REMARKS AND CORRECTIONS TO

INTRODUCTION TO THE SPECTRAL THEORY OF NON-COMMUTATIVE HARMONIC OSCILLATORS

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- **Page 12, line 8 below formula (2.2.1)**: $\mathrm{Ker}(H^* \pm i) = \{0\}$ (in fact, $\mathrm{Im}(H \mp i) = L^2(\mathbb{R})$).
- **Page 17, definition of confined symbols**: *With* $B^g_{X_0,r} = \{X;\ g_{X_0}(X - X_0) < r^2\}$*, following Bony and Lerner [2] we say that* $a \in C^\infty(\mathbb{R}^{2n})$ *is a* symbol of weight m confined *to the ball* $B^g_{X_0,r}$*, and write* $a \in \mathrm{Conf}(m, g, X_0, r)$*, if for all* $k \in \mathbb{Z}_+$

(1)
$$\|a\|_{k,\mathrm{Conf}(m,g,X_0,r)} := \sup_{\ell \le k,\ X \in \mathbb{R}^{2n}} \frac{|a|^{g_{X_0}}_\ell(X)}{m(X_0)} \big(1 + g^\sigma_{X_0}(X - B_{X_0,r})\big)^{k/2} < +\infty,$$

where $g^\sigma_Y(X - B) = \inf_{Z \in B} g^\sigma_Y(X - Z)$*. Hence the space of symbols confined to the ball* $B^g_{X_0,r}$ *coincides with* $\mathcal{S}(\mathbb{R}^{2n})$ *endowed with the seminorms (1). Any given* $\varphi \in C^\infty_0(B^g_{X_0,r})$ *is automatically confined to the ball* $B^g_{X_0,r}$*.*

- **Page 19, Formula (3.1.5)**:

$$c(X) = \pi^{-2n} \iint e^{-2i\sigma(X-Y,X-Z)} a(Y) b(Z) dY dZ =$$

$$= \pi^{-2n} \iint e^{-2i\sigma(T,Z)} a(X+T) b(X+Z) dT dZ = e^{i\sigma(D_X;D_Y)/2}(a(X)b(Y))\big|_{X=Y},$$

- **Page 31, Remark 3.2.10**: I have used here the notation ${}^{\mathrm{co}}a$ for the cofactor matrix of a, that is the matrix whose entries are *the cofactors of* ${}^t a$. Hence ${}^{\mathrm{co}}a\, a = a\, {}^{\mathrm{co}}a = \det a$.
- **Page 40, inner-product of B^s**: $(u, v)_s := (\Lambda^s u, \Lambda^s v)_0$.
- **Page 49, line 6 of the proof of Lemma 3.3.13**: dist should be intended here as

$$\inf_{\substack{\zeta \in \mathrm{Spec}(q_{\mu/2}(\omega)) \\ \zeta' \in \mathrm{Spec}(-q_{\mu/2}(\omega))}} |\zeta - \zeta'| \ge c_0 > 0, \quad \forall \omega \in \mathbb{S}^{2n-1}.$$

- **Page 50, Remark 3.3.15**: One has to take $\mu = 2$.
- **Page 61, line 5 of the proof of Lemma 4.3.3**: $\alpha \in \mathbb{Z}^n_+$.
- **Page 61, line -5 of the proof of Lemma 4.3.3**: $j > 1$.

- **Page 68, formula (5.2.1)**: the norm on the left-hand side of the inequality is $\|\varphi_j\|^2_{B^{\mu r}}$.
- **Page 69, definition of $D(f(A))$**: the condition is
$$\sum_{j\geq 1} |f(\lambda_j)|^2 |u_j|^2 < +\infty.$$
- **Page 80**: $\operatorname{Tr} B_{-2j}(t) = \beta'_{-2j}(t) - \beta''_{-2j}(t)$.
- **Page 93, Formula for $\operatorname{Res}(\zeta_A, s_j)$**: the integration contour Γ should be replaced by γ.
- **Page 93, Exercise 7.1.2**: "*Upone*" should be replaced by "*Upon*".
- **Page 94, statement of Thm. 7.2.1**: the given statement is somewhat imprecise. Here is the correct statement of the meromorphic extension of the spectral zeta:

Theorem 7.2.1. *There exist constants $c_{-2j,n}$, $0 \leq j \leq n-1$, and constants C_j, $j \geq n$, such that for any given integer $\nu \in \mathbb{Z}_+$ with $\nu \geq n$, $\zeta_A(s)$ is represented as*

(7.2.1)
$$\zeta_A(s) = \frac{1}{\Gamma(s)}\Big[\sum_{j=0}^{n-1} \frac{c_{-2j,n}}{s-n+j} + \sum_{j=n}^{\nu} \frac{C_j}{s-n+j} + H_\nu(s)\Big], \qquad \operatorname{Re} s > -(\nu-n)-1,$$

where $\Gamma(s)$ is the Euler gamma function, and H_ν is holomorphic in $\operatorname{Re} s > -(\nu-n)-1$. Consequently, the spectral zeta function $\zeta_A(s)$ is meromorphic in the whole complex plane $\mathbb{C}$ with at most simple poles at $s = n, n-1, n-2, \ldots, 1$. In particular, when $n = 1$, ζ_A has only one simple pole at $s = 1$. One has (recall (6.3.12))

(7.2.2)
$$c_{-2j,n} = (2\pi)^{-n} \int_0^{+\infty} \int_{\mathbb{S}^{2n-1}} \mathsf{Tr}\, b_{-2j}(1, \rho\omega) \rho^{2n-1} d\rho d\omega, \quad 0 \leq j \leq n-1,$$

where the b_{-2j} are the terms in the symbol of the parametrix $U_A \in \mathrm{OP}S_{\mathrm{cl}}(2,0)$ of Theorem 6.1.5, Remark 6.1.6 and (6.1.9),
$$U_A \sim \sum_{j\geq 0} B_{-2j}.$$

Moreover,
$$c_{0,n} = (2\pi)^{-n} \int_0^{+\infty} \int_{\mathbb{S}^{2n-1}} \mathsf{Tr}(e^{-a_2(\rho\omega)}) \rho^{2n-1} d\rho d\omega$$
is the leading coefficient in the Weyl asymptotics for $N_A(\lambda)$ (see (6.3.14) and Theorem 6.3.1).

- **Page 98, line -6**: $s = -k$ should be replaced by $s = -j-k$.
- **Page 99, formula (7.2.14)**: there is an extra summation in k. The correct formula reads as follows

(7.2.14)
$$Z_0(s) = \frac{1}{\Gamma(s)}\Big[\sum_{j=0}^{\nu} \frac{c_{-2j,n}}{s-n+j} + \sum_{j,k=0}^{\nu} \frac{f_j^{(j+k)}(0)}{(j+k)!} \frac{1}{s+j+k} + \sum_{k=0}^{\nu} \frac{f_R^{(k)}(0)}{k!} \frac{1}{s+k} +$$
$$+ \sum_{j=0}^{\nu} F_{j,\nu}(s) + F_{R,\nu}(s) + F_{\nu+1}(s)\Big] =$$

$$= \frac{1}{\Gamma(s)}\Big[\sum_{j=0}^{n-1} \frac{c_{-2j,n}}{s-n+j} + \sum_{j=n}^{\nu} \frac{C_j}{s-n+j} + \tilde{H}_\nu(s)\Big], \tag{7.2.15}$$

- **Page 101, line 2:** the factor e^{-tA} in front of the term $R(0)$ is missing.
- **Page 111, lines -8,-9:** one sets

  $$[u_1, \dots, u_{n-1}]^\perp := \mathrm{Span}(u_1, \dots, u_{n-1})^\perp \cap D(Q)$$

  the elements in the domain $D(Q)$ of Q which are orthogonal to the subspace spanned by the vectors $u_1, \dots, u_{n-1}$.
- **Page 120, line 1 below formula (9.1.4):** $a \in S_0^k(m^\mu; \mathsf{M}_N)$.
- **Page 126, bottom line:** $r_{N_0+1} \in S_0^0(m^{-\mu}; \mathsf{M}_N)$.
- **Page 140, line -6:** one clearly has h^{-k_1} instead of h^{-3k_0}, in the inequality.
- **Page 184, line 3 of the proof:** $\xi(\theta) = \sqrt{2}\,\rho_\pm(\theta)\cos\theta$.
- **Page 201, Hypothesis** $(H5')$: $c_0 > 0$ should be replaced by $c_0 \in \mathbb{R}$.
- **Page 203, line -2**: c_1 should be replaced by c_0.