REMARKS AND CORRECTIONS TO
INTRODUCTION TO THE SPECTRAL THEORY
OF NON-COMMUTATIVE HARMONIC OSCILLATORS

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• Page 12, line 8 below formula (2.2.1): \( \text{Ker}(H^* \pm i) = \{0\} \) (in fact, \( \text{Im}(H \mp i) = L^2(\mathbb{R}) \)).

• Page 17, definition of confined symbols: With \( B^g_{X_0,r} = \{X; \, g_{X_0}(X - X_0) < r^2\} \), following Bony and Lerner [2] we say that \( a \in C^\infty(\mathbb{R}^{2n}) \) is a symbol of weight \( m \) confined to the ball \( B^g_{X_0,r} \), and write \( a \in \text{Conf}(m,g,X_0,r) \), if for all \( k \in \mathbb{Z}_+^n \)

\[
\|a\|_{k,\text{Conf}(m,g,X_0,r)} := \sup_{\ell \leq k, \, X \in \mathbb{R}^{2n}} \frac{|a|^{g_{X_0}}(X)}{m(X_0)} (1 + g_{X_0}^2(X - B_{X_0,r}))^{k/2} < +\infty,
\]

where \( g_{\sigma Y}^2(X - B) = \inf_{Z \in B^g_{X_0,r}} g_{Y}(X - Z) \). Hence the space of symbols confined to the ball \( B^g_{X_0,r} \) coincides with \( S(\mathbb{R}^{2n}) \) endowed with the seminorms (1). Any given \( \varphi \in C^\infty_0(B^g_{X_0,r}) \) is automatically confined to the ball \( B^g_{X_0,r} \).

• Page 19, Formula (3.1.5):

\[
c(X) = \pi^{-2n} \int \int e^{-2i\sigma(X-Y,X-Z)}a(Y)b(Z)dYdZ = \pi^{-2n} \int \int e^{-2i\sigma(T,Z)}a(X+T)b(X+Z)dTdZ = e^{i\sigma(D_X;D_Y)/2}(a(X)b(Y))|_{X=Y},
\]

• Page 31, Remark 3.2.10: I have used here the notation \( \text{co}a \) for the cofactor matrix of \( a \), that is the matrix whose entries are the cofactors of \( a \). Hence \( \text{co}a a = a \text{co}a = \det a \).

• Page 40, inner-product of \( B^s \): \( (u,v)_s := \langle \Lambda^s u, \Lambda^s v \rangle_0 \).

• Page 49, line 6 of the proof of Lemma 3.3.13: dist should be intended here as

\[
\inf_{\zeta \in \text{Spec}(g_{\omega/2}(\omega))} |\zeta - \zeta'| \geq c_0 > 0, \quad \forall \omega \in S^{2n-1}.
\]

• Page 50, Remark 3.3.15: One has to take \( \mu = 2 \).

• Page 61, line 5 of the proof of Lemma 4.3.3: \( \alpha \in \mathbb{Z}_+^n \).

• Page 61, line -5 of the proof of Lemma 4.3.3: \( j > 1 \).
Page 68, formula (5.2.1): the norm on the left-hand side of the inequality is \( \| \varphi_j \|_{L^p_r}^2 \).

Page 69, definition of \( D(f(A)) \): the condition is 
\[
\sum_{j \geq 1} |f(\lambda_j)|^2 |u_j|^2 < +\infty.
\]

Page 80: \( \Tr B_{-2j}(t) = \beta'_{-2j}(t) - \beta''_{-2j}(t) \).

Page 93, Formula for \( \Res(\zeta_A, s_j) \): the integration contour \( \Gamma \) should be replaced by \( \gamma \).

Page 93, Exercise 7.1.2: "Upon" should be replaced by "Upone".

Page 94, statement of Thm. 7.2.1: the given statement is somewhat imprecise. Here is the correct statement of the meromorphic extension of the spectral zeta:

**Theorem 7.2.1.** There exist constants \( c_{-2j,n}, 0 \leq j \leq n - 1 \), and constants \( C_j, j \geq n \), such that for any given integer \( \nu \in \mathbb{Z}_+ \) with \( \nu \geq n \), \( \zeta_A(s) \) is represented as

\[
(7.2.1) 
\zeta_A(s) = \frac{1}{\Gamma(s)} \sum_{j=0}^{n-1} \frac{c_{-2j,n}}{s-n+j} + \sum_{j=n}^{\nu} \frac{C_j}{s-n+j} + H_\nu(s), \quad \Re s > -(\nu-n)-1,
\]

where \( \Gamma(s) \) is the Euler gamma function, and \( H_\nu \) is holomorphic in \( \Re s > -(\nu-n)-1 \). Consequently, the spectral zeta function \( \zeta_A(s) \) is meromorphic in the whole complex plane \( \mathbb{C} \) with at most simple poles at \( s = n, n-1, n-2, \ldots, 1 \). In particular, when \( n = 1 \), \( \zeta_A \) has only one simple pole at \( s = 1 \). One has (recall (6.3.12))

\[
(7.2.2) 
eq \quad c_{-2j,n} = (2\pi)^{-n} \int_0^{+\infty} \int_{\mathbb{S}^{2n-1}} \Tr b_{-2j}(1, \rho \omega) \rho^{2n-1} d\rho d\omega, \quad 0 \leq j \leq n - 1,
\]

where the \( b_{-2j} \) are the terms in the symbol of the parametrix \( U_A \in \text{OPS}_3(2,0) \) of Theorem 6.1.5, Remark 6.1.6 and (6.1.9),

\[
U_A \sim \sum_{j \geq 0} B_{-2j}.
\]

Moreover,

\[
c_{0,n} = (2\pi)^{-n} \int_0^{+\infty} \int_{\mathbb{S}^{2n-1}} \Tr (e^{-a_2(\omega)}) \rho^{2n-1} d\rho d\omega
\]

is the leading coefficient in the Weyl asymptotics for \( N_A(\lambda) \) (see (6.3.14) and Theorem 6.3.1).

Page 98, line -6: \( s = -k \) should be replaced by \( s = -j - k \).

Page 99, formula (7.2.14): there is an extra summation in \( k \). The correct formula reads as follows

\[
(7.2.14) 
Z_0(s) = \frac{1}{\Gamma(s)} \sum_{j=0}^{\nu} \frac{c_{-2j,n}}{s-n+j} + \sum_{j,k=0}^{\nu} \frac{f_{j+k}(0)}{(j+k)!} \frac{1}{s+j+k} + \sum_{k=0}^{\nu} \frac{f^{(k)}(0)}{k!} \frac{1}{s+k} + \sum_{j=0}^{\nu} F_{j,\nu}(s) + F_{R,\nu}(s) + F_{\nu+1}(s)
\]
\[(7.2.15) \quad = \frac{1}{\Gamma(s)} \left[ \sum_{j=0}^{n-1} \frac{c_{-2j,n}}{s-n+j} + \sum_{j=n}^{\nu} \frac{C_j}{s-n+j} + H_{\nu}(s) \right],\]

- **Page 101, line 2:** the factor $e^{-tA}$ in front of the term $R(0)$ is missing.
- **Page 111, lines -8,-9:** one sets \[\left[ u_1, \ldots, u_{n-1} \right]^\perp := \text{Span}(u_1, \ldots, u_{n-1})^\perp \cap D(Q)\]
  the elements in the domain $D(Q)$ of $Q$ which are orthogonal to the subspace spanned by the vectors $u_1, \ldots, u_{n-1}$.
- **Page 120, line 1 below formula (9.1.4):** $a \in S_k^0(m^\mu; M_N)$.
- **Page 126, bottom line:** $r_{N_0+1} \in S_0^0(m^{-\mu}; M_N)$.
- **Page 140, line -6:** one clearly has $h^{-k_1}$ instead of $h^{-3k_0}$, in the inequality.
- **Page 184, line 3 of the proof:** $\xi(\theta) = \sqrt{2} \rho_\pm(\theta) \cos \theta$.
- **Page 201, Hypothesis (H5):** $c_0 > 0$ should be replaced by $c_0 \in \mathbb{R}$.
- **Page 203, line -2:** $c_1$ should be replaced by $c_0$. 