SIAM J. MATH. ANAL.
Vol. 35, No. 3, pp. 579–595


# ON THE CAUCHY PROBLEM FOR A NONLINEAR KOLMOGOROV EQUATION*

ANDREA PASCUCCI† AND SERGIO POLIDORO†

**Abstract.** We consider the Cauchy problem related to the partial differential equation

$$Lu \equiv \Delta_x u + h(u)\partial_y u - \partial_t u = f(\cdot, u),$$

where $(x, y, t) \in \mathbb{R}^N \times \mathbb{R} \times \ ]0, T[$, which arises in mathematical finance and in the theory of diffusion processes. We study the regularity of solutions regarding $L$ as a perturbation of an operator of Kolmogorov type. We prove the existence of local classical solutions and give some sufficient conditions for global existence.

**Key words.** nonlinear degenerate Kolmogorov equation, interior regularity, Hörmander operators

**AMS subject classifications.** 35K57, 35K65, 35K70

**DOI.** 10.1137/S0036141001399349

## 1. Introduction.

In this paper we study the Cauchy problem

$$Lu = f(\cdot, u) \qquad \text{in } \ S_T \equiv \mathbb{R}^{N+1} \times \ ]0, T[, \tag{1.1}$$

$$u(\cdot, 0) = g \qquad \text{in } \ \mathbb{R}^{N+1}, \tag{1.2}$$

where $L$ is the nonlinear operator defined by

$$Lu = \Delta_x u + h(u)\partial_y u - \partial_t u, \tag{1.3}$$

$(x, y, t) = z$ denotes the point in $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$, and $\Delta_x$ is the Laplace operator acting in the variable $x \in \mathbb{R}^N$. We assume that $f, g$, and $h$ are globally Lipschitz continuous functions.

One of the main features of operator (1.3) is the strong degeneracy of its characteristic form due to the lack of diffusion in the $y$-direction, so that (1.1)–(1.2) may include the Cauchy problem for the Burgers equation, when $h(u) = u$, $g = g(y)$, and $f \equiv 0$. On the other hand, $L$ can be considered as nonlinear version of the operator

$$K = \Delta_x + x_1\partial_y - \partial_t, \tag{1.4}$$

which was introduced by Kolmogorov [17] and has been extensively studied by Kuptsov [12] and Lanconelli and Polidoro [14]. Among the known results of $K$, we recall that every solution to $Ku = 0$ is smooth; thus we may expect some regularity properties also for the solutions to (1.1).

Problem (1.1)–(1.2) arises in mathematical finance as well as in the study of nonlinear physical phenomena such as the combined effects of diffusion and convection of matter.

---

*Received by the editors December 10, 2001; accepted for publication (in revised form) March 21, 2003; published electronically October 2, 2003. This research was supported by University of Bologna funds for selected research topics.
http://www.siam.org/journals/sima/35-3/39934.html

†Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40127 Bologna, Italy (pascucci@dm.unibo.it, polidoro@dm.unibo.it).

Escobedo, Vazquez, and Zuazua [8] prove that there exists a unique distributional solution to (1.1)–(1.2) satisfying an entropy condition that generalizes the one by Kruzhkov [11]. This solution is characterized in the *vanishing viscosity* sense; i.e., it is the limit of a sequence of classical solutions to Cauchy problems related to the regularized operator

$$L_\varepsilon u = \Delta_x u + \varepsilon^2 \partial_y^2 u + h(u)\partial_y u - \partial_t u. \tag{1.5}$$

Vol'pert and Hudjaev [19] prove similar existence and uniqueness results in a space of bounded variation functions whose spatial derivatives are square integrable with respect to (w.r.t.) a suitable weight. In this framework, it is natural to consider bounded and integrable initial data $g$ and nonlinearities of the form $h(u) = u^{p-1}$ for $p \in \left]1, \frac{N+2}{N+1}\right[$.

Our paper is mainly motivated by the theory of agents' decisions under risk, arising in mathematical finance. The problem is the representation of agents' preferences over consumption processes. Antonelli, Barucci, and Mancino [1] propose a utility functional that takes into account aspects of decision making such as the agents' habit formation, which is described as a smoothed average of past consumption and expected utility. In that model the processes utility and habit are described by a system of backward-forward stochastic differential equations. The solution of such a system, as a function of consumption and time, satisfies the Cauchy problem (1.1)–(1.2). Our regularity assumption on $f, g, h$ is required by the financial model, since these functions appear in the backward-forward system as Lipschitz continuous coefficients.

In the paper by Antonelli and Pascucci [2] an existence result, in the case $N = 1$, is proved by probabilistic techniques that exploit the properties of the solutions to the system of backward-forward stochastic differential equations related to (1.1)–(1.2). In [2], the existence of a viscosity solution, in the sense of [7], is proved. The solution is defined in a suitably small strip $\mathbb{R}^2 \times [0, T]$ and satisfies the following conditions:

$$\begin{aligned} |u(x,y,t) - u(x',y',t)| &\le c_0(|x-x'| + |y-y'|), \\ |u(x,y,t) - u(x,y,t')| &\le c_0(1 + |(x,y)|)|t-t'|^{\frac{1}{2}} \end{aligned} \tag{1.6}$$

for every $(x,y), (x',y') \in \mathbb{R}^2, t, t' \in [0,T]$, where $c_0$ is a positive constant that depends on the Lipschitz constants of $f, g$, and $h$. Concerning the regularity of $u$, we remark that the results by Caffarelli and Cabré [3] and Wang [20, 21] do not apply to our operator.

In this paper we prove the existence of a classical solution $u$ to problem (1.1)–(1.2) by combining the analysis on Lie groups with the standard techniques for the Cauchy problem related to degenerate parabolic equations. We say that $u$ is a classical solution if $\partial_{x_j x_k} u$, $j, k = 1, \dots, N$, the directional derivative

$$z \longmapsto Yu(z) = \frac{\partial u}{\partial \nu_z}(z), \qquad \nu(z) = (0, h(u(z)), -1),$$

are continuous functions, and (1.1)–(1.2) are verified at every point. Our main result is the following.

THEOREM 1.1. *There exists a positive $T$ and a unique function $u \in C(S_T)$, verifying estimates* (1.6) *on $\overline{S}_T$, which is a classical solution to* (1.1)–(1.2).

We stress that the regularity stated above is natural for the problem under consideration. Indeed, although $Yu$ is the sum of the more simple terms $h(u)\partial_y u$ and

$\partial_t u$, it is not true in general that they are continuous functions. Further regularity properties of solutions can be obtained under additional conditions. For instance, in [5, 6] in collaboration with Citti, we considered the nonlinear equation in three variables,

$$\partial_{xx} u + u \partial_y u - \partial_t u = 0, \tag{1.7}$$

which is a special case of (1.1). Assuming a hypothesis formally analogous to the classical Hörmander condition, we proved that the viscosity solution $u$ of (1.7) constructed in [2] actually is a $C^\infty$ classical solution.

In this paper we give a direct proof of the existence of a classical solution to the Cauchy problem (1.1)–(1.2) by using analytical methods. The regularity part in Theorem 1.1 is based on a modification of the classical freezing method, introduced by Citti in [4] for the study of the Levi equation. More precisely, for any $\bar{z} \in S_T$, we approximate $L$ by the linear operator

$$L_{\bar{z}} = \Delta_x + (h(u(\bar{z})) + x_1 - \bar{x}_1)\, \partial_y - \partial_t, \tag{1.8}$$

and we represent a solution $u$ in terms of its fundamental solution. Note that up to a straightforward change of coordinates, $L_{\bar{z}}$ is the Kolmogorov operator (1.4), and hence an explicit expression of the fundamental solution of $L_{\bar{z}}$ is available. Also note that $L_{\bar{z}}$ is a good approximation of $L$ in the sense that, by (1.6), we have

$$|Lu(z) - L_{\bar{z}} u(z)| = |u(z) - u(\bar{z}) - (x_1 - \bar{x}_1)|\, |\partial_y u(z)| \le c_0 d(\bar{z}, z),$$

where $d(\bar{z}, z)$ is the standard parabolic distance.

The existence part of Theorem 1.1 relies on the Bernstein technique. We explicitly note that the nonlinearity in (1.3) is not monotone; therefore a maximum principle for the operator $Lv + h'(u)v^2$, which occurs when we differentiate both sides of (1.1) w.r.t. $y$, does not hold unless we assume condition (1.6).

We end this introduction with a remark about the existence of global solutions. We first note that the space of functions characterized by conditions (1.6) is, in some sense, optimal for the existence of local classical solutions. Indeed the linear growth of the initial data $g$ does not allow, in general, solutions that are defined at every time $t > 0$, as the following example given in [2] shows. Consider the problem (1.7), with $f \equiv 0$ and $g(x, y) = x + y$: a direct computation shows that $u(x, y, t) = \frac{x+y}{1-t}$ is the unique solution to the problem and blows up as $t \to 1$. This fact is expected since, if $u$ grows as a linear function, then its Cole–Hopf transformed function grows as $\exp(y^2)$, which is the critical case for the parabolic Cauchy problem. Next we give a simple sufficient condition for the global existence of classical solutions.

THEOREM 1.2. *Let $f, g$, and $h$ be globally Lipschitz continuous functions. Suppose that $g$ is nonincreasing w.r.t. $y$, that $f$ is nondecreasing w.r.t. $y$, and that there exists* $c_0 \in\ ]0, c_1]$ *such that*

$$c_0(u - v) \le h(u) - h(v) \tag{1.9}$$

*for every $u, v \in \mathbb{R}$. Then the Cauchy problem* (1.1)–(1.2) *has a solution $u$ that is defined in* $\mathbb{R}^{N+1} \times \mathbb{R}^+$.

This paper is organized as follows. In section 2 we prove Theorem 1.1, and in section 3 we prove the existence of a viscosity solution of (1.1)–(1.2). Section 4 is devoted to the proof of Theorem 1.2.

**2. Classical solutions.** In this section we prove Theorem 1.1. We first state an existence and uniqueness result of a strong solution $u$ to problem (1.1)–(1.2). And then we prove that $u$ is a solution in the classical sense. We say that a continuous function $u$ is a strong solution to (1.1)–(1.2) if $u \in H^1_{\text{loc}}(S_T)$, $\partial_{x_j x_k} u \in L^2_{\text{loc}}(S_T)$, $j, k = 1, \dots, N$, it satisfies equation (1.1) a.e., and it assumes the initial datum $g$.

THEOREM 2.1. *If $T$ is suitably small, there exists a unique strong solution of* (1.1)–(1.2) *verifying estimates* (1.6) *on* $\overline{S}_T$.

The proof of Theorem 2.1 is postponed to section 3. We remark that in the above statement, we consider the term

$$Yu = h(u)\partial_y u - \partial_t u$$

as a sum of weak derivatives. Here we aim to prove that $Yu$ is a continuous function and that it coincides with the directional derivative w.r.t. the vector $\nu_z = (0, h(u), -1)$, namely,

$$Y(u(z)) = \frac{\partial u}{\partial \nu_z}(z) \qquad \forall z \in S_T. \tag{2.1}$$

In what follows, when we consider a function $F$ that depends on many variables, to avoid any ambiguity we shall systematically write the directional derivative introduced in (2.1) as

$$Y(z)F(\cdot, \zeta) = \frac{\partial F(\cdot, \zeta)}{\partial \nu_z}(z).$$

Our technique is inspired by the recent paper [6], where, in collaboration with Citti, we developed some ideas for a general study of a nonlinear equation of the form (1.4). We recall the following lemma, which has been proved in Lemma 3.1 of [6], for the Cauchy problem (1.7). We state the lemma for the operator (1.3) and omit the proof, since it is analogous to the one given in [6].

LEMMA 2.2. *Let $v$ be a continuous function defined in $S_T$. Assume that its weak derivatives $v_y, v_t \in L^2_{loc}$ and that the limit*

$$\lim_{\delta \to 0} \frac{v(z + \delta \nu_z) - v(z)}{\delta}$$

*exists and is uniform w.r.t. $z$ in compact subsets of $S_T$. Then*

$$\frac{\partial v}{\partial \nu_z}(z) = (h(u)\partial_y v - \partial_t v)(z) \qquad a.e.\ z \in S_T.$$

We next prove Theorem 1.1 by using a representation formula of the strong solution $u$ in terms of the fundamental solution of the operator $L_{\bar{z}}$ introduced in (1.8). We define the first order operators (vector fields)

$$X_j = \partial_{x_j},\ j = 1, \dots, N, \qquad Y_{\bar{z}} = \left(h(u(\bar{z})) + x_1 - \bar{x}_1\right) \partial_y - \partial_t. \tag{2.2}$$

Thus we can rewrite the operator $L_{\bar{z}}$ in the standard form

$$L_{\bar{z}} = \sum_{j=1}^{N} X_j^2 + Y_{\bar{z}}. \tag{2.3}$$

Let us recall some preliminary facts about real analysis on nilpotent Lie groups. More details about this topic can be found in [15] and [18]. We define on $\mathbb{R}^{N+2}$ the composition law

$$\theta \oplus \theta' = \left(\theta_1 + \theta'_1, \dots, \theta_N + \theta'_N, \theta_{N+1} + \theta'_{N+1}, \theta_{N+2} + \theta'_{N+2} + \frac{1}{2}(\theta_1 \theta'_{N+1} - \theta_{N+1}\theta'_1)\right)$$

and the dilations group

$$\delta_\lambda(\theta) = (\lambda\theta_1, \dots, \lambda\theta_N, \lambda^2\theta_{N+1}, \lambda^3\theta_{N+2}), \qquad \lambda > 0.$$

We remark that $G = (\mathbb{R}^{N+2}, \oplus)$ is a nilpotent stratified Lie group which, in the case $N = 1$, coincides with the standard Heisenberg group. Since the Jacobian $J\delta_\lambda$ equals $\lambda^{N+5}$, the homogeneous dimension of $G$ w.r.t. $(\delta_\lambda)_{\lambda>0}$ is the natural number $Q = N + 5$. A norm which is homogeneous w.r.t. this dilations group is given by

$$\|\theta\| = \left(|\theta_1|^6 + \cdots + |\theta_N|^6 + |\theta_{N+1}|^3 + |\theta_{N+2}|^2\right)^{\frac{1}{6}}.$$

Let $\nabla_{\bar z} = (X_1, \dots, X_N, Y_{\bar z}, \partial_y)$ be the gradient naturally associated to $L_{\bar z}$ and consider any $z \in \mathbb{R}^{N+2}$. The exponential map

$$E_{\bar z}(\theta, z) = \exp(\langle \theta, \nabla_{\bar z}\rangle)(z)$$

is a global diffeomorphism and induces a Lie group structure on $\mathbb{R}^{N+2}$ whose product is defined by

$$\zeta \circ z = E_{\bar z}\left(\left(E_{\bar z}^{-1}(\zeta, 0) \oplus E_{\bar z}^{-1}(z, 0)\right), 0\right),$$

and it can be explicitly computed as

$$\zeta \circ z = (x + \xi, y + \eta - t\xi_1, t + \tau).$$

Moreover, a control distance $d_{\bar z}$ in $(\mathbb{R}^{N+2}, \circ)$ is defined by

$$\begin{aligned} d_{\bar z}(z, \zeta) &= \|E_{\bar z}^{-1}\left(\zeta^{-1} \circ z, 0\right)\| \\ &= \left(|x - \xi|^6 + |t - \tau|^3 + \left|y - \eta + (t - \tau)\left(h(u(\bar z)) + \frac{x_1 - \xi_1 - 2\bar x_1}{2}\right)\right|^2\right)^{\frac{1}{6}}, \end{aligned} \tag{2.4}$$

where $\zeta^{-1}$ is the inverse in the group law "$\circ$". We denote by $\Gamma_{\bar z}(z, \zeta)$ the fundamental solution of $L_{\bar z}$ with pole in $\zeta$ and evaluated at $z$. We refer to [12, 14, 13, 9] for known results about $\Gamma_{\bar z}$. The following bound holds:

$$\Gamma_{\bar z}(z, \zeta) = \Gamma_{\bar z}(\zeta^{-1} \circ z, 0) \le c\, d_{\bar z}(z, \zeta)^{-Q+2}, \tag{2.5}$$

where the constant $c$ continuously depends on $\bar z$. We are now in a position to prove Theorem 1.1.

*Proof of Theorem* 1.1. By Theorem 2.1 there exists a unique strong solution of (1.1)–(1.2) verifying (1.6) in $\overline{S}_T$ for $T$ suitably small. In order to prove that $u$ is a classical solution, we represent it in terms of the fundamental solution $\Gamma_{\bar z}$:

$$(u\varphi)(z) = \int_{S_T} \Gamma_{\bar z}(z, \zeta)\left(U_{1,\bar z}(\zeta) - U_{2,\bar z}(\zeta)\right) d\zeta \equiv I_1(z) - I_2(z) \tag{2.6}$$

for every $\varphi \in C_0^\infty(S_T)$, where

$$U_{1,\bar{z}} = \varphi f(\cdot, u) + uL_{\bar{z}}\varphi + 2\langle \nabla_x u, \nabla_x \varphi\rangle,$$
$$U_{2,\bar{z}} = (h(u) - h(u(\bar{z})) - (x_1 - \bar{x}_1))\, \partial_y u\varphi$$

are bounded functions with compact support. Therefore it is straightforward to prove that $u\varphi \in C^{1+\alpha}_{d_{\bar{z}}}$, $\alpha \in ]0,1[$, where $C^{k+\alpha}_{d_{\bar{z}}}$ denotes the space of Hölder continuous functions w.r.t. the control distance $d_{\bar{z}}$. In particular, by choosing $\varphi \equiv 1$ in a compact neighborhood $K$ of $\bar{z}$, we have that

$$X_j u(z) = \int X_j(z)\Gamma_{\bar{z}}(\cdot, \zeta)\,(U_{1,\bar{z}}(\zeta) - U_{2,\bar{z}}(\zeta))\, d\zeta, \qquad z \in K,\ j = 1, \dots, N,$$

and

$$|X_j u(z) - X_j u(\zeta)| \le c d_{\bar{z}}(z, \zeta)^\alpha \qquad \forall z, \zeta \in K,\ \alpha \in ]0,1[. \tag{2.7}$$

This proves the Hölder continuity of the first order derivatives of $u$. Let us now consider the second order derivatives $X_j X_h u$, $j, k, = 1, \dots, N$, and $Yu$.

We next prove the existence of the directional derivative $Yu(\bar{z})$ by studying separately the terms $I_1, I_2$. Since $Y$ is the unique nonlinear vector field to be considered, the proof of our result for the derivatives $X_j X_h u$ is simpler and will be omitted.

*The term $I_2$.* We set

$$J(\bar{z}) = \int_{S_T} Y(\bar{z})\Gamma_{\bar{z}}(\cdot, \zeta) U_{2,\bar{z}}(\zeta) d\zeta.$$

We remark that $J$ is well defined and continuous since, by (1.6), we have

$$|U_{2,\bar{z}}(\zeta)| \le c\ d_{\bar{z}}(\bar{z}, \zeta). \tag{2.8}$$

We denote by $\chi \in C^\infty([0, +\infty[, [0,1])$ a cut-off function such that

$$\chi(s) = 0 \ \text{ for } \ 0 \le s \le \frac{1}{2}, \qquad \chi(s) = 1 \ \text{ for } \ s \ge 1,$$

and we set

$$I_{2,\delta}(z) = \int_{S_T} \Gamma_{\bar{z}}(z, \zeta)\chi\left(\frac{d_{\bar{z}}(\bar{z}, \zeta)}{\bar{c}\ \delta^{\frac{1}{2}}}\right) U_{2,\bar{z}}(\zeta) d\zeta, \qquad \bar{c}, \delta > 0.$$

In what follows we shall assume $d_{\bar{z}}(\bar{z}, z) \le \delta^{\frac{1}{2}}$; then by the triangular inequality

$$d_{\bar{z}}(\bar{z}, \zeta) \le c\ \left(d_{\bar{z}}(\bar{z}, z) + d_{\bar{z}}(z, \zeta)\right), \tag{2.9}$$

we can choose $\bar{c}$ suitably large so that

$$\chi\left(\frac{d_{\bar{z}}(\bar{z}, \zeta)}{\bar{c}\ \delta^{\frac{1}{2}}}\right) = 0 \qquad \text{if } \ d_{\bar{z}}(z, \zeta) < \delta^{\frac{1}{2}},$$

and, as a consequence, $I_{2,\delta}$ is smooth for any $\delta > 0$. We claim that

$$\sup_{d_{\bar{z}}(\bar{z}, z) \le \delta^{\frac{1}{2}}} |I_{2,\delta}(z) - I_2(z)| \le c\ \delta^{\frac{3}{2}}, \tag{2.10}$$

$$\sup_{d_{\bar{z}}(\bar{z}, z) \le \delta^{\frac{1}{2}}} |Y_{\bar{z}} I_{2,\delta}(z) - J(\bar{z})| \le c\ \delta^{\frac{1}{2}} |\log(\delta)| \tag{2.11}$$

for some positive constant $c$. We postpone the proof of (2.10)–(2.11) to the end.

Let us now compute the derivative $\frac{\partial I_2}{\partial \nu_{\bar z}}(\bar z)$. For every positive $\delta$ we have

$$\left|\frac{I_2(\bar z+\delta\nu_{\bar z})-I_2(\bar z)}{\delta}-J(\bar z)\right|\le\left|\frac{I_{2,\delta}(\bar z+\delta\nu_{\bar z})-I_{2,\delta}(\bar z)}{\delta}-J(\bar z)\right|$$
$$+\left|\frac{I_2(\bar z+\delta\nu_{\bar z})-I_{2,\delta}(\bar z+\delta\nu_{\bar z})}{\delta}\right|+\left|\frac{I_2(\bar z)-I_{2,\delta}(\bar z)}{\delta}\right|.$$

We first note that, using the expression (2.4), we find $d_{\bar z}(\bar z,\bar z+\delta\nu_{\bar z})=\delta^{\frac12}$. Thus, by (2.10) and by the mean value theorem, there exists a $\delta_0\in\ ]0,\delta[$ such that

$$\left|\frac{I_2(\bar z+\delta\nu_{\bar z})-I_2(\bar z)}{\delta}-J(\bar z)\right|\le\left|\left(h(u(\bar z))\partial_y I_{2,\delta}-\partial_t I_{2,\delta}\right)(\bar z+\delta_0\nu_{\bar z})-J(\bar z)\right|+c\delta^{\frac12}$$
$$=|Y_{\bar z}I_{2,\delta}(\bar z+\delta_0\nu_{\bar z})-J(\bar z)|+c\delta^{\frac12}\le c\ \delta^{\frac12}|\log\delta|,$$

where the last inequality follows from (2.11). Therefore we have

$$\frac{\partial I_2}{\partial\nu_z}(z)=J(z),$$

and, by Lemma 2.2, we get (2.1).

We are left with the proof of (2.10)–(2.11). We assume $d_{\bar z}(\bar z,z)\le\delta^{\frac12}$. By (2.8) and (2.5), we have

$$|I_{2,\delta}(z)-I_2(z)|\le c\int_{d_{\bar z}(z,\zeta)<\delta^{\frac12}}d_{\bar z}(z,\zeta)^{-Q+2}d_{\bar z}(\bar z,\zeta)d\zeta$$

(since, by (2.9), $d_{\bar z}(\bar z,\zeta)<c\delta^{\frac12}$, and by using the homogeneous polar coordinates)

$$\le\ \delta^{\frac12}\int_{\varrho<\delta^{\frac12}}\varrho^{-Q+2+Q-1}d\varrho=c\delta^{\frac32}.$$

This proves (2.10). Next we recall the following estimate which immediately follows by the mean value theorem:

$$|Y_{\bar z}(z)\Gamma_{\bar z}(\cdot,\zeta)-Y_{\bar z}(\bar z)\Gamma_{\bar z}(\cdot,\zeta)|\le c d_{\bar z}(\bar z,z)d_{\bar z}(\bar z,\zeta)^{-Q-1}\tag{2.12}$$

for $d_{\bar z}(\bar z,\zeta)\ge\bar c d_{\bar z}(\bar z,z)$. Then we have

$$|Y_{\bar z}I_{2,\delta}(z)-J(\bar z)|\le\int|Y_{\bar z}(z)\Gamma_{\bar z}(\cdot,\zeta)-Y_{\bar z}(\bar z)\Gamma_{\bar z}(\cdot,\zeta)|\,\chi\left(\frac{d_{\bar z}(\bar z,\zeta)}{\bar c\ \delta^{\frac12}}\right)|U_{2,\bar z}(\zeta)|d\zeta$$
$$+\int_{d_{\bar z}(\bar z,\zeta)<\delta^{\frac12}}|Y_{\bar z}(\bar z)\Gamma_{\bar z}(\cdot,\zeta)U_{2,\bar z}(\zeta)|d\zeta$$

(by (2.12) and since the second term can be estimated as before)

$$\le c\delta^{\frac12}\int_{d_{\bar z}(\bar z,\zeta)>\bar c\delta^{\frac12}}d_{\bar z}(\bar z,\zeta)^{-Q-1}|U_{2,\bar z}(\zeta)|d\zeta+c\delta^{\frac12}=c\delta^{\frac12}|\log(\delta)|.$$

This concludes the proof of (2.11).

*The term* $I_1$. Let $G(z,\zeta) = g(\zeta^{-1} \circ z)$, where $g$ is a smooth function. A direct computation gives

$$Y_{\bar{z}}(z)G(\cdot,\zeta) = R_{\bar{z}}(\zeta)G(z,\cdot), \tag{2.13}$$

where

$$R_{\bar{z}}(\zeta) = -Y_{\bar{z}}(\zeta) - (x_1 - \xi_1)\partial_\eta$$

(see [16, p. 295] for a general statement of this result). We aim to prove that

$$Y(\bar{z})I_1 = \int_\Omega Y(\bar{z})\Gamma_{\bar{z}}(\cdot,\zeta)\left(U_{1,\bar{z}}(\zeta) - U_{1,\bar{z}}(\bar{z})\right)d\zeta + U_{1,\bar{z}}(\bar{z})\int_{\partial\Omega}\Gamma_{\bar{z}}(\bar{z},\zeta)\langle R_{\bar{z}}(\zeta),\nu(\zeta)\rangle d\sigma, \tag{2.14}$$

where $\nu$ is the outer normal to the set $\Omega = \operatorname{supp}(\varphi)$, for which we assume that the divergence theorem holds. By (2.5), the homogeneity of the fundamental solution, and the Hölder continuity of $U_{1,\bar{z}}$, the function

$$V(z) = \int_\Omega Y(z)\Gamma_{\bar{z}}(\cdot,\zeta)\left(U_{1,\bar{z}}(\zeta) - U_{1,z}(z)\right)d\zeta + U_{1,\bar{z}}(z)\int_{\partial\Omega}\Gamma_{\bar{z}}(z,\zeta)\langle R_{\bar{z}}(\zeta),\nu(\zeta)\rangle d\sigma(\zeta) \tag{2.15}$$

is well defined. Let $K$ be a compact subset of $\Omega$. We set, for $\delta > 0$,

$$I_{1,\delta}(z) = \int_\Omega \Gamma_{\bar{z}}(z,\zeta)\chi\left(\frac{d_{\bar{z}}(z,\zeta)}{\delta}\right)U_{1,\bar{z}}(\zeta)d\zeta,$$

where $\chi$ is the cut-off function previously introduced. We choose $\delta$ suitably small so that

$$\chi\left(\frac{d_{\bar{z}}(z,\zeta)}{\delta}\right) = 1 \tag{2.16}$$

for any $z \in K, \zeta \in \partial\Omega$. Clearly $I_{1,\delta}$ is a smooth function, and differentiating we get

$$\begin{aligned} Y_{\bar{z}}(z)I_{1,\delta} = &\int_\Omega Y_{\bar{z}}(z)\left(\Gamma_{\bar{z}}(\cdot,\zeta)\chi\left(\frac{d_{\bar{z}}(\cdot,\zeta)}{\delta}\right)\right)\left(U_{1,\bar{z}}(\zeta) - U_{1,\bar{z}}(z)\right)d\zeta \\ &+ U_{1,\bar{z}}(z)\int_\Omega Y_{\bar{z}}(z)\left(\Gamma_{\bar{z}}(\cdot,\zeta)\chi\left(\frac{d_{\bar{z}}(\cdot,\zeta)}{\delta}\right)\right)d\zeta. \end{aligned} \tag{2.17}$$

By (2.13) and the divergence theorem, we have

$$\begin{aligned} \int_\Omega Y_{\bar{z}}(z)\left(\Gamma_{\bar{z}}(\cdot,\zeta)\chi\left(\frac{d_{\bar{z}}(\cdot,\zeta)}{\delta}\right)\right)d\zeta &= \int_\Omega R_{\bar{z}}(\zeta)\left(\Gamma_{\bar{z}}(z,\cdot)\chi\left(\frac{d_{\bar{z}}(z,\cdot)}{\delta}\right)\right)d\zeta \\ &= \int_{\partial\Omega}\Gamma_{\bar{z}}(z,\zeta)\chi\left(\frac{d_{\bar{z}}(z,\zeta)}{\delta}\right)\langle R_{\bar{z}}(\zeta),\nu(\zeta)\rangle d\sigma(\zeta). \end{aligned} \tag{2.18}$$

Then, by (2.18) and (2.16), the last terms in (2.17) and (2.15) are equal. Hence we get

$$\begin{aligned} &|V(z) - Y_{\bar{z}}(z)I_{1,\delta}| \\ &= \left| \int_{\delta_{\bar{z}}(z,\zeta)\le\delta} Y_{\bar{z}}(z)\left(\Gamma_{\bar{z}}(\cdot,\zeta)\left(1 - \chi\left(\frac{d_{\bar{z}}(\cdot,\zeta)}{\delta}\right)\right)\right)\left(U_{1,\bar{z}}(\zeta) - U_{1,\bar{z}}(z)\right)d\zeta \right| \\ &\le C\int_{\delta_{\bar{z}}(z,\zeta)\le\delta}\left(d_{\bar{z}}(z,\zeta)^{-Q} + \Gamma_{\bar{z}}(z,\zeta)\frac{d_{\bar{z}}(z,\zeta)^{-1}}{\delta}\right)d_{\bar{z}}(z,\zeta)^\alpha d\zeta \le C\delta^\alpha. \end{aligned}$$

Since the constant $C$ continuously depends on $\bar{z}$, we have that $Y_{\bar{z}}(z)I_{1,\delta}$ converges to $V$ as $\delta \to 0$ uniformly on $K$. Since $I_{1,\delta}$ converges to $I_1$ we get (2.14). This completes the proof of Theorem 1.1. $\square$

**3. A priori estimates.** In this section we prove Theorem 2.1 by using a modification of the classical Bernstein method. Here we adopt the notation of [10, Chap. 3], which we briefly recall for the reader's convenience. Given a bounded domain $\Omega$ in $\mathbb{R}^{N+2}$ and $\alpha \in \ ]0,1[$, $\overline{C}_\alpha(\Omega)$ denotes the space of Hölder continuous functions w.r.t. the parabolic distance

$$d(z,z') \equiv |x-x'| + |y-y'| + |t-t'|^{\frac{1}{2}},$$

i.e., the family of all functions $u$ on $\Omega$ for which

$$\overline{|u|}_\alpha^\Omega = \overline{|u|}_\alpha = |u|_0 + \sup_\Omega \frac{|u(z)-u(z')|}{d(z,z')^\alpha} < \infty,$$

where $|u|_0^\Omega = |u|_0 = \sup_\Omega |u|$. The spaces of Hölder continuous functions $\overline{C}_{k+\alpha}$, $k \in \mathbb{N}$, are defined straightforwardly. We set

$$B_r = \{(x,y) \in \mathbb{R}^N \times \mathbb{R} \ \mid \ |(x,y)| < r\}, \qquad S_{r,T} = B_r \times \ ]0,T[, \quad T, r > 0. \tag{3.1}$$

The "parabolic" boundary of the cylinder $S_{r,T}$ is defined by

$$\partial_p S_{r,T} = (B_r \times \{0\}) \cup (\partial B_r \times [0,T])\,. \tag{3.2}$$

Given two points $z, z' \in S_{r,T}$ in (3.1), we denote by $d_z$ the distance from $z$ to the parabolic boundary $\partial_p S_{r,T}$ (cf. (3.2)), and $d_{zz'} = \min\{d_z, d_{z'}\}$. We set

$$|u|_\alpha^{S_{r,T}} = |u|_\alpha = |u|_0 + \sup_{S_{r,T}} d_{zz'}^\alpha \frac{|u(z)-u(z')|}{d(z,z')^\alpha}.$$

The space of all functions $u$ with finite norm $|u|_\alpha^{S_{r,T}}$ is denoted by $C_\alpha(S_{r,T})$. The spaces $C_{k+\alpha}$ of Hölder continuous functions of higher order are defined analogously. We say that $u \in C_{k+\alpha,\mathrm{loc}}(S_T)$ if $u \in C_{k+\alpha}(S_{r,T})$ for every $r > 0$.

We consider the Cauchy problem

$$L^\varepsilon u = f(\cdot, u) \qquad \text{in } \ S_T \equiv \mathbb{R}^{N+1} \times \ ]0,T[, \tag{3.3}$$

$$u(\cdot, 0) = g \qquad \text{in } \ \mathbb{R}^{N+1}, \tag{3.4}$$

where $L^\varepsilon$, $\varepsilon > 0$, is the regularized operator in (1.5). We assume that the functions $f, g, h$ are globally Lipschitz continuous; then there exists a positive constant $c_1$ such that

$$\begin{aligned} & c_1 \geq \max\{\text{Lipschitz constants of } f, g, h\}, \\ & |h(v)| \leq c_1\sqrt{1+v^2}, \qquad |g(x,y)| \leq c_1 \sqrt{1+|(x,y)|^2}, \\ & |f(x,y,t,v)| \leq c_1\sqrt{1+|(x,y,t,v)|^2}, \qquad (x,y,t,v) \in S_T \times \mathbb{R}. \end{aligned} \tag{3.5}$$

The following result holds.

THEOREM 3.1. *There exist two positive constants $T, c$ that depend only on the constant $c_1$ in (3.5) such that for every $\varepsilon > 0$ and $\alpha \in \ ]0,1[$ the Cauchy problem*

(3.3)–(3.4) *has a unique solution* $u^\varepsilon \in C_{2+\alpha,\mathrm{loc}}(S_T) \cap C\left(\overline{S}_T\right)$ *verifying the following* $\varepsilon$*-uniform estimates:*

$$|u^\varepsilon_{x_i}|_0, |u^\varepsilon_y|_0 \le 4c_1, \qquad i = 1, \dots, N, \tag{3.6}$$

$$|u^\varepsilon(x,y,t+s) - u^\varepsilon(x,y,t)| \le c\sqrt{1 + |(x,y)|^2}\ |s|^{\frac{1}{2}}, \tag{3.7}$$

$$|u^\varepsilon(x,y,t)| \le 2c_1\sqrt{1 + |(x,y,t)|^2} \qquad \forall (x,y,t) \in \overline{S}_T. \tag{3.8}$$

Before proving Theorem 3.1, we introduce some further notation. If $\chi = \chi(x,y) \in C_0^\infty\left(\mathbb{R}^{N+1}\right)$ is a cut-off function such that $\chi = 1$ in $B_{\frac{1}{2}}$ and $\mathrm{supp}(\chi) \subset B_1$, we set

$$\chi_n(x,y) = \chi\left(\frac{x}{n}, \frac{y}{n}\right), \quad f_n = f\chi_n, \quad g_n(\cdot,t) = g\chi_n, \quad h_n(\cdot,v) = h(v)\chi_n, \quad n \in \mathbb{N}, \tag{3.9}$$

so that, by (3.5) and readjusting the constant $c_1$ if necessary, we have

$$|\nabla\chi_n|_0 \le \frac{|\nabla\chi|_0}{n}, \qquad |\nabla g_n| \le c_1,$$

$$|\nabla_{x,y} f_n(x,y,t,v)| \le |\chi_n \nabla_{x,y} f| + \frac{c_1\,|\nabla\chi|_0}{n}\sqrt{1 + n^2 + T^2 + v^2} \le c_1$$

if $\frac{|v|}{n}$ is bounded and $t \in [0,T]$.

Finally, fixing $n \in \mathbb{N}$ and $\varepsilon > 0$, we consider the linearized Cauchy–Dirichlet problem

$$L^{\varepsilon,n}_v u \equiv \Delta_x u + \varepsilon^2 u_{yy} + h_n(\cdot,v)\partial_y u - \partial_t u = f_n(\cdot,v) \qquad \text{in } S_{n,T}, \tag{3.10}$$

$$u = g_n \qquad \text{in } \partial_p S_{n,T}. \tag{3.11}$$

Given $\alpha \in ]0,1[$, we assume that the coefficient $v$ in (3.10)–(3.11) belongs to the space $\overline{C}_{1+\alpha}(S_{n,T})$ and satisfies the estimates

$$|v(x,y,t)| \le 2c_1\sqrt{1 + |(x,y)|^2} \qquad \text{in } S_{n,T}, \tag{3.12}$$

$$|v_{x_i}|_0 \le 4c_1, \qquad i = 1, \dots, N, \tag{3.13}$$

$$|v_y|_0 \le 4c_1. \tag{3.14}$$

Then a classical solution $u \in \overline{C}_{2+\alpha}(S_{n,T})$ to (3.10)–(3.11) exists by known results (see, for example, [10, Chap. 3, Thm. 7], since $h_n(\cdot,v), f_n(\cdot,v) \in \overline{C}_{1+\alpha}(S_{n,T})$, $g_n \in C^\infty\left(\overline{S}_{n,T}\right)$, and the compatibility condition $L^{\varepsilon,n}_v g_n = f_n = 0$ holds on $\partial B_n$. Once we have given the following $\varepsilon$-uniform a priori estimates, the proof of Theorem 3.1 is rather standard.

LEMMA 3.2. *Under the above assumptions, there exists* $T > 0$ *such that, for any* $n \in \mathbb{N}$*, every classical solution of* (3.10)–(3.11) *verifies* (3.12)–(3.14).

*Proof.* Let $u$ be a classical solution of (3.10)–(3.11). We prove estimate (3.12) for $u$ by applying the maximum principle to the functions $H \pm u$, where $H$ is defined as

$$H(x,y,t) = (c_1 + \mu t)\sqrt{1 + |(x,y)|^2}$$

and $\mu$ is to be suitably fixed. Keeping in mind (3.5) and (3.12), it is easily verified that

$$\begin{aligned} L^{\varepsilon,n}_v H(x,y,t) &\le \frac{(1+\varepsilon^2)(c_1 + \mu T)}{\sqrt{1 + |(x,y)|^2}} + ((c_1 + \mu T)\,c_1 - \mu)\sqrt{1 + |(x,y)|^2} \\ &\le -\,|f_n(x,y,t,v(x,y,t))| \end{aligned}$$

if $\mu, \frac{1}{T}$ are suitably large. On the other hand, by (3.5), $H|_{\partial_p S_{n,T}} \geq |g_n|$. Therefore, by the maximum principle, we infer that

$$|u| \leq H \leq 2c_1\sqrt{1+|(x,y)|^2} \qquad \text{if } T \leq \frac{c_1}{\mu}.$$

Next we prove estimate (3.14) for the $y$-derivative of $u$. Our method is based on the maximum principle. We start by proving a gradient estimate for $u$ on the parabolic boundary of $S_{n,T}$. Since $u \in \overline{C}_{2+\alpha}(S_{n,T})$, it is clear that $\nabla_{x,y}u = \nabla_{x,y}g_n$ in $B_n \times \{0\}$. In order to estimate $\nabla_{x,y}u$ on $\partial B_n \times\, ]0,T[$, we employ the classical argument of the barrier functions on the cylinder $Q \equiv S_{n,T} \setminus S_{\frac{n}{2},T}$. More precisely, given $(x_0,y_0,t_0) \in \partial B_n \times\, ]0,T[$, we set

$$w(x,y) = 4c_1\langle (x-x_0, y-y_0), \nu\rangle,$$

where $\nu$ is the inner normal to $Q$ at $(x_0,y_0,t_0)$. Then we have

$$L_v^{\varepsilon,n}(w \pm u) = \pm f_n(\cdot, v) = 0 \qquad \text{in } Q,$$

since $f_n$ and $h_n$ vanish on $Q$. On the other hand, it is straightforward to verify that $|u| \leq w$ on $\partial_p Q$. Therefore, by the maximum principle, we get $|u| \leq w$ and, in particular,

$$|\nabla_{x,y}u(x_0,y_0,t_0)| \leq |\nabla_{x,y}w(x_0,y_0)| \leq 4c_1. \tag{3.15}$$

Now we are in a position to prove estimate (3.14) for $u$. We differentiate equation (3.10) w.r.t. the variable $y$ and then multiply it by $e^{-2\lambda t}u_y$. Denoting $\omega = \left(e^{-\lambda t}u_y\right)^2$, we obtain

$$\begin{aligned} L_v^\varepsilon \omega &= e^{-2\lambda t}L_v^\varepsilon u_y^2 + 2\lambda\omega \\ &= 2\left(e^{-2\lambda t}\left(|\nabla_x u_y|^2 + \varepsilon^2 u_{yy}^2 + u_y\left((f_n)_y + (f_n)_v v_y\right)\right) + \omega\left(\lambda - h'(v)v_y\right)\right) \\ &\geq 2\left(e^{-2\lambda t}u_y\left((f_n)_y + (f_n)_v v_y\right) + \omega\left(\lambda - h'(v)v_y\right)\right). \end{aligned} \tag{3.16}$$

Hence, by setting $w = \omega - (4c_1)^2$, we get from (3.16)

$$L_v^\varepsilon w \geq 2\sqrt{\omega}\left(-|(f_n)_y| - |v_y(f_n)_v| + \sqrt{\omega}\left(\lambda - |h'v_y|\right)\right)$$

(by (3.5), (3.14), and by the elementary inequality $\sqrt{\omega} \geq \frac{\sqrt{2}}{2}(4c_1 + \operatorname{sgn}(w)\sqrt{|w|})$)

$$\geq \sqrt{2\omega}\left(\sqrt{2}c_1\left(2\sqrt{2}\left(\lambda - 4c_1^2\right) - 4c_1 - 1\right) + \left(\lambda - 4c_1^2\right)\operatorname{sgn}(w)\sqrt{|w|}\right)$$

(for $\lambda = \lambda(c_1)$ suitably large)

$$\geq c\sqrt{\omega|w|}\operatorname{sgn}(w) \tag{3.17}$$

for some positive constant $c = c(c_1)$. By contradiction, we want to prove that $w \leq 0$ in $S_{n,T}$. It will follow that

$$|u_y| \leq c_1 e^{\lambda t},$$

which implies (3.16) if $T = T(c_1) > 0$ is sufficiently small. Let $z_0$ be the maximum of $w$ on $\overline{Q}_T$. If $w(z_0) > 0$, then $z_0 \in S_{n,T} \setminus \partial_p S_{n,T}$, since by (3.15) $w \leq 0$ on $\partial_p S_{n,T}$. This leads to a contradiction, since by (3.17)

$$0 \geq L_v^\varepsilon w(z_0) \geq c\sqrt{\omega(z_0)w(z_0)} > 0.$$

This concludes the proof of (3.14). By a similar technique, we prove estimate (3.13) of the $x$-derivatives of $u$:

$$|u_{x_k}|_0 \le 4c_1, \qquad k = 1, \dots, N.$$

We set

$$\omega = \left(e^{-\lambda t} u_{x_k}\right)^2, \qquad w = \omega - (4c_1)^2.$$

Differentiating (3.10) w.r.t. $x_k$ and multiplying it by $e^{-2\lambda t} u_{x_k}$, we get

$$\begin{aligned} L_v^\varepsilon w &= e^{-2\lambda t} L_v^\varepsilon u_{x_k}^2 + 2\lambda\omega \\ &= 2\left(e^{-2\lambda t} u_{x_k}\left((f_n)_{x_k} + v_{x_k}\left((f_n)_v - u_y h'\right)\right) + \lambda\omega\right) \end{aligned}$$

(by (3.5), (3.13), and estimate (3.14) of $u_y$ previously proved)

$$\ge c\sqrt{\omega|w|}\,\mathrm{sgn}(w),$$

if $\lambda = \lambda(c_1)$ is suitably large, for some positive constant $c$ which depends only on $c_1$. As before, we infer that $w \le 0$, which yields (3.13). $\square$

We are in a position to prove Theorem 3.1.

*Proof of Theorem* 3.1. In order to prove the existence of a unique classical solution to (3.3)–(3.4), we consider, for every $\varepsilon > 0$ and $n \in \mathbb{N}$, the Cauchy–Dirichlet problem

$$\Delta_x u + \varepsilon^2 u_{yy} + h_n(\cdot, u)\partial_y u - \partial_t u = f_n(\cdot, u) \qquad \text{in } S_{n,T}, \tag{3.18}$$

$$u = g_n \qquad \text{in } \partial_p S_{n,T}. \tag{3.19}$$

We split the proof into four steps: We first use Schauder's fixed point theorem to solve the above problem. Then we let $n$ go to infinity under the assumption that the coefficients are smooth. Next we prove estimates (3.6), (3.7), and (3.8). Finally we remove the smoothness assumption.

*First step.* Assume that $f, g, h$ are $C^\infty$ functions. We fix $\alpha \in ]0,1[$, $n \in \mathbb{N}$ and denote by $\mathcal{W}$ the family of functions $v \in \overline{C}_{1+\alpha}(S_{n,T})$ such that

$$\overline{|v|}_{1+\alpha} \le M, \tag{3.20}$$

$$|v(x,y,t)| \le 2c_1\sqrt{1 + |(x,y)|^2} \qquad \text{in } S_{n,T}, \tag{3.21}$$

$$|v_{x_i}|_0 \le 4c_1, \qquad i = 1, \dots, N, \tag{3.22}$$

$$|v_y|_0 \le 4c_1, \tag{3.23}$$

where the positive constants $M, T$ will be suitably chosen later. Clearly, $\mathcal{W}$ is a closed, convex subset of $\overline{C}_{1+\alpha}(S_{n,T})$. We define a transformation $u \equiv \mathcal{Z}v$ on $\mathcal{W}$ by choosing $u$ as the unique classical solution of the linear Cauchy–Dirichlet problem (3.10)–(3.11). If we show that

(i) $\mathcal{Z}(\mathcal{W})$ is precompact in $\overline{C}_{1+\alpha}(S_{n,T})$;

(ii) $\mathcal{Z}$ is a continuous operator;

(iii) $\mathcal{Z}(\mathcal{W}) \subseteq \mathcal{W}$,

then we are done. The proof of (i) and (ii) is quite standard and relies on the following two estimates of $u$ (see, for example, [10, Chap. 3, Thm. 6 and Chap. 7, Thm. 4]:

$$\overline{|u|}_{2+\alpha} \le c\left(\overline{|g_n|}_{2+\alpha} + \overline{|f_n(\cdot, v)|}_\alpha\right) \le \bar{c}\left(\overline{|g_n|}_{2+\alpha} + \overline{|v|}_\alpha\right) \tag{3.24}$$

for some constant $\bar{c} > 0$ dependent on $\varepsilon, n, M, \alpha$;

$$\overline{|u|}_{1+\delta} \le \widetilde{c}\left(|f_n|_0 + |L_v^\varepsilon g_n|_0 + \overline{|g_n|}_{1+\delta}\right), \qquad \delta \in ]0,1[, \tag{3.25}$$

for some positive constant $\widetilde{c}$ dependent on $\varepsilon, n, \delta$ but not on $M$. Besides, (iii) is exactly the content of Lemma 3.2. Therefore, by Schauder's theorem, the operator $\mathcal{Z}$ has a fixed point $u$ in $\mathcal{W}$.

Note that, by (3.6), a comparison principle in the space $\mathcal{W}$ does hold; therefore $u$ is the unique classical solution of problem (3.18)–(3.19) verifying estimates (3.6) and (3.8). Moreover, by a standard bootstrap argument, $u \in C^\infty(S_{n,T})$.

*Second step.* We fix $\varepsilon > 0$ and denote by $u^n$ the solution of the Cauchy–Dirichlet problem (3.18)–(3.19), whose existence has been proved in the previous step. We now want to obtain the solution of the Cauchy problem (3.3)–(3.4) letting $n$ go to infinity.

Fixing $k \in \mathbb{N}$, we consider the sequence $(u^n\chi_{4k})_{n\ge 4k}$, where $\chi$ is the cut-off function introduced in (3.9). Then we have

$$\begin{aligned} L^\varepsilon_{u^n}\left(u^n\chi_{4k}\right) &= f_{4k}(\cdot, u_n) + 2\left(\langle\nabla_x u^n, \nabla_x\chi_{4k}\rangle + \varepsilon^2\partial_y u^n\partial_y\chi_{4k}\right) + u^n L^\varepsilon_{u_n}\chi_{4k} \\ &\equiv F_{n,4k} \qquad \text{on } S_{4k,T}, \\ \left(u^n\chi_{4k}\right)|_{\partial_p S_{4k,T}} &= g_{4k}. \end{aligned}$$

By classical Hölder estimates, we deduce

$$\overline{|u^n|}^{S_{2k,T}}_\delta \le \overline{|u^n\chi_{4k}|}^{S_{4k,T}}_{1+\delta} \le c\left(|F_{n,4k}|_0^{S_{4k,T}} + |L^\varepsilon_{u^n}g_{4k}|_0^{S_{4k,T}} + \overline{|g_{4k}|}^{S_{4k,T}}_{1+\delta}\right) \le \bar{c}$$

for every $n \ge 4k$ and $\delta \in ]0,1[$, where $\bar{c} = \bar{c}(\delta, \varepsilon, c_1, k)$ does not depend on $n$. Moreover, since

$$\begin{aligned} L^\varepsilon_{u^n}\left(u^n\chi_{2k}\right) &= F_{n,2k} \qquad \text{on } S_{4k,T}, \\ \left(u^n\chi_{2k}\right)|_{\partial_p S_{2k,T}} &= g_{2k}, \end{aligned}$$

we obtain

$$\overline{|u^n|}^{S_{k,T}}_{2+\delta} \le \overline{|u^n\chi_{2k}|}^{S_{2k,T}}_{2+\delta} \le c\left(\overline{|F_{n,2k}|}^{S_{2k,T}}_\delta + \overline{|g_{2k}|}^{S_{2k,T}}_{2+\delta}\right) \le \bar{\bar{c}} \qquad \forall n \ge 4k,$$

where $\bar{\bar{c}} = \bar{\bar{c}}(\delta, \varepsilon, c_1, k)$ does not depend on $n$.

Then, by the Ascoli–Arzelà theorem and Cantor's diagonal argument, we can extract from $u^n$ a subsequence $\overline{|\ |}_{2+\alpha}$-convergent on compacts of $S_T$ for every $\alpha \in ]0,1[$ to the solution $u^\varepsilon$ of (3.18)–(3.19) verifying estimates (3.6) and (3.8). The uniqueness of $u^\varepsilon$ follows again from standard results.

*Third step.* We still assume $f, g, h \in C^\infty \cap \mathrm{Lip}$. We aim to prove estimate (3.7) for the solution $u^\varepsilon$ found in the previous step. We fix $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}$ and set

$$w(x,y,t) = u^\varepsilon(x, \varepsilon y, t)\bar{\chi}(x, \varepsilon y), \qquad \varepsilon > 0,\ (x,y,t) \in S_T,$$

where $\bar{\chi}(x,y) = \chi(x - \bar{x}, y - \bar{y})$ and $\chi$ is the cut-off function in (3.9). We have

$$(\Delta_x + \partial_{yy} - \partial_t)w = \Psi^\varepsilon \qquad \text{on } S_T,$$

where

$$\begin{aligned} \Psi^\varepsilon(x,y,t) = &\Big[\bar{\chi}\left(f(\cdot, u^\varepsilon) - h(u^\varepsilon)u^\varepsilon_y\right) + u^\varepsilon\left(\Delta_x\bar{\chi} + \varepsilon^2\bar{\chi}_{yy}\right) \\ &+ 2\left(\langle\nabla_x u^\varepsilon, \nabla_x\bar{\chi}\rangle + \varepsilon^2 u^\varepsilon_y\bar{\chi}_y\right)\Big](x, \varepsilon y, t), \qquad (x,y,t) \in S_T. \end{aligned}$$

Denoting by $\Gamma_H(z;\zeta)$ the fundamental solution of the heat operator in $\mathbb{R}^{N+2}$ with pole at $\zeta = (\xi,\eta,\tau)$ and evaluated in $z = (x,y,t)$, we have the following representation of $w$:

$$
\begin{aligned}
w(z) = &\int_0^t \int_{\mathbb{R}^{N+1}} \Gamma_H(z;\zeta)\Psi^\varepsilon(\zeta)d(\xi,\eta)d\tau \\
&- \int_{\mathbb{R}^{N+1}} \Gamma_H(z;\xi,\eta,0)g\bar{\chi}(\xi,\varepsilon\eta)d(\xi,\eta) \equiv I_1(z) - I_2(z).
\end{aligned} \tag{3.26}
$$

In order to estimate $I_1$, it suffices to note that, by (3.5), (3.6), and (3.8), we have that

$$
|\Psi^\varepsilon|_0 \le c\sqrt{1+|(\bar{x},\bar{y})|^2}, \tag{3.27}
$$

with $c$ dependent only on $c_1$. Hence, by an elementary argument, we get

$$
|I_1(x,y,t+s) - I_1(x,y,t)| \le c\sqrt{1+|(\bar{x},\bar{y})|^2}\ |s|^{\frac{1}{2}} \qquad \forall (x,y,t)\in S_T,\ s\in[-t,T-t], \tag{3.28}
$$

where $c$ depends only on $c_1$.

To estimate $I_2$, we begin by noting that a simple change of variables gives

$$
I_2(x,y,t) = \int_{\mathbb{R}^{N+1}} \Gamma_H(\xi,\eta,1;0)g\bar{\chi}\left(x-\xi\sqrt{t},\varepsilon(y-\eta\sqrt{t})\right)d\xi d\eta.
$$

Then

$$
\begin{aligned}
|I_2(x,y,t+s) - I_2(x,y,t)| \le &\int_{\mathbb{R}^{N+1}} \Gamma_H(\xi,\eta,1;0) \\
\cdot &\left| g\bar{\chi}\left(x-\xi\sqrt{t+s},\varepsilon(y-\eta\sqrt{t+s})\right) - g\bar{\chi}\left(x-\xi\sqrt{t},\varepsilon(y-\eta\sqrt{t})\right)\right| d\xi d\eta
\end{aligned}
$$

(by the mean value theorem, for some constant $c = c(c_1) > 0$)

$$
\begin{aligned}
&\le c\sqrt{1+|(\bar{x},\bar{y})|^2}\ \left|\sqrt{t+s}-\sqrt{t}\right| \int_{\mathbb{R}^{N+1}} \Gamma_H(\xi,\eta,1;0)\left(|\xi|+\varepsilon|\eta|\right)d\xi d\eta \\
&\le c\sqrt{1+|(\bar{x},\bar{y})|^2}\ \sqrt{2|s|} \qquad \forall (x,y,t)\in S_T,\ s\in[-t,T-t],
\end{aligned} \tag{3.29}
$$

where $c$ depends only on $c_1$.

Then, by the definition of $w$ and by (3.26), we obtain

$$
u^\varepsilon(\bar{x},\bar{y},t) = I_1\left(\bar{x},\frac{\bar{y}}{\varepsilon},t\right) - I_2\left(\bar{x},\frac{\bar{y}}{\varepsilon},t\right),
$$

and estimate (3.7) follows from (3.28), (3.29).

*Fourth step.* We finally consider the general case where $f,g,h$ are only assumed to be globally Lipschitz continuous. We use the standard mollifiers to approximate $f,g,h$ uniformly on compacts by some sequences $(f_n)$, $(g_n)$, $(h_n)$ in $C^\infty\cap\mathrm{Lip}$ that verify the estimates (3.5). Since the interval $[0,T]$ of existence of the solution constructed in the second step does not depend on the regularity of the coefficients, we may employ the usual density argument to find a function $u^\varepsilon$ which is the unique classical solution of (3.3)–(3.4). $\square$

*Proof of Theorem* 2.1. By Theorem 3.1, there exists a sequence

$$u^{\varepsilon_n} \in C_{2+\alpha,\mathrm{loc}}\left(S_T\right) \cap C\left(\overline{S}_T\right),$$

with $\varepsilon_n \downarrow 0$, such that every function $u^{\varepsilon_n}$ is a solution of (3.3)–(3.4) with $\varepsilon = \varepsilon_n$ and verifies (1.6) for a constant $c_0$ that does not depend on $n$, and $(u^{\varepsilon_n})$ converges uniformly on compact subsets of $\overline{S}_T$ to a function $u$.

Arguing as in [6, Lem. 2.4], we can prove the following a priori estimates of Caccioppoli type for the derivatives of the functions $(u^{\varepsilon_n})$: if $\varphi \in C_0^\infty\left(S_T\right)$, there exists a positive constant $c$ which depends only on $f, \varphi$ and on the constant $c_0$ in (1.6) such that

$$\sum_{j=1}^{N}\left(\|u^{\varepsilon_n}_{x_j x_j}\varphi\|_2 + \|u^{\varepsilon_n}_{x_j y}\varphi\|_2\right) + \varepsilon_n\|u^{\varepsilon_n}_{yy}\varphi\|_2 + \|u^{\varepsilon_n}_t\varphi\|_2 \le c \tag{3.30}$$

for every $n$. Therefore, up to a subsequence, $\partial_{x_j,x_k}u^{\varepsilon_n}$, $\varepsilon_n^2\partial_{yy}u^{\varepsilon_n}$, $\partial_y u^{\varepsilon_n}$, and $\partial_t u^{\varepsilon_n}$ weakly converge in $L^2_{\mathrm{loc}}(S_T)$ to $\partial_{x_j,x_k}u$, 0, $\partial_y u$, and $\partial_t u$, respectively. Hence $u \in H^1_{\mathrm{loc}}(S_T), \partial_{x_j x_k}u \in L^2_{\mathrm{loc}}(S_T)$ for $j,k = 1,\dots,N$, and (1.1) is satisfied a.e.

The uniqueness of the solution can be proved as in [2, Prop. 5.1]. Indeed, since $(u^{\varepsilon_n})$ converges uniformly on compact sets, it is standard to prove that the limit $u$ is a viscosity solution of (1.1)–(1.2) satisfying (1.6). Then the uniqueness of $u$ follows by the comparison principle for viscosity solutions. □

**4. Global existence.** The main purpose of this section is to prove Theorem 1.2 by a simple continuation argument which relies on a bound of the gradient of $u$.

*Proof of Theorem* 1.2. The local existence result stated in Theorem 3.1 and a standard argument ensure that there exist an interval $I = [0,\overline{T}[$, where $\overline{T} \in \mathbb{R}^+$ or $\overline{T} = +\infty$, and a solution $u \in C^2(\mathbb{R}^{N+1} \times I)$ to problem (1.1)–(1.2), which cannot be defined for $t \ge \overline{T}$. We claim that our assumptions on $f, g$, and $h$ yield $\overline{T} = +\infty$. To this end, we consider the local solution $u \in C^2(\mathbb{R}^{N+1} \times [0,T])$, which has been constructed in Theorem 3.1, and we denote by $c_T$ the spatial Lipschitz constant corresponding to the strip $S_T$:

$$c_T = \inf\Big\{c > 0 : |u(x,y,t) - u(x',y',t)| \le c(|x-x'|^2 + |y-y'|^2)^{1/2} \\ \forall (x,y,t),(x',y',t) \in \mathbb{R}^{N+1} \times [0,T]\Big\}.$$

We explicitly note that if $\overline{T} \ne +\infty$, then $c_t \to +\infty$ as $t \to \overline{T}$; hence a bound of the form

$$c_t \le ce^{kt} \tag{4.1}$$

for some positive constants $c, k$ will prove our claim.

In order to prove (4.1) we first observe that, as in the proof of Theorem 3.1, it is not restrictive to assume that $f, g$, and $h$ are smooth and that $u$ is the classical solution of the regularized equation (1.5). We next show that

$$0 \le -u_y \le \frac{c_1}{c_0} + 1, \tag{4.2}$$

$$|u_{x_j}| \le c_1 e^{kt} \quad \text{for} \quad j = 1,\dots,N \tag{4.3}$$

for every $(x, y, t) \in S_T$, where $c_1$ is the Lipschitz constant defined in (3.5) and $k > 0$ does not depend on $\varepsilon$. To prove the first inequality in (4.2) we set $w(x, y, t) = e^{-\lambda t} u_y(x, y, t)$ for some $\lambda > 0$, and we note that since $u$ is smooth, $w$ is a solution to

$$\begin{cases} L_\varepsilon w = e^{-\lambda t} f_y + (\lambda - h'(u) u_y + f_u) w & \text{in } S_T, \\ w(\,\cdot\,, 0) = g_y. \end{cases}$$

By our assumptions $f_y \geq 0, g_y \leq 0, c_0 \leq h' \leq c_1$ and also by Theorem 3.1, $h'(u)u_y + f_u$ is bounded in $S_T$. Then $\lambda - h'(u)u_y + f_u > 0$ for suitably large $\lambda$ and, as a consequence, $w \leq 0$ by the maximum principle. This proves the first inequality in (4.2). To prove the second one we set $w(x, y, t) = \frac{1}{2}\big(u_y^2(x, y, t) - \lambda^2\big)$, $\lambda > 0$, and argue as in the proof of Theorem 3.1: $w$ is a solution to

$$\begin{cases} L_\varepsilon w = |\nabla_x u_y|^2 + \varepsilon^2 u_{yy}^2 - u_y (h'(u) u_y^2 - u_y f_u - f_y) & \text{in } S_T, \\ w(\,\cdot\,, 0) = \frac{1}{2}\big(g_y^2 - \lambda^2\big). \end{cases}$$

Since $u_y \leq 0$, we may choose $\lambda$ sufficiently large (for instance, $\lambda = \frac{c_1}{c_0} + 1$) so that the right-hand side of the above differential equation is positive when $w > 0$. Then the second inequality in (4.2) follows again from the maximum principle.

We finally consider the function $w(x, y, t) = e^{-2\lambda t} \frac{u_{x_j}^2}{2} - \frac{c_1^2}{2}$ for $j = 1, \ldots, N$. Clearly $w(x, y, 0) \leq 0$ and

$$\begin{aligned} L_\varepsilon w &= e^{-2\lambda t} \left( u_{x_j}^2 (\lambda - h'(u) u_y + f_u) + |\nabla_x u_{x_j}|^2 + \varepsilon^2 u_{x_j, y}^2 + u_{x_j} f_{x_j} \right) \\ &\geq e^{-2\lambda t} \left( u_{x_j}^2 (\lambda + f_u) + u_{x_j} f_{x_j} \right) \end{aligned}$$

by (4.2). Since $w \geq 0$ implies $|u_{x_j} f_{x_j}| \leq u_{x_j}^2$ then, for suitably large $\lambda$, we find $L_\varepsilon w \leq 0$ for $w \geq 0$ and prove (4.3) as above. This gives (4.1) and concludes the proof of Theorem 1.2. ∎

*Remark* 4.1. Hypothesis (1.9) on $h$ is related to the natural assumptions in the theory of conservation laws. A simple counterexample shows that we cannot drop this condition. Indeed if $h(u) = -u, f \equiv 0$ and $g(x, y) = x - y$, then $u(x, y, t) = \frac{x-y}{1-t}$.

REFERENCES


[1] F. Antonelli, E. Barucci, and M. E. Mancino, *A comparison result for FBSDE with applications to decisions theory*, Math. Methods Oper. Res., 54 (2001), pp. 407–423.

[2] F. Antonelli and A. Pascucci, *On the viscosity solutions of a stochastic differential utility problem*, J. Differential Equations, 186 (2002), pp. 69–87.

[3] L. A. Caffarelli and X. Cabré, *Fully Nonlinear Elliptic Equations*, Amer. Math. Soc. Colloq. Publ. 43, AMS, New York, 1995.

[4] G. Citti, *$C^\infty$ regularity of solutions of a quasilinear equation related to the Levi operator*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 23 (1996), pp. 483–529.

[5] G. Citti, A. Pascucci, and S. Polidoro, *On the regularity of solutions to a nonlinear ultraparabolic equation arising in mathematical finance*, Differential Integral Equations, 14 (2001), pp. 701–738.

[6] G. Citti, A. Pascucci, and S. Polidoro, *Regularity properties of viscosity solutions of a non-Hörmander degenerate equation*, J. Math. Pures Appl., 80 (2001), pp. 901–918.

[7] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), 27 (1992), pp. 1–67.

[8] M. Escobedo, J. L. Vazquez, and E. Zuazua, *Entropy solutions for diffusion-convection equations with partial diffusivity*, Trans. Amer. Math. Soc., 343 (1994), pp. 829–842.

[9] C. Fefferman and A. Sánchez-Calle, *Fundamental solutions for second order subelliptic operators*, Ann. of Math., 124 (1986), pp. 247–272.

[10] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[11] S. N. Kruzhkov, *First order quasilinear equations in several independent variables*, Math. USSR-Sb., 10 (1970), pp. 217–243.

[12] L. P. Kuptsov, *Fundamental solutions of certain degenerate second-order parabolic equations*, Math. Notes, 31 (1982), pp. 283–289.

[13] S. Kusuoka and D. Stroock, *Applications of the Malliavin calculus.* III, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 34 (1987), pp. 391–442.

[14] E. Lanconelli and S. Polidoro, *On a class of hypoelliptic evolution operators*, Rend. Sem. Mat. Univ. Politec. Torino, 52 (1994), pp. 29–63.

[15] A. Nagel, E. M. Stein, and S. Wainger, *Balls and metrics defined by vector fields* I: *Basic properties*, Acta Math., 155 (1985), pp. 103–147.

[16] L. P. Rothschild and E. M. Stein, *Hypoelliptic differential operators on nilpotent groups*, Acta Math., 137 (1977), pp. 247–320.

[17] A. N. Shiryayev, *Selected Works of A.N. Kolmogorov. Vol.* II. *Probability Theory and Mathematical Statistics*, Kluwer Academic, Dordrecht, Boston, London, 1991.

[18] N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge Tracts in Math. 100, Cambridge University Press, Cambridge, UK, 1992.

[19] A. I. Vol'pert and S. I. Hudjaev, *Cauchy's problem for degenerate second order quasilinear parabolic equations*, Math. USSR-Sb., 7 (1970), pp. 365–387.

[20] L. Wang, *On the regularity theory of fully nonlinear parabolic equations.* I, Comm. Pure Appl. Math., 45 (1992), pp. 27–76.

[21] L. Wang, *On the regularity theory of fully nonlinear parabolic equations.* II, Comm. Pure Appl. Math., 45 (1992), pp. 141–178.