REMARKS AND CORRECTIONS TO

SPECTRAL THEORY OF NON-COMMUTATIVE HARMONIC OSCILLATORS: AN INTRODUCTION

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- **Page 8, Exercise 2.1.1:** replace "the" by "that" so as to obtain "Show that formula (2.3) indeed holds."
- **Page 90, formula on line 3 of Remark 6.3.3:** remove $(2\pi)^{-n}$ from the right-hand side of the formula so as to obtain
$$\iint_{a_\mu(x,\xi)\le 1} dxd\xi = \frac{1}{2n}\int_{\mathbb{S}^{2n-1}} \frac{1}{a_\mu(\omega)^{2n/\mu}} d\omega$$
- **Page 90, formula (6.37):** add $(2\pi)^{-n}$ so as to obtain
$$(2\pi)^{-n}\frac{2n}{\mu}\frac{\Gamma(\frac{2n}{\mu})}{\Gamma(\frac{2n}{\mu}+1)}\sum_{j=1}^{N}\iint_{\ell_j(x,\xi)\le 1} dxd\xi$$
- **Pages 203–205:** Of course, (12.36) of Proposition 12.1.17 has to be split into two parts:
$$E_N^{\mathrm{w}}(h)E_N^{\mathrm{w}}(h)^* = I + h^{N+1}R_N^{\mathrm{w}}(h)$$
and
$$E_N^{\mathrm{w}}(h)^*E_N^{\mathrm{w}}(h) = I + h^{N+1}{R'_N}^{\mathrm{w}}(h)$$
with all the same properties then listed for $R_N^{\mathrm{w}}(h)$ and ${R'_N}^{\mathrm{w}}(h)$, and for $\tilde{R}_N^{\mathrm{w}}(h)$. Hence, in particular,
$$\|{R'_N}^{\mathrm{w}}\|_{L^2\to L^2} \le C_N,\ \forall h\in(0,h_0],$$
and $R'_N \in S_0^0(m^{-2(N+1)}, g; \mathsf{M}_2)$.

It follows that (12.40) of Theorem 12.1.18 has to be changed in
$$E(h)E(h)^* = I,\ E(h)^*E(h) > 0 \textit{ and invertible with a bounded inverse},\ \forall h\in(0,h_0].$$

One in fact proceeds as in the proof of Thm. 12.1.18 and sees that, after having constructed $E(h)$ and $E(h)^*$ such that $E(h)E(h)^* = I$ and the other properties, one then has
$$E(h)^*E(h) = I + h^5R'(h),\ \ h\in(0,h_0],$$
with $\|R'(h)\|_{L^2\to L^2} \le C$ for all $0 < h \le h_0$ and for some $C > 0$ independent of h. Therefore by suitably shrinking h_0 we have that

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$E(h)^*E(h)$ is positive and invertible with a bounded inverse, for all $h \in (0, h_0]$.

Hence the eigenvalue problem for $A_r^{\mathrm{w}}(x, hD) =: A_r^{\mathrm{w}}(h)$ is equivalent to the eigenvalue problem for $\Lambda^{\mathrm{w}}(h)$ modulo $O_{L^2 \to L^2}(h^5)$ for all $h \in (0, h_0]$. In fact, setting $F(h) = E(h)^*E(h)$, if $A_r^{\mathrm{w}}(h)u = \lambda u$ (some $0 \neq u \in B^2$) then $E(h)\Lambda^{\mathrm{w}}(h)E(h)^*u + h^5\tilde{R}^{\mathrm{w}}(h)u = \lambda u$, i.e., on putting $w = E(h)^*u$ (then $0 \neq w \in B^2$),

$$E(h)\Big(\Lambda^{\mathrm{w}}(h) - \lambda + h^5 E(h)^* \tilde{R}^{\mathrm{w}}(h) E(h)\Big) E(h)^* u = 0,$$

which gives, since $\operatorname{Ker} E(h) = \{0\}$,

$$\Lambda^{\mathrm{w}}(h)w = \lambda w + O_{L^2 \to L^2}(h^5)w.$$

Conversely, if $\Lambda^{\mathrm{w}}(h)u = \lambda u$ (some $0 \neq u \in B^2$), then we put $u = E(h)^*w$ (then $0 \neq w \in B^2$) and write

$$E(h)\Lambda^{\mathrm{w}}(h)E(h)^*w = \lambda w,$$

so that

$$A_r^{\mathrm{w}}(h)w - h^5\tilde{R}^{\mathrm{w}}(h)w = \lambda w$$

which concludes the proof of the claim.