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A Gaussian upper bound for the fundamental solutions of a class of ultraparabolic equations ☆

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Abstract

We prove Gaussian estimates from above of the fundamental solutions to a class of ultraparabolic equations. These estimates are independent of the modulus of continuity of the coefficients and generalize the classical upper bounds by Aronson for uniformly parabolic equations.

Keywords: Ultraparabolic equations; Fundamental solution; Gaussian estimates

1. Introduction

We consider the second-order partial differential equation in divergence form

$$Lu(x,t) \equiv \sum_{i,j=1}^{m} \partial_{x_i}\big(a_{ij}(x,t)\partial_{x_j}u(x,t)\big) + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j}u(x,t) - \partial_t u(x,t) = 0, \tag{1.1}$$

where $(x,t) = (x_1,\ldots,x_N,t) = z$ denotes the point in $\mathbb{R}^{N+1}$, and $1 \leqslant m \leqslant N$.

In this paper, under some structural conditions which ensure the existence of a fundamental solution Γ of (1.1), we aim to prove a global upper bound for Γ independent of the regularity of the coefficients. This bound is given in terms of the fundamental solution of the "constant coefficients" operator

$$L_1 \equiv \Delta_m + Y, \tag{1.2}$$

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where Δ_m is the Laplace operator in $\mathbb{R}^m$ and Y denotes the first-order part in (1.1),

$$Y = \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t. \tag{1.3}$$

This justifies the word "Gaussian" in the title. Indeed, we recall that an explicit fundamental solution of Gaussian type for (1.2) has been constructed by Hörmander in [6] assuming the classical rank condition that, in our setting, reads

$$\operatorname{rank} \operatorname{Lie}(\partial_{x_1}, \dots, \partial_{x_m}, Y)(z) = N+1, \quad \forall z \in \mathbb{R}^{N+1},$$

where $\operatorname{Lie}(\partial_{x_1}, \dots, \partial_{x_m}, Y)$ denotes the Lie algebra generated by the first-order differential operators $\partial_{x_1}, \dots, \partial_{x_m}, Y$. If we set

$$A_1 = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = (b_{ij})_{i,j=1,\dots,N},$$

where I_m is the identity matrix in $\mathbb{R}^m$ and we define

$$\mathcal{C}(t) = \int_0^t \exp(-sB^T) A_1 \exp(-sB)\, ds, \tag{1.4}$$

it is not difficult to see that the Hörmander's condition is satisfied if and only if $\mathcal{C}(t) > 0$ for any $t > 0$ (cf. Proposition A.1 in [10], see also [8]). In that case, the fundamental solution Γ_1 of L_1, with pole at the origin, is defined as follows:

$$\Gamma_1(x,t) = \begin{cases} \frac{(4\pi)^{-N/2}}{\sqrt{\det \mathcal{C}(t)}} \exp\left(-\frac{1}{4}\langle \mathcal{C}^{-1}(t)x, x\rangle - t\,\mathrm{tr}(B)\right), & \text{if } t > 0, \\ 0, & \text{if } t \leqslant 0. \end{cases}$$

In the case of Hölder continuous coefficients, Eq. (1.1) has been studied by many authors assuming the following basic conditions:

(H1) $a_{ij} = a_{ji}$, $1 \leqslant i, j \leqslant m$, and there exists a positive constant λ such that

$$\lambda^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^{m} a_{ij}(z)\xi_i\xi_j \leqslant \lambda|\xi|^2 \tag{1.5}$$

for every $z \in \mathbb{R}^{N+1}$ and $\xi \in \mathbb{R}^m$. The matrix B is constant;

(H2) The "constant coefficients" operator L_1 in (1.2) verifies the Hörmander's rank condition and it is homogeneous of degree two with respect to some dilations group in $\mathbb{R}^{N+1}$.

We remark that (H2) is a requirement only on the coefficients b_{ij}'s of (1.1). Indeed, by Propositions 2.1 and 2.2 of [10], hypothesis (H2) is equivalent to assume that for some basis on $\mathbb{R}^N$, the matrix B has the form

$$\begin{pmatrix} 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_q \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{1.6}$$

where B_k is an $m_{k-1} \times m_k$ matrix of rank m_k, $k = 1, 2, \dots, q$, with

$$m \equiv m_0 \geqslant m_1 \geqslant \cdots \geqslant m_q \geqslant 1 \quad \text{and} \quad \sum_{k=0}^{q} m_k = N.$$

Assuming (H1), (H2), and the Hölder continuity of the coefficients a_{ij}'s, the Levi parametrix method has been used to prove the existence of a fundamental solution Γ in the papers by Il'in [7], Weber [26], Sonin [25], Polidoro [21], Eidelman et al. [4]. In [21] the following upper bound has been given:

$$\Gamma(x, t, \xi, \tau) \leqslant C \Gamma_\mu(x, t, \xi, \tau), \quad \forall x, \xi \in \mathbb{R}^N, \ t > \tau, \tag{1.7}$$

where $\Gamma_\mu(\cdot, \cdot, \xi, \tau)$ denotes the fundamental solution, with pole at (ξ, τ), of the "constant coefficients" operator

$$L_\mu = \mu \Delta_m + Y, \tag{1.8}$$

and μ is any positive number greater than λ appearing in (H1). The constant C in (1.7) depends upon the Hölder norms of the coefficients and on μ.

Other known results concerning the case of *continuous coefficients* are the Harnack inequality and mean value formulas for the solutions to (1.1) by Polidoro [21], a lower bound for the fundamental solution by Polidoro [22], Schauder type estimates by Šatyro [24], Manfredini [15], Lunardi [14] and Pascucci [19], a theory for the Dirichlet problem for linear equations by Manfredini [15] and for quasilinear equations Lanconelli and Lascialfari [9], Lascialfari and Morbidelli [11].

In this note we prove an upper bound analogous to (1.7), with the constant C independent of the moduli of continuity of the coefficients. It is well known that, in the case of classical parabolic operators in divergence form, *uniform* global upper (and lower) bounds for the fundamental solution have been proved by Nash [18], Aronson [1], and Davies [2]. The proofs of the lower bound by Aronson and Davies rely on the Moser's parabolic Harnack inequality [16,17]. However, as Fabes and Stroock emphasized in [5], the upper bound is an important tool for using the ideas of Nash in order to directly obtain the lower bound and then to derive the Harnack inequality and the local Hölder continuity of weak solutions. The main motivation of our work is to follow the same procedure and prove analogous results for (1.1).

The interest in the study of Eq. (1.1) is motivated by the kinetic theory and by the theory of stochastic processes. For instance, (1.1) contains the family of kinetic equations of the form

$$\partial_t f - \langle v, \nabla_x \rangle f = Q(f), \quad t \geqslant 0, \ x \in \mathbb{R}^m, \ v \in \mathbb{R}^m, \tag{1.9}$$

where f is a density function and $Q(f)$ is a quadratic operator which describes some kind of collisions. Meaningful examples are the linear Fokker–Planck operator (cf. [3,23])

$$Q(f) = \sum_{j=1}^{m} \partial_{v_j v_j} f,$$

the Boltzmann–Landau operator (cf. [12,13])

$$Q(f) = \sum_{i,j=1}^{m} \partial_{v_i}\big(a_{ij}(\cdot, f)\partial_{v_j}\big),$$

where the coefficients a_{ij} depend on the unknown function through some integral expressions.

Our main result is the following

Theorem 1.1. *Under hypotheses* (H1) *and* (H2)*, there exist two positive constants C and μ, only dependent on λ in* (1.5) *and on B, such that*

$$\Gamma(x,t,\xi,\tau) \leqslant C\Gamma_\mu(x,t,\xi,\tau), \quad \forall x,\xi \in \mathbb{R}^N,\ t > \tau.$$

Here Γ_μ is the fundamental solution of the operator in (1.8)*.*

We remark that if (1.1) is an uniformly parabolic equation (i.e., $m = N$ and $B \equiv 0$), then (H2) is clearly satisfied. Indeed, (1.2) simply becomes the heat operator which is hypoelliptic and homogeneous with respect to the parabolic dilations $\delta_r(x,t) = (rx, r^2 t)$. Then our result recovers Aronson's upper bound proved in [1].

This note is organized as follows. In Section 2 we set the notations and we describe the natural geometry underlying operator L, which is determined by a suitable homogeneous Lie group structure on $\mathbb{R}^{N+1}$. In Section 3 we recall the main results on L that are needed in the sequel and we prove a Nash type upper bound. Section 4 is devoted to the proof of Theorem 1.1.

2. The geometric framework

In this section we set the notations and recall some known facts about Eq. (1.1).

We denote by $\nabla = (\partial_{x_1}, \dots, \partial_{x_N})$, $\langle\cdot,\cdot\rangle$, respectively, the gradient and the inner product in $\mathbb{R}^N$ and we recall notation (1.3). For greater convenience, we rewrite operator L in (1.1) in the compact form

$$L = \operatorname{div}(A\nabla) + Y, \tag{2.1}$$

where $A = (a_{ij})_{1\leqslant i,j\leqslant N}$ is the $N \times N$ matrix with $a_{ij} \equiv 0$ if $i > m$ or $j > m$.

The constant coefficients operator L_μ in (1.8) has the remarkable property of being invariant with respect to a Lie product in $\mathbb{R}^{N+1}$. More precisely, we let

$$E(s) = \exp(-sB^T), \quad s \in \mathbb{R}, \tag{2.2}$$

and we denote by ℓ_ζ, $\zeta \in \mathbb{R}^{N+1}$, the left translation $\ell_\zeta(z) = \zeta \circ z$ in the group law

$$(x,t) \circ (\xi,\tau) = \bigl(\xi + E(\tau)x, t+\tau\bigr), \quad (x,t), (\xi,\tau) \in \mathbb{R}^{N+1}, \tag{2.3}$$

then we have

$$L_\mu(u \circ \ell_\zeta) = (L_\mu u) \circ \ell_\zeta.$$

Let us explicitly note that the Lie product "$\circ$" does not depend on $\mu > 0$.

If L satisfies hypothesis (H2), it is not restrictive to assume that the matrix B has the form (1.6). Then the dilations associated to L_μ are given by

$$\delta_r(x,t) = \bigl(D(r)x, r^2 t\bigr), \quad r > 0, \ (x,t) \in \mathbb{R}^{N+1}, \tag{2.4}$$

where

$$D(r) = \operatorname{diag}(r I_{m_0}, r^3 I_{m_1}, \ldots, r^{2q+1} I_{m_q}) \tag{2.5}$$

and I_{m_k} denotes the $m_k \times m_k$ identity matrix. In the sequel we shall need the following simple identity:

$$r^2 D(r) B = B D(r), \quad \forall r > 0. \tag{2.6}$$

By reader's convenience, we write more explicitly the second assertion in hypothesis (H2):

$$L_\mu(u \circ \delta_r) = r^2 (L_\mu u) \circ \delta_r, \quad \forall r, \mu > 0. \tag{2.7}$$

Remark 2.1. A transformation of the form

$$\zeta \mapsto z_0 \circ \delta_r(\zeta), \quad r > 0, \ z_0 \in \mathbb{R}^{N+1}, \tag{2.8}$$

preserves the class of differential equations considered. More precisely, if u is a weak solution of L then the function

$$v(\zeta) = u\bigl(z_0 \circ \delta_r(\zeta)\bigr)$$

is a solution of $L_{\tilde{A}}$ where $\tilde{A}(\zeta) = A(z_0 \circ \delta_r(\zeta))$. Note that $L_{\tilde{A}}$ satisfies hypothesis (H1) and (H2) with the same constant λ of L and it has the same first-order part Y.

We next give the explicit expression of the fundamental solution Γ_μ of L_μ with pole at the origin. By using notation (1.4) we have

$$\Gamma_\mu(x,t) = \begin{cases} \frac{(4\pi\mu)^{-N/2}}{\sqrt{\det \mathcal{C}(t)}} \exp\bigl(-\frac{1}{4\mu}\langle \mathcal{C}^{-1}(t)x, x\rangle\bigr), & \text{if } t > 0, \\ 0, & \text{if } t \leqslant 0. \end{cases}$$

In view of the invariance properties of L_μ, it is not difficult to show that

$$\Gamma_\mu(z,\zeta) = \Gamma_\mu(\zeta^{-1} \circ z) = r^{-Q} \Gamma_\mu\bigl(\delta_r(\zeta^{-1} \circ z)\bigr) \tag{2.9}$$

for every $z \in \mathbb{R}^{N+1} \setminus \{0\}$, $r, \mu > 0$, where

$$Q = m + 3m_1 + \cdots + (2q+1)m_q. \tag{2.10}$$

The natural number $Q + 2$ is usually called the *homogeneous dimension of* $\mathbb{R}^{N+1}$ *with respect to* $(\delta_r)_{r>0}$. This denomination is proper since the Jacobian $J\delta_r$ equals r^{Q+2}.

We next show that any Gaussian function which is homogeneous with respect to the dilation group $(\delta_r)_{r>0}$ can be bounded by a suitable fundamental solution Γ_μ.

Remark 2.2. A simple consequence of the $(\delta_r)_{r>0}$-invariance of Γ_μ is the following identity (see Eq. (1.22) in [10]):

$$\mathcal{C}(t) = D(\sqrt{t})\mathcal{C}(1)D(\sqrt{t}), \quad \forall t > 0.$$

Then, since $\mathcal{C}(1)$ is a strictly positive symmetric matrix, we find that for every positive k there exist two positive constants C and μ, only dependent on k and B, such that

$$t^{-Q/2}\exp\left(-k\left|D\left(\frac{1}{\sqrt{t}}\right)x\right|^2\right) \leqslant t^{-Q/2}\exp\left(-\frac{1}{4\mu}\left\langle \mathcal{C}^{-1}(1)D\left(\frac{1}{\sqrt{t}}\right)x, D\left(\frac{1}{\sqrt{t}}\right)x\right\rangle\right) \leqslant C\Gamma_\mu(x,t)$$

for any $(x,t) \in \mathbb{R}^{N+1}$.

We finally introduce a norm which is homogeneous of degree one with respect to $(\delta_r)_{r>0}$. Let $\alpha_1, \ldots, \alpha_N$ be the positive integers such that

$$D(r) = \mathrm{diag}(r^{\alpha_1}, \ldots, r^{\alpha_N}) \tag{2.11}$$

with $D(\cdot)$ defined in (2.5). We set $\|z\| = \varrho$ if $z \neq 0$ and ϱ is the unique positive solution to the equation

$$\left|\delta_{1/\varrho}(z)\right|^2 = \frac{x_1^2}{\varrho^{2\alpha_1}} + \frac{x_2^2}{\varrho^{2\alpha_2}} + \cdots + \frac{x_N^2}{\varrho^{2\alpha_N}} = 1.$$

It is clear that $\|z\| = 1$ if and only if $|z| = 1$, moreover

$$\left\|\delta_r(z)\right\| = r\|z\|, \quad r > 0,\ z \in \mathbb{R}^{N+1}.$$

3. Some known results

In this section we recall the main known results concerning the operator L, with variable coefficients, that are needed in the sequel. We also prove an upper bound for Γ analogous to the one by Nash [18].

The first result, proved in [21], concerns the case of Hölder continuous coefficients.

Theorem 3.1. *Let L be as in* (2.1) *verifying hypotheses* (H1), (H2), *and assume that the matrix A has Hölder continuous entries with respect to the homogeneous norm $\|\cdot\|$. Then there exists a fundamental solution Γ for L, which is a function defined on* $(\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}) \setminus \{(z,z)\colon z \in \mathbb{R}^{N+1}\}$ *which satisfies the following conditions*:

(i) *For fixed $\zeta \in \mathbb{R}^{N+1}$, $\Gamma(\cdot,\zeta)$ is a solution to* (2.1) *in* $\mathbb{R}^{N+1} \setminus \{\zeta\}$;
(ii) *For every continuous function φ in $\mathbb{R}^N$, if $x \in \mathbb{R}^N$ then*

$$\lim_{t\to\tau^+} \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\varphi(\xi)\,d\xi = \varphi(x).$$

Moreover Γ has the following properties:

$$\int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\,d\xi = 1, \quad t > \tau, \tag{3.1}$$

$$\Gamma(x,t,\xi,\tau) = \int_{\mathbb{R}^N} \Gamma(x,t,y,s)\Gamma(y,s,\xi,\tau)\,dy, \quad \tau < s < t. \tag{3.2}$$

The function $\Gamma^*(\zeta, z) = \Gamma(z, \zeta)$ is the fundamental solution to the adjoint operator L^* of L, defined by

$$L^* = \operatorname{div}(A\nabla) - Y,$$

and it satisfies the dual statements of (i), (ii), (3.1), *and* (3.2).

The second result was proved in [20] and it is a L^∞ bound for the solutions to $Lu = 0$. For the next statement, we have to introduce a family of cylinders defined in terms of the Lie product (2.3) and the dilations (2.4) on $\mathbb{R}^{N+1}$ naturally associated to L. Consider the Euclidean cylinder

$$R_1 = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid |x| < 1,\ |t| < 1\}.$$

For every $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$ and $r > 0$, we set

$$R_r(z_0) \equiv z_0 \circ \big(\delta_r(R_1)\big) = \{z \in \mathbb{R}^{N+1} \mid z = z_0 \circ \delta_r(\zeta),\ \zeta \in R_1\} \tag{3.3}$$

and

$$R_r^-(z_0) = R_r(z_0) \cap \{t < t_0\}. \tag{3.4}$$

We recall that a *weak solution* of (1.1) is a function u such that $u, \partial_{x_1}u, \ldots, \partial_{x_m}u, Yu \in L^2_{\mathrm{loc}}$ and

$$\int -\langle A\nabla u, \nabla\varphi\rangle + \varphi Yu = 0, \quad \forall \varphi \in C_0^\infty. \tag{3.5}$$

In this note we only need to consider classical solutions which are, obviously, solutions also in the weak sense.

Theorem 3.2. *Let u be a weak solution of* (1.1) *in a domain Ω. Let $z_0 \in \Omega$ and r, ϱ, $0 < r/2 \leqslant \varrho < r$, be such that $\overline{R_r(z_0)} \subseteq \Omega$. Then there exists a positive constant C which depends only on λ and on the matrix B, such that*

$$\sup_{R_\varrho(z_0)} |u| \leqslant \left(\frac{C}{(r-\varrho)^{Q+2}} \int_{R_r(z_0)} |u|^p \right)^{1/p}, \quad \forall p \geqslant 1. \tag{3.6}$$

As a straightforward application of Theorem 3.2, we get the following

Theorem 3.3 (Nash upper bound). *Let Γ be a fundamental solution of the operator L satisfying the hypotheses* (H1) *and* (H2). *Then there exists a positive constant C, dependent only on λ and B, such that*

$$\Gamma(x,t,\xi,\tau) \leqslant \frac{C}{(t-\tau)^{Q/2}}, \quad \forall x,\xi \in \mathbb{R}^N,\ t>\tau. \tag{3.7}$$

Proof. We simply rely on Theorem 3.2 and on (3.1) of Theorem 3.1. Indeed,

$$\begin{aligned}\Gamma(x,t,\xi,\tau) &\leqslant \sup_{R_{\sqrt{t-\tau}/2}(z)} \Gamma(\cdot,\cdot,\xi,\tau) \quad \text{(by Theorem 3.2)}\\ &\leqslant \frac{C}{(t-\tau)^{(Q+2)/2}} \iint\limits_{R_{\sqrt{t-\tau}}(z)} \Gamma(x',t',\xi,\tau)\,dx'\,dt'\\ &\leqslant \frac{C}{(t-\tau)^{(Q+2)/2}} \iint\limits_{\mathbb{R}^N\times\,]\tau,\tau+2(t-\tau)[} \Gamma(x',t',\xi,\tau)\,dx'\,dt'\\ &= \frac{2C(t-\tau)}{(t-\tau)^{(Q+2)/2}}.\end{aligned}$$

This completes the proof. □

Corollary 3.4. *There exists a positive constant C, dependent only on λ and B, such that*

$$\int\limits_{\mathbb{R}^N} \Gamma^2(x,t,\xi,\tau)\,d\xi \leqslant \frac{C}{(t-\tau)^{Q/2}}, \qquad \int\limits_{\mathbb{R}^N} \Gamma^2(x,t,\xi,\tau)\,dx \leqslant \frac{C}{(t-\tau)^{Q/2}},$$

for any $x,\xi \in \mathbb{R}^N$, $t>\tau$.

4. The Aronson type bound

In this section we adapt the Aronson's method to prove Theorem 1.1. In the sequel, the letter C denotes a positive constant, dependent only on λ and on the matrix B, which is not always the same. Then, to avoid confusion, we use the symbols $\simeq$ (respectively, $\lesssim$) instead of $=$ (respectively, $\leqslant$), to warn the reader of the change of value of C in subsequent expressions.

We first give some preliminary results.

Theorem 4.1. *Let u_0 be an $L^2(\mathbb{R}^N)$ function such that $u_0(x)=0$ for $|x-y|<\sigma$, where $y \in \mathbb{R}^N$ and $\sigma>0$ are fixed. Suppose that u is a bounded solution to* (1.1) *in $\mathbb{R}^N \times\,]\eta,\eta+\sigma^2]$ with initial value $u(x,\eta)=u_0(x)$. Then, for any s which satisfies $0<s-\eta \leqslant \min\{1,\sigma^2\}$, we have*

$$\big|u\big((y,\eta)\circ(0,s-\eta)\big)\big| \leqslant C(s-\eta)^{-Q/4}\exp\left(-\frac{\sigma^2}{C(s-\eta)}\right)\|u_0\|_{L^2(\mathbb{R}^N)}, \tag{4.1}$$

where the constant $C>0$ depends only upon λ and B.

Proof. We first prove the thesis in the case $(y,\eta)=(0,0)$.

Proceeding as in the proof of Theorem 2 in paper [1] by Aronson, our first goal is to show that

$$\int_{\mathbb{R}^N} e^{2h}u^2|_{t=\tau}\,dx - 2\iint_{\mathbb{R}^N\times]0,\tau[} e^{2h}u^2\big(2\langle A\nabla_m h,\nabla_m h\rangle - Yh\big)\,dx\,dt \leqslant \int_{\mathbb{R}^N} e^{2h}u^2|_{t=0}\,dx, \tag{4.2}$$

where the function h is defined as follows (recall notation (2.11)):

$$h(x,t) = -\big|D\big(\big(k\varphi(t)\big)^{-1/2}\big)x\big|^2 = -\sum_{j=1}^{N}\frac{x_j^2}{(k\varphi(t))^{\alpha_j}},\quad 0<t\leqslant s,$$

and $\varphi(t)=2s-t$. To prove (4.2), we consider, for $R\geqslant 2$, a function $\gamma_R\in C_0^\infty(\mathbb{R}^N,[0,1])$ such that $\gamma_R(x)\equiv 1$ for $|x|\leqslant R-1$, $\gamma_R(x)\equiv 0$ for $|x|\geqslant R$, and $|\nabla\gamma_R|$ is bounded by a constant independent of R. Then we multiply both sides of (1.1) by $\gamma_R e^{2h}u$ and we integrate over $\mathbb{R}^N\times]0,\tau[$, $0<\tau\leqslant s$. After some standard computations, we get

$$\int_{\mathbb{R}^N}\gamma_R^2 e^{2h}u^2|_{t=\tau}\,dx - 2\iint_{\mathbb{R}^N\times]0,\tau[}\gamma_R^2 e^{2h}u^2\big(2\langle A\nabla_m h,\nabla_m h\rangle - Yh\big)\,dx\,dt \leqslant \int_{\mathbb{R}^N}\gamma_R^2 e^{2h}u^2|_{t=0}\,dx + \iint_{\mathbb{R}^N\times]0,\tau[} e^{2h}u^2\big(4\lambda|\nabla_m\gamma_R|^2 + \big|Y\gamma_R^2\big|\big)\,dx\,dt.$$

We next let R go to infinity in the above equation. Since u is bounded and $e^{2h(x,t)}\leqslant e^{-|x|^2/(2ks)}$, the last integral tends to zero and we get (4.2).

We now claim that, by a suitable choice of $k>0$ only dependent on λ and B, we have

$$2\langle A\nabla_m h,\nabla_m h\rangle - Yh\leqslant 0,\quad \text{in } \mathbb{R}^N\times]0,s]. \tag{4.3}$$

Indeed, since $\alpha_1=\cdots=\alpha_m=1$ and

$$\partial_{x_j}h(x,t) = -\frac{2x_j}{k\varphi(t)},\quad j=1,\ldots,m,$$

we have

$$\langle A\nabla_m h,\nabla_m h\rangle \leqslant \frac{4\lambda}{k\varphi(t)}\sum_{j=1}^{m}\frac{x_j^2}{k\varphi(t)} \leqslant \frac{4\lambda}{k\varphi(t)}\big|h(x,t)\big|.$$

On the other hand,

$$-Yh(x,t) = -\big\langle x, B\nabla h(x,t)\big\rangle + \partial_t h(x,t) = 2\big\langle x, BD\big(\big(k\varphi(t)\big)^{-1}\big)x\big\rangle + \partial_t h(x,t).$$

By (2.6), we have

$$\big\langle x, BD\big(\big(k\varphi(t)\big)^{-1}\big)x\big\rangle = \frac{1}{k\varphi(t)}\big\langle D\big(\big(k\varphi(t)\big)^{-1/2}\big)x, BD\big(\big(k\varphi(t)\big)^{-1/2}\big)x\big\rangle,$$

then

$$
\begin{aligned}
-Yh(x,t) &= \frac{2}{k\varphi(t)}\big\langle D\big(\big(k\varphi(t)\big)^{-1/2}\big)x,\, BD\big(\big(k\varphi(t)\big)^{-1/2}\big)x\big\rangle - \partial_t \sum_{j=1}^{N} \frac{x_j^2}{(k\varphi(t))^{\alpha_j}} \\
&\leqslant \frac{1}{k\varphi(t)}\left(2\|B\|\big|D\big(\big(k\varphi(t)\big)^{-1/2}\big)x\big|^2 - k\sum_{j=1}^{N}\frac{\alpha_j x_j^2}{(k\varphi(t))^{\alpha_j}}\right) \\
&\leqslant \frac{|h(x,t)|}{k\varphi(t)}\big(2\|B\| - k\big).
\end{aligned}
$$

Consequently, we get

$$
2\langle A\nabla_m h, \nabla_m h\rangle - Yh \leqslant \frac{|h(x,t)|}{k\varphi(t)}\big(8\lambda + 2\|B\| - k\big),
$$

and therefore (4.3) obviously follows with $k = k(\lambda, B)$.

From (4.3) and (4.2), we derive the inequality

$$
\begin{aligned}
\max_{t\in\,]0,s[} \int\limits_{|D(2/\sqrt{s})x|\leqslant 1} e^{2h(x,t)}u^2(x,t)\,dx &\leqslant \max_{t\in\,]0,s[} \int\limits_{\mathbb{R}^N} e^{2h(x,t)}u^2(x,t)\,dx \\
&\leqslant \int\limits_{|x|\geqslant\sigma} e^{2h(x,0)}u_0^2(x)\,dx. \qquad (4.4)
\end{aligned}
$$

As a consequence, we obtain the following L^2_{loc} estimate which is a weak version of (4.1):

$$
\max_{t\in\,]0,s[} \int\limits_{|D(2/\sqrt{s})x|\leqslant 1} u^2(x,t)\,dx \leqslant e^{1/(2k)}\exp\left(-\frac{C\sigma^2}{ks}\right)\|u_0\|^2_{L^2(\mathbb{R}^N)}. \qquad (4.5)
$$

To prove (4.5) we observe that, if $|D(2/\sqrt{s})x| \leqslant 1$, then

$$
-2h(x,t) \leqslant 2\left|D\left(\frac{1}{\sqrt{ks}}\right)x\right|^2 = 2\left|D\left(\frac{1}{2\sqrt{k}}\right)D\left(\frac{2}{\sqrt{s}}\right)x\right|^2 \leqslant \frac{1}{2k} \qquad (4.6)
$$

for every $t \in\,]0,s]$. On the other hand, if $|x| \geqslant \sigma$, we have

$$
-2h(x,0) = 2\left|D\left(\frac{1}{\sqrt{2ks}}\right)x\right|^2
$$

(since, by our assumption, $s \leqslant 1$, there exists a constant $C = C(\lambda, B) > 0$ such that)

$$
\geqslant \frac{C|x|^2}{ks} \geqslant \frac{C\sigma^2}{ks}. \qquad (4.7)
$$

Plugging (4.6) and (4.7) in (4.4), we easily obtain (4.5). We next rely on Theorem 3.2 in order to get the desired estimate (4.1):

$$\begin{aligned}\left|u(0,s)\right|^2 &\leqslant \sup_{R^-_{\sqrt{s}/4}(0,s)} |u|^2 \leqslant \frac{C}{s^{(Q+2)/2}} \iint\limits_{R^-_{\sqrt{s}/2}(0,s)} u^2(x,t)\,dx\,dt \\ &= \frac{C}{s^{(Q+2)/2}} \int_{3s/4}^{s} \int\limits_{|D(2/\sqrt{s})x|\leqslant 1} u^2(x,t)\,dx\,dt \quad \text{(by (4.5))} \\ &\lesssim \frac{C}{s^{Q/2}} \exp\left(-\frac{\sigma^2}{Cs}\right) \|u_0\|^2_{L^2(\mathbb{R}^N)},\end{aligned}$$

where the constant $C = C(\lambda, B)$. This yields (4.1) in the case $(y,\eta) = (0,0)$.

For the general case, fixed u and u_0 as in the statement and (y,η), we set

$$v(x,t) = u\big((y,\eta)\circ(x,t)\big), \quad v_0(x) = u_0(x+y), \quad x \in \mathbb{R}^N,\ t > 0.$$

We observe that $v_0(x) = 0$ for $|x - y| \leqslant \sigma$. Moreover, by Remark 2.1, if $\tilde{A} = A \circ \ell_{(y,\eta)}$, we have

$$L_{\tilde{A}} v = 0, \quad \text{and} \quad v(\cdot,0) = v_0.$$

Thus, as in the preceding case, we get

$$\left|u\big((y,\eta)\circ(0,s-\eta)\big)\right| = \left|v(0,s-\eta)\right| \leqslant \frac{C}{(s-\eta)^{Q/4}} \exp\left(-\frac{\sigma^2}{C(s-\eta)}\right) \|v_0\|_{L^2(\mathbb{R}^N)}$$

and this yields the thesis. □

Theorem 4.1 has the following dual version. The proof follows exactly the same lines and, for this reason, will be omitted.

Theorem 4.2. *Let u_0, y, and σ be as in the previous statement. Suppose that u is a bounded solution to the adjoint operator L^* in $\mathbb{R}^N \times]\eta - \sigma^2, \eta]$ with final value $u(x,\eta) = u_0(x)$. Then, for any s which satisfies $0 < \eta - s \leqslant \min\{1, \sigma^2\}$, we have*

$$\left|u\big((y,\eta)\circ(0,s-\eta)\big)\right| \leqslant C(\eta - s)^{-Q/4} \exp\left(-\frac{\sigma^2}{C(\eta - s)}\right) \|u_0\|_{L^2(\mathbb{R}^N)}, \tag{4.8}$$

where the constant $C > 0$ depends only upon λ and B.

As a simple consequence of the above Theorems 4.1 and 4.2 we obtain the following

Lemma 4.3. *If $\sigma > 0$ and $0 < s - \eta \leqslant \min\{1, \sigma^2\}$, then*

$$\int\limits_{|\xi - E(\eta - s)x| \geqslant \sigma} \Gamma^2(x,s,\xi,\eta)\,d\xi \leqslant \frac{C}{(s-\eta)^{Q/2}} \exp\left(-\frac{\sigma^2}{C(s-\eta)}\right), \quad x \in \mathbb{R}^N, \tag{4.9}$$

for some constant $C = C(\lambda, B) > 0$. Analogously, if $0 < \eta - s \leqslant \min\{1, \sigma^2\}$, we have

$$\int\limits_{|\xi - E(\eta - s)x| \geqslant \sigma} \Gamma^2(\xi,\eta,x,s)\,d\xi \leqslant \frac{C}{(\eta - s)^{Q/2}} \exp\left(-\frac{\sigma^2}{C(\eta - s)}\right), \quad x \in \mathbb{R}^N. \tag{4.10}$$

Proof. We only give the proof of (4.9), since the one of (4.10) is analogous. Setting $(y,\eta) = (x,s) \circ (0,\eta - s)$, estimate (4.9) reads

$$\int\limits_{|\xi - y| \geqslant \sigma} \Gamma^2\big((y,\eta)\circ(0,s-\eta),\xi,\eta\big)\,d\xi \leqslant \frac{C}{(s-\eta)^{Q/2}}\exp\left(-\frac{\sigma^2}{C(s-\eta)}\right). \tag{4.11}$$

We consider the function

$$u(x',t) = \int\limits_{|\xi - y| \geqslant \sigma} \Gamma(x',t,\xi,\eta)\Gamma\big((y,\eta)\circ(0,s-\eta),\xi,\eta\big)\,d\xi,$$

which is a non-negative solution to (1.1) in the set $\{t > \eta\}$, with initial condition $u(x',\eta) = 0$ for $|x' - y| < \sigma$ and $u(x',\eta) = \Gamma((y,\eta)\circ(0,s-\eta),x',\eta)$ for $|x'-y| \geqslant \sigma$. By choosing $(x',t) = (y,\eta)\circ(0,s-\eta)$ and by Theorem 4.1, we obtain

$$\int\limits_{|\xi - y| \geqslant \sigma} \Gamma^2\big((y,\eta)\circ(0,s-\eta),\xi,\eta\big)\,d\xi = u\big((y,\eta)\circ(0,s-\eta)\big)$$
$$\leqslant \frac{C}{(s-\eta)^{Q/4}}\exp\left(-\frac{\sigma^2}{C(s-\eta)}\right)\left(\int \Gamma^2\big((y,\eta)\circ(0,s-\eta),\xi,\eta\big)\,d\xi\right)^{1/2},$$

and we get the thesis by Corollary 3.4. □

We are now in position to give the

Proof of Theorem 1.1. We start by proving that

$$\Gamma(x,1) \leqslant Ce^{-|x|^2/C} \tag{4.12}$$

for every $x \in \mathbb{R}^N$ with $C = C(\lambda, B) > 0$. As noticed in Remark 2.2, it follows that there exist two positive constant C and μ such that

$$\Gamma(x,1) \leqslant C\Gamma_\mu(x,1), \quad \forall x \in \mathbb{R}^N. \tag{4.13}$$

Here, Γ_μ denotes the fundamental solution to L_μ in (1.8) with pole at the origin. We first prove (4.12), then we will conclude the proof of Theorem 1.1 by using the invariance of L_μ and Γ_μ with respect to the dilations and the translations groups.

Fixed $x \in \mathbb{R}^N$, we set

$$\sigma = \frac{|x|}{2\|E(1/2)\|},$$

where $E(\cdot)$ is defined in (2.2) and we assume that $\sigma \geqslant 1$. By the reproduction property (3.2) we have

$$\Gamma(x,1) = \int\limits_{\mathbb{R}^N} \Gamma(x,1,\xi,1/2)\Gamma(\xi,1/2,0,0)\,d\xi.$$

We split the integral over $\mathbb{R}^N$ into an integral J_1 over $|\xi - E(-1/2)x| > \sigma$ and an integral J_2 over $|\xi - E(-1/2)x| \leqslant \sigma$. By the Schwartz inequality, we have

$$(J_1)^2 \leqslant \int\limits_{|\xi - E(-1/2)x| > \sigma} \Gamma^2(x, 1, \xi, 1/2)\, d\xi \int\limits_{|\xi - E(-1/2)x| > \sigma} \Gamma^2(\xi, 1/2, 0, 0)\, d\xi$$

(using (4.9) of Lemma 4.3 to estimate the first integral on the right and Corollary 3.4 to estimate the second one)

$$\leqslant C e^{-\sigma^2/C} \lesssim C e^{-|x|^2/C}.$$

Aiming to use (4.10) of Lemma 4.3 to estimate J_2, we first remark that

$$\left|\xi - E(-1/2)x\right| \leqslant \sigma \quad \Rightarrow \quad |\xi| \geqslant \sigma. \tag{4.14}$$

Indeed, we have

$$|x| = \left|E(1/2)E(-1/2)x\right| \leqslant \left\|E(1/2)\right\| \left|E(-1/2)x\right|,$$

so that

$$\left|E(-1/2)x\right| \geqslant \frac{|x|}{\|E(1/2)\|} = 2\sigma,$$

and this proves (4.14). Thus J_2 is dominated by the integral over $|\xi| \geqslant \sigma$ which can be estimated with the same argument used above, by means of (4.10) of Lemma 4.3 and by Corollary 3.4. Therefore we have completed the proof of the bound of Γ in the case $|x| \geqslant 2\|E(1/2)\|$. On the other hand, if $|x| < 2\|E(1/2)\|$, then (4.12) is a direct consequence of Theorem 3.3.

The above argument proves (4.13). We next use (4.13) to deduce that

$$\Gamma(x, t) \leqslant C\Gamma_\mu(x, t), \quad \forall x \in \mathbb{R}^N, \ t > 0. \tag{4.15}$$

Set

$$\Gamma^{(r)} = r^Q \Gamma \circ \delta_r, \quad r > 0.$$

By Remark 2.1, $\Gamma^{(r)}$ is a fundamental solution of the operator

$$\operatorname{div}(A^{(r)}\nabla) + Y, \quad A^{(r)} = A \circ \delta_r,$$

which satisfies (H1) with the same constant λ. Therefore we have, by (4.13) and (2.9),

$$\Gamma(x, t) = t^{-Q/2}\Gamma^{(\sqrt{t})}\big(D(t^{-1/2})x, 1\big) \leqslant C t^{-Q/2}\Gamma_\mu\big(D(t^{-1/2})x, 1\big) = C\Gamma_\mu(x, t),$$

and then (4.15) follows.

We next conclude the proof of the theorem by using the invariance with respect to the translations. Let $z = (x, t)$, $\zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$ with $t > \tau$. We set

$$\tilde{\Gamma}(\tilde{z}, \tilde{\zeta}) = \Gamma(\zeta \circ \tilde{z}, \zeta \circ \tilde{\zeta}).$$

Then $\tilde{\Gamma}(\cdot, \tilde{\zeta})$ is a fundamental solution of the operator

$$\operatorname{div}(\tilde{A}\nabla) + Y, \quad \tilde{A} = A \circ \ell_\zeta,$$

which satisfies (H1) with the same constant λ. Therefore, by (4.15) and (2.9) we have

$$\Gamma(z, \zeta) = \tilde{\Gamma}(\zeta^{-1} \circ z, 0) \leqslant C\Gamma_\mu(\zeta^{-1} \circ z, 0) = C\Gamma_\mu(z, \zeta),$$

and this completes the proof of Theorem 1.1. □

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References

[1] D.G. Aronson, Bounds for the fundamental solution of a parabolic equation, Bull. Amer. Math. Soc. 73 (1967) 890–896.
[2] E.B. Davies, Explicit constants for Gaussian upper bounds on heat kernels, Amer. J. Math. 109 (1987) 319–333.
[3] L. Desvillettes, C. Villani, On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker–Planck equation, Comm. Pure Appl. Math. 54 (2001) 1–42.
[4] S.D. Eidelman, S.D. Ivasyshen, H.P. Malytska, A modified Levi method: development and application, Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki 5 (1998) 14–19.
[5] E.B. Fabes, D.W. Stroock, A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash, Arch. Rational Mech. Anal. 96 (1986) 327–338.
[6] L. Hörmander, Hypoelliptic second order differential equations, Acta Math. 119 (1967) 147–171.
[7] A.M. Il'in, On a class of ultraparabolic equations, Dokl. Akad. Nauk SSSR 159 (1964) 1214–1217.
[8] L.P. Kupcov, The fundamental solutions of a certain class of elliptic-parabolic second order equations, Differ. Uravn. 8 (1972) 1649–1660, 1716.
[9] E. Lanconelli, F. Lascialfari, A boundary value problem for a class of quasi-linear operators of Fokker–Planck type, in: Proceedings of the Conference "Differential Equations" (Italian) (Ferrara, 1996), Vol. 41, 1997, pp. 65–84.
[10] E. Lanconelli, S. Polidoro, On a class of hypoelliptic evolution operators, Rend. Sem. Mat. Univ. Politec. Torino 52 (1994) 29–63.
[11] F. Lascialfari, D. Morbidelli, A boundary value problem for a class of quasilinear ultraparabolic equations, Comm. Partial Differential Equations 23 (1998) 847–868.
[12] E.M. Lifschitz, L.P. Pitaevskii, Teoreticheskaya Fizika ("Landau-Lifshits"), Tom 10, Nauka, Moscow, 1979.
[13] P.-L. Lions, On Boltzmann and Landau equations, Philos. Trans. Roy. Soc. London Ser. A 346 (1994) 191–204.
[14] A. Lunardi, Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in $\mathbb{R}^N$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 24 (1997) 133–164.
[15] M. Manfredini, The Dirichlet problem for a class of ultraparabolic equations, Adv. Differential Equations 2 (1997) 831–866.
[16] J. Moser, A Harnack inequality for parabolic differential equations, Comm. Pure Appl. Math. 17 (1964) 101–134.
[17] J. Moser, Correction to: A Harnack inequality for parabolic differential equations, Comm. Pure Appl. Math. 20 (1967) 231–236.
[18] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. 80 (1958) 931–954.
[19] A. Pascucci, Holder regularity for a Kolmogorov equation, Trans. Amer. Math. Soc. 355 (2003) 901–924.
[20] A. Pascucci, S. Polidoro, The Moser's iterative method for a class of ultra-parabolic equations, 2002, preprint.
[21] S. Polidoro, On a class of ultraparabolic operators of Kolmogorov–Fokker–Planck type, Matematiche (Catania) 49 (1994) 53–105.
[22] S. Polidoro, A global lower bound for the fundamental solution of Kolmogorov–Fokker–Planck equations, Arch. Rational Mech. Anal. 137 (1997) 321–340.
[23] H. Risken, The Fokker–Planck Equation: Methods of Solution and Applications, 2nd ed., Springer-Verlag, Berlin, 1989.
[24] J.I. Šatyro, The smoothness of the solutions of certain degenerate second order equations, Mat. Zametki 10 (1971) 101–111.
[25] I.M. Sonin, A class of degenerate diffusion processes, Teor. Veroyatnost. i Primenen. 12 (1967) 540–547.
[26] M. Weber, The fundamental solution of a degenerate partial differential equation of parabolic type, Trans. Amer. Math. Soc. 71 (1951) 24–37.