators and relations we refer to [Joh76]. If  $X$  is any set, let  $F(X)$  denote the free group generated by  $X$ .

An  $(n + 1)$ -coloured graph  $\Gamma$  is a connected  $(n + 1)$ -regular graph (without loops) with a proper<sup>3</sup> edge- $(n + 1)$ -colouration. We assume that the set of colours is always  $\Delta_n = \{0, 1, 2, \dots, n\}$ ; further define  $\Delta_{-1} = \emptyset$  and  $\hat{k} = \Delta_n - \{k\}$ , for each  $k \in \Delta_n$ . If  $\mathcal{C} \subseteq \Delta_n$ , a  $\mathcal{C}$ -residue of  $\Gamma$  is a connected component of the subgraph obtained by deleting all  $c$ -edges, with  $c \in \Delta_n - \mathcal{C}$ . An  $h$ -residue ( $0 \leq h \leq n + 1$ ) is a  $\mathcal{C}$ -residue such that  $|\mathcal{C}| = h$ . We can think of an  $h$ -residue as an  $h$ -coloured graph. Observe that 0-residues are vertices, 1-residues are edges and 2-residues are bicoloured polygons of even length (also called *bigons*) of  $\Gamma$ . For each  $k \in \Delta_n$ , let  $E_k$  denote the set of  $k$ -edges and  $\Gamma_{\hat{k}}$  the set of  $\hat{k}$ -residues of  $\Gamma$ ; moreover let  $\rho_k = |\Gamma_{\hat{k}}|$ .

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<sup>3</sup>Adjacent edges have different colouring.

There is a standard construction which associates a closed quasi- $n$ -manifold<sup>4</sup>  $S(\Gamma)$  to an  $(n + 1)$ -coloured graph  $\Gamma$  (see [FGG86]) and every quasi-manifold arises in this way. We shall say that  $S(\Gamma)$  is *represented* by  $\Gamma$ . An  $(n + 1)$ -coloured graph is said to be an  $n$ -gem iff it represents a manifold; a *crystallization* is a gem with  $\rho_k = 1$ , for all  $k \in \Delta_n$ . Every manifold also arises from a crystallization by the above construction. All 2- or 3-coloured graphs are 1- or 2-gems; whereas a 4-coloured graph is a 3-gem iff the number of bigons is equal to the number of vertices plus the number of 3-residues ([LM85]). In general, an  $(n + 1)$ -coloured graph is an  $n$ -gem iff each  $n$ -residue represents the  $(n - 1)$ -dimensional sphere  $\mathbf{S}^{n-1}$  (see [Fer76]).

## 2 The Vertex Group

The *vertex group* of an  $(n + 1)$ -coloured graph  $\Gamma$  is the group

$$\mathcal{H}(\Gamma) = \langle \mathcal{V}(\Gamma) \mid \mathcal{R}(\Gamma) \rangle$$

generated by the vertices of  $\Gamma$  and with a relator for each bigon of  $\Gamma$ . The relator  $r_\beta \in \mathcal{R}(\Gamma)$  associated to an arbitrary  $\{i, j\}$ -residue  $\beta$  is constructed as follows. Suppose  $i < j$  and let  $v_0, v_1, \dots, v_{2l-1}$  be the sequence of vertices on  $\beta$ , obtained starting from a chosen vertex  $v_0$  with the  $i$ -edge; then

$$r_\beta = \prod_{h=0}^{2l-1} (v_h)^{\varepsilon_h}, \quad (1)$$

where  $\varepsilon_h = (-1)^h$ . Further define  $\bar{r}_\beta$  by the same rule of  $r_\beta$ , but reversing the sense of traversal of the bigon (i.e. start from  $v_0$  with the  $j$ -edge).

Let  $B_k^+$  [resp.  $B_k^-$ ] be the set of  $\{k, c\}$ -residues such that  $k < c$  [resp.  $k > c$ ]; define  $R_k = \{r_\beta \mid \beta \in B_k^+\}$  and  $\bar{R}_k = \{\bar{r}_\beta \mid \beta \in B_k^-\}$ . Thus we have the presentation

$$\mathcal{H}(\Gamma) = \langle \mathcal{V}(\Gamma) \mid R_0 \cup R_1 \cup \dots \cup R_{n-1} \rangle .$$

**Remark 1** *Note that the definition of  $\mathcal{H}(\Gamma)$  depends on various choices. The choice of the starting vertex of a bigon is irrelevant, for a change of  $v_0$  leads to a new relator  $r'_\beta$ , which is a conjugate of  $r_\beta$  or of  $r_\beta^{-1}$ . Apparently, the group depends on the ordering of the colour-set. However, Corollary 7 will prove the independence, up to isomorphism, in case of gems.*

The vertex group was first defined in [Lin89], only for crystallizations; there, the group has been proved to be an invariant of the represented manifold. In particular,

<sup>4</sup>A quasi-manifold is a pseudo-manifold in which the star of every simplex is strongly connected ([Gag79]).

L. Kauffman and S. Lins showed ([KL92]) that the vertex group of a 3-crystallization is isomorphic to the free product of three copies of the fundamental group of the manifold and a copy of the free group on one generator. This result is a partial answer to the following conjecture ([Lin89], [KL92]):

**Conjecture 1** *If  $\Omega$  is an  $n$ -crystallization representing the  $n$ -manifold  $M$ , then:*

$$\mathcal{H}(\Omega) \cong \underbrace{\pi_1(M) * \pi_1(M) * \cdots * \pi_1(M)}_{n \text{ times}} * \mathbf{Z}.$$

Our paper completely solves this conjecture, obtaining a more general result involving gems.

To end this section, we prove a technical lemma on oriented graphs, which will be useful later. Let  $\vec{\Omega} = (V, A)$  be an oriented graph and  $H = \langle A \mid C \rangle$  be a group generated by the arcs of  $\vec{\Omega}$ , with each relator corresponding to a closed cycle in  $\vec{\Omega}$ . Let  $\vec{\Theta} = (V, T)$  be any spanning tree of  $\vec{\Omega}$ , and  $\tilde{T} = \{\tilde{t} \mid t \in T\}$  be a set of symbols such that  $|\tilde{T}| = |T|$ ,  $\tilde{T} \cap A = \emptyset$ . For each  $v \in V$ , denote by  $p_v$  the word of  $F(A)$  corresponding to the path in  $\vec{\Theta}$  from a fixed vertex  $\bar{v}$  to  $v$ , and by  $\tilde{p}_v$  the word of  $F(\tilde{T})$  obtained by replacing  $t$  with  $\tilde{t}$  in  $p_v$ . Finally, let  $a_-$  and  $a_+$  be respectively the tail and the head of the arc  $a \in A$ .

**Lemma 1** *The map  $\phi : A \rightarrow \langle A \mid C \cup T \rangle * F(\tilde{T})$ , defined by  $\phi(a) = \tilde{p}_{a_-}^{-1} a \tilde{p}_{a_+}$ , uniquely extends to an isomorphism*

$$f : \langle A \mid C \rangle \xrightarrow{\cong} \langle A \mid C \cup T \rangle * F(\tilde{T}).$$

**Proof:** Let  $\Phi : F(A) \rightarrow \langle A \mid C \cup T \rangle * F(\tilde{T})$  be the extension of  $\phi$  to the free group  $F(A)$ . If  $w \in F(A)$  corresponds to a path in  $\vec{\Omega}$  from  $v'$  to  $v''$ , then  $\Phi(w) = \tilde{p}_{v'}^{-1} w \tilde{p}_{v''}$ . Therefore, if  $c \in C$  and  $v_c$  is the starting vertex of the corresponding cycle, then we obtain  $\Phi(c) = \tilde{p}_{v_c}^{-1} c \tilde{p}_{v_c} = \tilde{p}_{v_c}^{-1} \tilde{p}_{v_c} = 1$ . Thus, there exists a unique homomorphism  $f : \langle A \mid C \rangle \rightarrow \langle A \mid C \cup T \rangle * F(\tilde{T})$  extending  $\phi$ . Observe that, for every  $t \in T$ ,  $f(t) = \tilde{p}_{t_-}^{-1} t \tilde{p}_{t_+} = \tilde{p}_{t_-}^{-1} \tilde{p}_{t_+} = \tilde{t}$ . We show that  $f$  is an isomorphism by finding its inverse. Define  $\Psi : F(A \cup \tilde{T}) \rightarrow \langle A \mid C \rangle$ , by  $\Psi(a) = p_{a_-} a p_{a_+}^{-1}$ , for each  $a \in A$ , and  $\Psi(\tilde{t}) = t$ , for each  $\tilde{t} \in \tilde{T}$ . Since  $\Psi(t) = p_{t_-} t p_{t_+}^{-1} = 1$  (for  $t \in T$ ) and  $\Psi(c) = p_{v_c} c p_{v_c}^{-1} = p_{v_c} p_{v_c}^{-1} = 1$  (for  $c \in C$ ), there exists a unique homomorphism  $g : \langle A \mid C \cup T \rangle * F(\tilde{T}) \rightarrow \langle A \mid C \rangle$  such that  $g(x) = \Psi(x)$ , for every  $x \in A \cup \tilde{T}$ . We get:

$$g \circ f(a) = g(\tilde{p}_{a_-}^{-1} a \tilde{p}_{a_+}) = p_{a_-}^{-1} p_{a_-} a p_{a_+}^{-1} p_{a_+} = a, \text{ for all } a \in A;$$

$$f \circ g(a) = f(p_{a_-} a p_{a_+}^{-1}) = \tilde{p}_{a_-} \tilde{p}_{a_-}^{-1} a \tilde{p}_{a_+} \tilde{p}_{a_+}^{-1} = a, \text{ for all } a \in A;$$

$$f \circ g(\tilde{t}) = f(t) = \tilde{t}, \text{ for all } \tilde{t} \in \tilde{T}.$$

Therefore  $g = f^{-1}$ ; so  $f$  is an isomorphism. ■

### 3 $k$ -Groups and the Fundamental Group

The  $k$ -group  $G_k(\Gamma)$  of an  $(n + 1)$ -coloured graph  $\Gamma$  is the group generated by the set  $\vec{E}_k$  of the  $k$ -edges of  $\Gamma$ , arbitrarily oriented, and with relator-set in 1-1 correspondence with the set  $B_k = B_k^+ \cup B_k^-$  of all  $\{k, c\}$ -residues. The relator  $s_\beta$ , associated to  $\beta \in B_k$ , is obtained by walking along the bigon, starting from a fixed vertex with a chosen direction, and giving the exponent  $+1$  or  $-1$  to each  $k$ -edge, according to whether the sense of traversal of the bigon is coherent or not with the orientation of the generator:

$$G_k(\Gamma) = \langle \vec{E}_k \mid S_k \rangle,$$

where  $S_k = \{s_\beta \mid \beta \in B_k\}$ .

Note that  $G_k(\Gamma)$  is independent on the choice of the starting vertex and on the sense of traversal of every bigon (see Remark 1). Moreover, this group is independent of the chosen orientation on the  $k$ -edges, up to a canonical isomorphism.

For  $k \in \Delta_n$ , define the oriented graph  $\vec{\Gamma}^{(k)} = (\Gamma_{\widehat{k}}, \vec{E}_k)$ , having the  $\widehat{k}$ -residues of  $\Gamma$  as vertices, and arcs corresponding to the  $k$ -edges of  $\Gamma$  by the following rule: for each  $k$ -edge, with vertices  $v'$  and  $v''$  respectively in the two  $\widehat{k}$ -residues  $t'$  and  $t''$ , we join  $t'$  and  $t''$  in  $\vec{\Gamma}^{(k)}$  by an arc with the same orientation used for defining  $G_k$  (observe that if  $t' = t''$  then the arc is a loop). Evidently, each relator  $s_\beta \in S_k$ , corresponds to a certain closed cycle  $\bar{\beta}$  in  $\vec{\Gamma}^{(k)}$ ; so we can apply Lemma 1 to  $G_k(\Gamma)$ .

When the coloured graph is a gem, representing the manifold  $M$ , the  $k$ -groups are strictly related to the fundamental group of  $M$ . In fact,  $k$ -groups have been introduced in [BM87] and [Don83] as an useful tool for obtaining a presentation of  $\pi_1(M)$ .

**Proposition 1** *If  $\Omega$  is an  $n$ -gem representing  $M$  ( $n \geq 2$ ), then*

$$G_k(\Omega) \cong \pi_1(M) * L_k,$$

where  $L_k$  is a free group with  $(\rho_k - 1)$  generators. For crystallizations, in particular,

$$G_k(\Omega) \cong \pi_1(M).$$

**Proof:** Let  $\vec{\Theta} = (\Omega_{\widehat{k}}, T)$  be any spanning tree of the oriented graph  $\vec{\Omega}^{(k)}$ ; since  $\Omega$  has  $\rho_k$   $\widehat{k}$ -residues,  $|T| = \rho_k - 1$ . From Lemma 1, we obtain  $G_k(\Omega) \cong \langle \vec{E}_k \mid S_k \cup T \rangle$

$*F(\vec{T}) \cong \langle \vec{E}_k \mid S_k \cup T \rangle *L_k$ . As well known,  $\langle \vec{E}_k \mid S_k \cup T \rangle$  is a presentation, called *k-edge presentation*, of the fundamental group of  $M$  (see [BM87] or [Chi86]). So the statement follows.  $\blacksquare$

For each path  $\sigma$  in an  $n$ -gem  $\Omega$  and for each  $k \in \Delta_n$ , define  $\sigma^{(k)} \in G_k(\Omega)$  as the product of the oriented  $k$ -edges in the order in which they are traversed along  $\sigma$ . Observe that  $(\sigma^{-1})^{(k)} = (\sigma^{(k)})^{-1}$ ; besides, if the terminus of  $\sigma$  coincides with the origin of  $\tau$ , then  $(\sigma \cdot \tau)^{(k)} = \sigma^{(k)}\tau^{(k)}$ .

**Lemma 2** *If  $\Omega$  is a  $n$ -gem representing a simply-connected manifold  $M$ , then  $\gamma^{(k)}$  is the identity element of  $G_k(\Omega)$ , for every closed cycle  $\gamma$  in  $\Omega$  and for every  $k \in \Delta_n$ .*

**Proof:** Choose a spanning tree  $\vec{\Theta} = (\Omega_{\widehat{k}}, T)$  of  $\vec{\Omega}^{(k)}$  and let  $f : G_k(\Omega) = \langle \vec{E}_k \mid S_k \rangle \rightarrow \langle \vec{E}_k \mid S_k \cup T \rangle *F(\vec{T})$  be the isomorphism of Lemma 1. Since  $M$  is simply-connected, the group  $\langle \vec{E}_k \mid S_k \cup T \rangle$  is the trivial group; so, if  $w \in F(\vec{E}_k)$  is a word corresponding to a path in  $\vec{\Omega}^{(k)}$  from  $r'$  to  $r''$ , we have  $f(w) = \tilde{p}_{r'}^{-1}w\tilde{p}_{r''} = \tilde{p}_{r'}^{-1}\tilde{p}_{r''}$ . The word  $\gamma^{(k)}$  corresponds to a closed cycle in  $\vec{\Omega}^{(k)}$ , which starts from a certain  $\widehat{k}$ -residue  $r$ ; thus  $f(\gamma^{(k)}) = \tilde{p}_r^{-1}\tilde{p}_r = 1$ . Since  $f$  is an isomorphism,  $\gamma^{(k)}$  is the identity element of  $G_k(\Omega)$ .  $\blacksquare$

**Lemma 3** *Let  $\Lambda$  be a  $\widehat{c}$ -residue of an  $n$ -gem  $\Omega$ . For every  $k \in \Delta_n$ , we have the properties:*

- a) *if  $\gamma$  is a closed cycle in  $\Lambda$ , then  $\gamma^{(k)}$  is the identity element of  $G_k(\Omega)$ ;*
- b) *if  $\sigma$  and  $\tau$  are paths in  $\Lambda$  from  $v$  to  $v'$ , then  $\sigma^{(k)} = \tau^{(k)}$ ;*
- c) *if  $v''$  is a vertex belonging to the same  $(\Delta_n - \{c, k\})$ -residue of  $v'$  and  $\sigma_1$  is a path from  $v$  to  $v''$ , then  $\sigma_1^{(k)} = \sigma^{(k)}$ .*

**Proof:** The proof is trivial when  $k = c$ , since every path in  $\Lambda$  does not contain any  $k$ -edge. Now suppose  $k \neq c$ .

(a) It is enough to show that  $\gamma^{(k)}$  is the identity element of  $G_k(\Lambda) = \langle \vec{E}'_k \mid S'_k \rangle$ . Indeed,  $\vec{E}'_k \subseteq \vec{E}_k$  and  $S'_k \subset S_k$ ; so the normal closure of  $S'_k$  in  $F(\vec{E}'_k)$  is contained in the normal closure of  $S_k$ . If  $n = 1$  the proof is immediate, for  $\Lambda$  is a  $k$ -edge and each closed cycle in  $\Lambda$  corresponds to the trivial word of  $F(\vec{E}'_k)$ . If  $n = 2$ ,  $\Lambda$  is a 2-residue; so  $\gamma^{(k)}$  is a power of the unique relator of  $S'_k$ , up to conjugation in  $F(\vec{E}'_k)$ . If  $n > 2$ , then  $\Lambda$  represents  $\mathbf{S}^{n-1}$ , which is simply-connected. Now the statement follows from Lemma 3.

(b) The join  $\sigma \cdot \tau^{-1}$  of  $\sigma$  with the inverse of  $\tau$  is a closed path in  $\Lambda$ ; by the previous point we get  $\sigma^{(k)}(\tau^{(k)})^{-1} = (\sigma \cdot \tau^{-1})^{(k)} = 1$ .

(c) We can take a path  $\xi$  from  $v'$  to  $v''$ , avoiding any  $k$ -edge. So  $\xi^{(k)} = 1$  and therefore  $\sigma_1^{(k)} = (\sigma \cdot \xi)^{(k)} = \sigma^{(k)}\xi^{(k)} = \sigma^{(k)}$ . ■

## 4 Decomposition of the Vertex Group of Gems

Let  $\Omega$  be an  $n$ -gem; for each  $k \in \Delta_n$  let  $X_k = \{x_i^{(k)} \mid i \in I_k\}$  denote the set of all  $\Delta_{k-1}$ -residues of  $\Omega$ . Observe that  $X_0 = \mathcal{V}(\Omega)$ ,  $X_1 = E_0$  and  $X_n = \Omega_n^\wedge$ . Further, let  $F_k(\Omega)$  denote the free group generated by  $X_k$ :

$$F_k(\Omega) = F(X_k).$$

For each  $\{i, j\}$ -residue  $\beta$ , we define the word  $r_\beta^{(k)} \in F_k(\Omega)$  from (1), by replacing each vertex  $v_h$  with its  $\Delta_{k-1}$ -residue  $x_{i_h}^{(k)}$ . By performing the same substitutions on  $\bar{r}_\beta$ , we define the word  $\bar{r}_\beta^{(k)}$ . Further, let  $\bar{R}_k^{(k)} = \{\bar{r}_\beta^{(k)} \mid \beta \in B_k^-\}$ . Note that every  $\bar{r}_\beta^{(k)} \in \bar{R}_k^{(k)}$  is the identity element of  $F_k(\Omega)$ , for vertices of a  $c$ -edge belong to the same  $\Delta_{k-1}$ -residue (recall:  $c < k$ ) and they appear with opposite exponents in  $\bar{r}_\beta$ .

For each  $k \in \Delta_{n-1}$ , let  $R_i^{(k)} = \{r_\beta^{(k)} \mid \beta \in B_i^+\}$ ,  $W_k = R_k^{(k)} \cup R_{k+1}^{(k)} \cup \dots \cup R_{n-1}^{(k)}$  and  $\tilde{W}_{k+1} = W_k - R_k^{(k)}$ ; further define  $W_n = \emptyset$ . Note that  $R_i^{(0)} = R_i$ ,  $W_0 = \mathcal{R}(\Omega)$  and  $\tilde{W}_n = \emptyset$ .

Now we introduce the group  $H_k(\Omega)$  as a quotient of  $F_k(\Omega)$ :

$$H_k(\Omega) = \langle X_k \mid W_k \rangle.$$

Clearly,  $H_0(\Omega) = \mathcal{H}(\Omega)$  and  $H_n(\Omega) = F_n(\Omega)$ ; thus  $H_n(\Omega)$  is a free group of rank  $\rho_n$ .

$k$ -groups and  $H_k$ 's groups of a gem are strongly related, as showed by the following proposition:

**Proposition 2** *For every  $k \in \Delta_{n-1}$ ,*

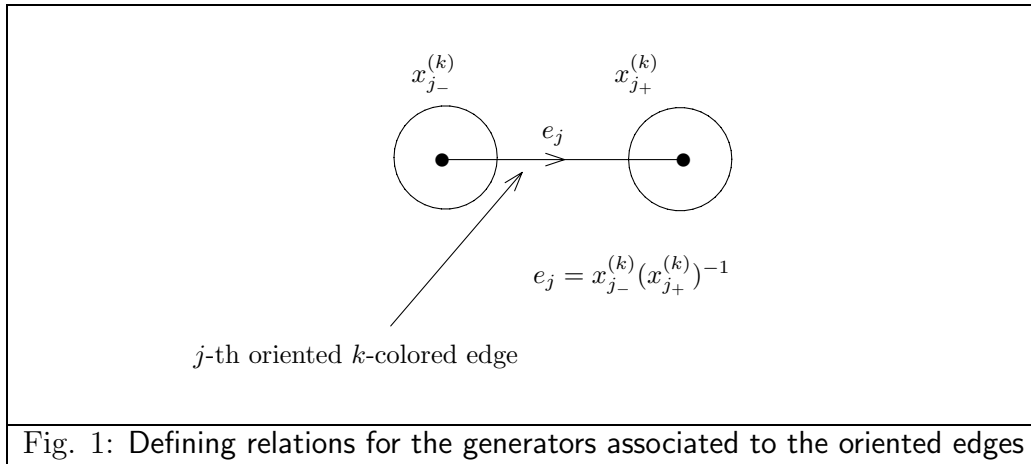
$$H_k(\Omega) \cong G_k(\Omega) * H_{k+1}(\Omega).$$

**Proof:** We obtain the proof by several steps involving a sequence of Tietze transformations ([Joh76]) on  $\langle X_k \mid W_k \rangle$ .

1) — Add redundant relators  $\bar{R}_k^{(k)}$  — Each word of  $\bar{R}_k^{(k)}$  is trivial in  $H_k(\Omega)$ ; so we can include  $\bar{R}_k^{(k)}$  in the relator-set:

$$\begin{aligned} H_k(\Omega) &= \langle X_k \mid W_k \rangle; \\ &\cong \langle X_k \mid R_k^{(k)} \cup \tilde{W}_{k+1} \rangle; \\ &\cong \langle X_k \mid \bar{R}_k^{(k)} \cup R_k^{(k)} \cup \tilde{W}_{k+1} \rangle. \end{aligned}$$

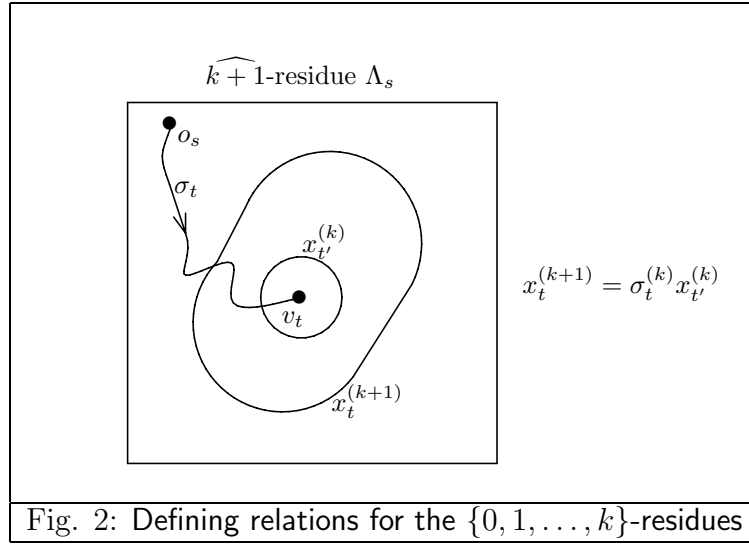
2) — Add  $(\vec{E}_k, D_k)$  and replace  $\bar{R}_k^{(k)} \cup R_k^{(k)}$  by  $S_k$  — Give to every  $k$ -edge of  $\Omega$  the orientation used for defining  $G_k(\Omega)$ . If the tail [resp. the head] of  $e_j \in \vec{E}_k$  belongs to  $x_{j-}^{(k)}$  [resp. belongs to  $x_{j+}^{(k)}$ ], take  $e_j^{-1}x_{j-}^{(k)}(x_{j+}^{(k)})^{-1}$  as defining relator for the new generator  $e_j$ ; besides, define  $D_k = \{e_j^{-1}x_{j-}^{(k)}(x_{j+}^{(k)})^{-1} \mid e_j \in \vec{E}_k\}$ .



So we can add  $\vec{E}_k$  and  $D_k$  respectively to the generators and to the relators. By replacing  $x_{j-}^{(k)}(x_{j+}^{(k)})^{-1}$  by  $e_j$ , the relator  $r_\beta^{(k)} \in R_k^{(k)}$ , for  $\beta \in B_k^+$ , or  $\bar{r}_\beta^{(k)} \in \bar{R}_k^{(k)}$ , for  $\beta \in B_k^-$ , becomes a conjugate of  $s_\beta$  or of  $s_\beta^{-1}$  in  $\langle \vec{E}_k \cup X_k \mid D_k \cup \bar{R}_k^{(k)} \cup R_k^{(k)} \cup \tilde{W}_{k+1} \rangle$ . Therefore we can substitute  $\bar{R}_k^{(k)} \cup R_k^{(k)}$  with  $S_k$  in the presentation:

$$\begin{aligned} H_k(\Omega) &\cong \langle X_k \mid \bar{R}_k^{(k)} \cup R_k^{(k)} \cup \tilde{W}_{k+1} \rangle; \\ &\cong \langle \vec{E}_k \cup X_k \mid D_k \cup \bar{R}_k^{(k)} \cup R_k^{(k)} \cup \tilde{W}_{k+1} \rangle; \\ &\cong \langle \vec{E}_k \cup X_k \mid D_k \cup S_k \cup \tilde{W}_{k+1} \rangle. \end{aligned}$$

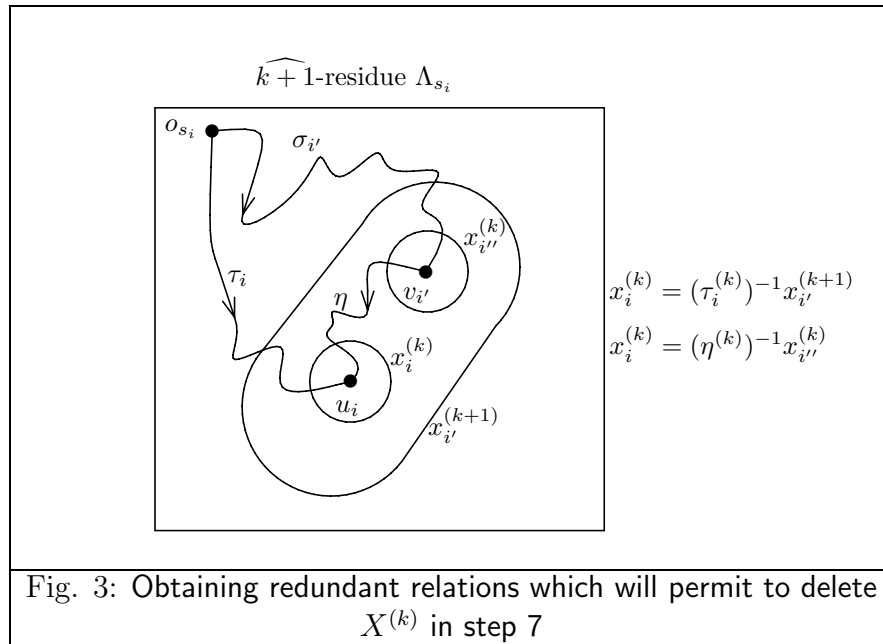
3) — Add  $(X_{k+1}, Y_{k+1})$  — For each  $(\Delta_n - \{k+1\})$ -residue  $\Lambda_s \in \Omega_{k+1}$ , fix a distinguished vertex  $o_s$ . Moreover, for each  $\Delta_k$ -residue  $x_t^{(k+1)}$  belonging to  $\Lambda_s$ , choose a vertex  $v_t$  and let  $x_{t'}^{(k)}$  be the  $\Delta_{k-1}$ -residue containing  $v_t$ . Take a path  $\sigma_t$  in  $\Lambda_s$  from  $o_s$  to  $v_t$ .



For each  $t \in I_{k+1}$ , we use  $(x_t^{(k+1)})^{-1} \sigma_t^{(k)} x_t^{(k)}$  as defining relator for the new generator  $x_t^{(k+1)}$ , and let  $Y_{k+1} = \{(x_t^{(k+1)})^{-1} \sigma_t^{(k)} x_t^{(k)} \mid t \in I_{k+1}\}$  be the set of all these relators. We have:

$$\begin{aligned} H_k(\Omega) &\cong \langle \vec{E}_k \cup X_k \mid D_k \cup S_k \cup \tilde{W}_{k+1} \rangle; \\ &\cong \langle \vec{E}_k \cup X_k \cup X_{k+1} \mid D_k \cup S_k \cup \tilde{W}_{k+1} \cup Y_{k+1} \rangle. \end{aligned}$$

4) — Add redundant relator set  $Z_k$  — On each  $\Delta_{k-1}$ -residue  $x_i^{(k)}$  choose a vertex  $u_i$  and a path  $\tau_i$  from  $o_{s_i}$  to  $u_i$ , in the residue  $\Lambda_{s_i} \in \Omega_{\widehat{k+1}}$  containing  $x_i^{(k)}$ ;



If  $x_{i'}^{(k+1)}$  is the  $\Delta_k$ -residue containing  $x_i^{(k)}$ , we claim that  $(x_{i'}^{(k+1)})^{-1}\tau_i^{(k)}x_i^{(k)}$  is the element 1 of the group  $H = \langle \vec{E}_k \cup X_k \cup X_{k+1} \mid D_k \cup S_k \cup \tilde{W}_{k+1} \cup Y_{k+1} \rangle$ . We take a path  $\eta$  in  $x_{i'}^{(k+1)}$  from  $v_{i'}$  to  $u_i$ . By the relators of  $D_k$ , we get  $x_i^{(k)} = (\eta^{(k)})^{-1}x_{i''}^{(k)}$ , where  $x_{i''}^{(k)}$  is  $\Delta_{k-1}$ -residue containing  $v_{i'}$ . Thus  $(x_{i'}^{(k+1)})^{-1}\tau_i^{(k)}x_i^{(k)} = (x_{i''}^{(k)})^{-1}(\sigma_{i'}^{(k)})^{-1}\tau_i^{(k)}(\eta^{(k)})^{-1}x_{i''}^{(k)}$ . But  $\sigma_{i'}^{-1} \cdot \tau_i \cdot \eta^{-1}$  is a closed path in  $\Lambda_{s_i}$  and therefore  $(x_{i'}^{(k+1)})^{-1}\tau_i^{(k)}x_i^{(k)} = 1$ , by Lemma 3.a. So we can add the set  $Z_k = \{(x_{i'}^{(k+1)})^{-1}\tau_i^{(k)}x_i^{(k)} \mid i \in I_k\}$  to the relators:

$$\begin{aligned} H_k(\Omega) &\cong \langle \vec{E}_k \cup X_k \cup X_{k+1} \mid D_k \cup S_k \cup \tilde{W}_{k+1} \cup Y_{k+1} \rangle; \\ &\cong \langle \vec{E}_k \cup X_k \cup X_{k+1} \mid D_k \cup S_k \cup \tilde{W}_{k+1} \cup Y_{k+1} \cup Z_k \rangle. \end{aligned}$$

5) — Delete redundant relator set  $Y_{k+1}$  — At this stage the relators of  $Y_{k+1}$  are redundant. To prove this claim, define  $z_i = (x_{i'}^{(k+1)})^{-1}\tau_i^{(k)}x_i^{(k)} \in Z_k$  (for  $i \in I_k$ ) and  $y_t = (x_t^{(k+1)})^{-1}\sigma_t^{(k)}x_{t'}^{(k)} \in Y_{k+1}$  (for  $t \in I_{k+1}$ ). Since the vertices  $v_t$  and  $u_{t'}$  belong to the same  $\Delta_{k-1}$ -residue  $x_{t'}^{(k)}$ , they also belong to the same  $(\Delta_n - \{k, k+1\})$ -residue; so, by Lemma 3.c,  $\sigma_t^{(k)} = \tau_{t'}^{(k)}$  and we obtain  $y_t^{-1}z_{t'} = (x_{t'}^{(k)})^{-1}(\sigma_t^{(k)})^{-1}x_t^{(k+1)}(x_t^{(k+1)})^{-1}\tau_{t'}^{(k)}x_{t'}^{(k)} = 1$ , for every  $t \in I_{k+1}$  (note that  $(t')' = t$ ). By dropping  $Y_{k+1}$  from the relators, we get:

$$\begin{aligned} H_k(\Omega) &\cong \langle \vec{E}_k \cup X_k \cup X_{k+1} \mid D_k \cup S_k \cup \tilde{W}_{k+1} \cup Y_{k+1} \cup Z_k \rangle; \\ &\cong \langle \vec{E}_k \cup X_k \cup X_{k+1} \mid D_k \cup S_k \cup \tilde{W}_{k+1} \cup Z_k \rangle. \end{aligned}$$

6) — Delete the redundant relator set  $D_k$  — We show that the relators of  $D_k$  are also redundant. Take a relator of  $D_k$ ,  $d_j = e_j^{-1}x_{j_-}^{(k)}(x_{j_+}^{(k)})^{-1}$ ; we have  $x_{j_-}^{(k+1)} = x_{j_+}^{(k+1)}$  and, again by Lemma 3,  $\tau_{j_-}^{(k)}e_j(\tau_{j_+}^{(k)})^{-1} = 1$ . We obtain  $d_j = e_j^{-1}x_{j_-}^{(k)}(x_{j_+}^{(k)})^{-1} = e_j^{-1}(\tau_{j_-}^{(k)})^{-1}x_{j_-}^{(k+1)}z_{j_-}z_{j_+}^{-1}(x_{j_+}^{(k+1)})^{-1}\tau_{j_+}^{(k)} = (\tau_{j_+}^{(k)})^{-1}x_{j_+}^{(k+1)}z_{j_-}z_{j_+}^{-1}((\tau_{j_+}^{(k)})^{-1}x_{j_+}^{(k+1)})^{-1}$  and therefore  $d_j$  is a conjugate of  $z_{j_-}z_{j_+}^{-1}$  in  $\langle \vec{E}_k \cup X_k \cup X_{k+1} \mid D_k \cup S_k \cup \tilde{W}_{k+1} \cup Z_k \rangle$ . Hence, we can drop  $D_k$ :

$$\begin{aligned} H_k(\Omega) &\cong \langle \vec{E}_k \cup X_k \cup X_{k+1} \mid D_k \cup S_k \cup \tilde{W}_{k+1} \cup Z_k \rangle; \\ &\cong \langle \vec{E}_k \cup X_k \cup X_{k+1} \mid S_k \cup \tilde{W}_{k+1} \cup Z_k \rangle. \end{aligned}$$

7) — Delete  $(X_k, Z_k)$  and replace  $\tilde{W}_{k+1}$  by  $W_{k+1}$  — Each element of  $X_k$  appears exactly in one relator of  $Z_k$  and each relator of  $Z_k$  contains only one element of  $X_k$ . So we can use  $Z_k$  to eliminate  $X_k$ . Now we have to replace  $x_i^{(k)}$  by  $(\tau_i^{(k)})^{-1}x_{i'}^{(k+1)}$  in all the relators of  $\tilde{W}_{k+1}$ . If  $r_\beta^{(k)} \in \tilde{W}_{k+1}$  is the relator corresponding to an  $\{i, j\}$ -residue  $\beta$  (of length  $2l$ ), then we claim that, after the above replacing,  $r_\beta^{(k)}$  becomes a relator equivalent to  $r_\beta^{(k+1)}$ . Let  $r_\beta = v_0v_1^{-1}v_2v_3^{-1} \cdots v_{2h-1}^{-1}v_{2h} \cdots v_{2l-3}^{-1}v_{2l-2}v_{2l-1}^{-1}$ ; if  $x_{i_m}^{(k)}$  is the  $\Delta_{k-1}$ -residue containing  $v_m$  ( $0 \leq m \leq 2l-1$ ), then

$$r_\beta^{(k)} = x_{i_0}^{(k)} (x_{i_1}^{(k)})^{-1} x_{i_2}^{(k)} (x_{i_3}^{(k)})^{-1} \cdots (x_{i_{2h-1}}^{(k)})^{-1} x_{i_{2h}}^{(k)} \cdots (x_{i_{2l-3}}^{(k)})^{-1} x_{i_{2l-2}}^{(k)} (x_{i_{2l-1}}^{(k)})^{-1}.$$

Let us suppose that  $i < j$ . Since  $i, j > k$ , then  $j > k + 1$ . Hence, each  $j$ -edge  $d_h$  of  $\beta$  ( $1 \leq h \leq l$ ) has vertices  $v_{2h-1}$  and  $v_{2h}$  belonging to the same  $(\Delta_n - \{k, k+1\})$ -residue; so  $\tau_{i_{2h-1}}^{(k)} = \tau_{i_{2h}}^{(k)}$ , by Lemma 3.c. By substitution, the subword  $(x_{i_{2h-1}}^{(k)})^{-1} x_{i_{2h}}^{(k)}$  becomes  $(x_{i'_{2h-1}}^{(k+1)})^{-1} \tau_{i_{2h-1}}^{(k)} (\tau_{i_{2h}}^{(k)})^{-1} x_{i'_{2h}}^{(k+1)} = (x_{i'_{2h-1}}^{(k+1)})^{-1} x_{i'_{2h}}^{(k+1)}$  and therefore  $r_\beta^{(k)}$  becomes  $(\tau_{i_0}^{(k)})^{-1} r_\beta^{(k+1)} \tau_{i_{2l-1}}^{(k)} = (\tau_{i_0}^{(k)})^{-1} r_\beta^{(k+1)} \tau_{i_0}^{(k)}$ . So  $r_\beta^{(k)}$  is equivalent to  $r_\beta^{(k+1)}$ ; we can drop  $X_k, Z_k$  and replace  $\tilde{W}_{k+1}$  with  $W_{k+1}$ :

$$\begin{aligned} H_k(\Omega) &\cong \langle \vec{E}_k \cup X_k \cup X_{k+1} \mid S_k \cup \tilde{W}_{k+1} \cup Z_k \rangle; \\ &\cong \langle \vec{E}_k \cup X_{k+1} \mid S_k \cup W_{k+1} \rangle. \end{aligned}$$

8) — Decomposition in a free product — Since every relator of  $S_k$  [resp. of  $W_{k+1}$ ] only contains elements of  $\vec{E}_k$  [resp. of  $X_{k+1}$ ], we can decompose the group as a free product:

$$\begin{aligned} H_k(\Omega) &\cong \langle \vec{E}_k \cup X_{k+1} \mid S_k \cup W_{k+1} \rangle; \\ &\cong \langle \vec{E}_k \mid S_k \rangle * \langle X_{k+1} \mid W_{k+1} \rangle; \\ &\cong G_k(\Omega) * H_{k+1}(\Omega). \end{aligned}$$

This completes the proof. ■

Now we give the decomposition theorem for the vertex group of gems and, as a consequence, we obtain the desired correlation with the fundamental group of the represented manifolds.

**Theorem 1 (Decomposition Theorem)** *For an  $n$ -gem  $\Omega$  we have:*

$$\mathcal{H}(\Omega) \cong G_0(\Omega) * G_1(\Omega) * \cdots * G_{n-1}(\Omega) * F_n(\Omega).$$

**Proof:** Repeatedly apply Proposition 2:

$$\begin{aligned} \mathcal{H}(\Omega) &= H_0(\Omega); \\ &\cong G_0(\Omega) * H_1(\Omega); \\ &\cong G_0(\Omega) * G_1(\Omega) * H_2(\Omega); \\ &\quad \dots \\ &\cong G_0(\Omega) * G_1(\Omega) * \cdots * G_{n-1}(\Omega) * H_n(\Omega); \\ &\cong G_0(\Omega) * G_1(\Omega) * \cdots * G_{n-1}(\Omega) * F_n(\Omega). \end{aligned}$$

This establishes the Theorem. ■

**Corollary 1** *If  $\Omega$  is an  $n$ -gem representing the  $n$ -manifold  $M$  ( $n \geq 2$ ), then:*

$$\mathcal{H}(\Omega) \cong \underbrace{\pi_1(M) * \pi_1(M) * \cdots * \pi_1(M)}_{n \text{ times}} * F, \quad (2)$$

where  $F$  is a free group with

$$\text{rank}(F) = \left( \sum_{k \in \Delta_n} \rho_k \right) - n.$$

In particular, if  $\Omega$  is a crystallization,

$$\mathcal{H}(\Omega) \cong \underbrace{\pi_1(M) * \pi_1(M) * \cdots * \pi_1(M)}_{n \text{ times}} * \mathbf{Z}$$

and the vertex group only depends on the topology of  $M$ .

**Proof:** By applying Decomposition Theorem and Proposition 2 (recall:  $F_n(\Omega)$  is a free group of rank  $\rho_n$ ). ■

Corollary 6 definitely solves the conjecture quoted in Section 2 and gives another proof that, in the category of crystallizations, the vertex group is an invariant of the represented manifolds (see [Lin89]).

**Remark 2** *Formula (2) does not hold for arbitrary coloured graphs. In fact, let us consider the 4-coloured graph  $\Gamma$  of Figure 4, representing the suspension  $\Sigma(\mathbf{RP}^2)$  of the real projective plane. Then  $\mathcal{H}(\Gamma) \cong \mathbf{Z} * \mathbf{Z}_2$ , whereas  $\pi_1(\Sigma(\mathbf{RP}^2))$  is the trivial group.*

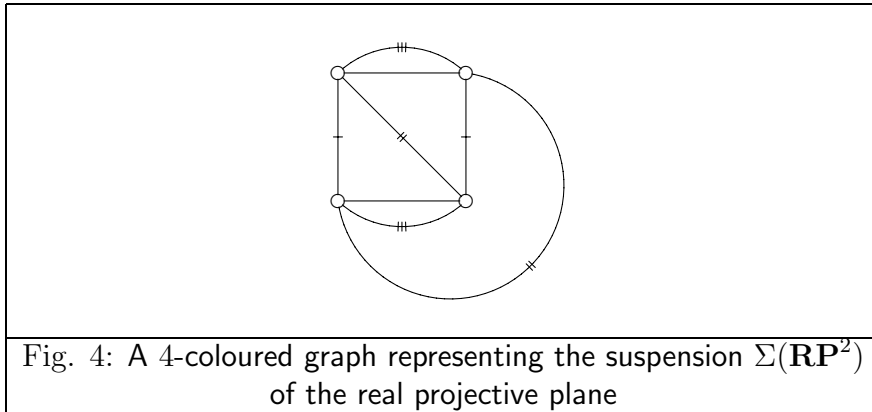


Fig. 4: A 4-coloured graph representing the suspension  $\Sigma(\mathbf{RP}^2)$  of the real projective plane

The total number of  $n$ -residues of an  $n$ -gem does not depend on the ordering of its colour-set. Therefore, by Corollary 6, the vertex group is independent too, up to isomorphism.

**Corollary 2** *Let  $\Omega$  be an  $n$ -gem, then the isomorphism class of  $\mathcal{H}(\Omega)$  does not depend on the ordering of the colour-set of  $\Omega$ .* ■

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