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ON THE HARNACK INEQUALITY FOR A CLASS OF HYPOELLIPTIC EVOLUTION EQUATIONS

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ABSTRACT. We give a direct proof of the Harnack inequality for a class of degenerate evolution operators which contains the linearized prototypes of the Kolmogorov and Fokker-Planck operators. We also improve the known results in that we find explicitly the optimal constant of the inequality.

1. INTRODUCTION

We consider the second order partial differential operator

$$L = \operatorname{div}(AD) + \langle x, BD\rangle - \partial_t, \qquad z = (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.1}$$

where D and div respectively denote the gradient and the divergence in $\mathbb{R}^N$. We assume that $A = (a_{ij})$ and $B = (b_{ij})$ are $N \times N$ real constant matrices in the form

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_r \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{1.2}$$

where A_0 is a $m_0 \times m_0$ symmetric and strictly positive definite matrix, and B_k is a $m_{k-1} \times m_k$ matrix of rank m_k, $k = 1, 2, \dots, r$, with

$$m_0 \geq m_1 \geq \cdots \geq m_r \geq 1 \qquad \text{and} \qquad \sum_{k=0}^{r} m_k = N.$$

We recall that (1.1) arises in the description of wide classes of stochastic processes and kinetic models (we refer to the classical monographies [4], [7] and [5]) and in mathematical finance (see [2] and [17]).

Before stating our results, it is worthwhile to make some few comments on the structural condition (1.2). We recall that, by Propositions 2.1 and 2.2 of [14], hypothesis (1.2) implies that the operator L verifies the classical Hörmander's rank condition [10]:

$$\text{rank Lie}\left(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y\right)(x,t) = N + 1, \qquad \forall (x,t) \in \mathbb{R}^{N+1}, \tag{1.3}$$

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where $\mathrm{Lie}\big(\partial_{x_1},\dots,\partial_{x_{m_0}},Y\big)$ denotes the Lie algebra generated by the first order differential operators $\partial_{x_1},\dots,\partial_{x_{m_0}}$ and $Y=\langle x,BD\rangle-\partial_t$, which is the first order part of L. Hence L is a hypoelliptic operator.

Moreover, (1.2) implies that L is homogeneous with respect to the dilations group on $\mathbb{R}^N$ defined by

$$\delta_\lambda=\mathrm{diag}(\lambda I_{m_0},\lambda^3 I_{m_1},\dots,\lambda^{2r+1}I_{m_r}),\qquad \lambda>0, \tag{1.4}$$

where I_{m_k} is the $m_k\times m_k$ identity matrix; that is, we have

$$L(u(\delta_\lambda x,\lambda^2 t))=\lambda^2(Lu)(\delta_\lambda x,\lambda^2 t) \tag{1.5}$$

for any smooth function u, $(x,t)\in\mathbb{R}^{N+1}$ and $\lambda>0$. We recall that (1.2) is also a necessary condition: indeed, any hypoelliptic and homogeneous operator L of the form (1.1) verifies condition (1.2) for a suitable basis of $\mathbb{R}^{N+1}$ (see [14]).

In view of expression (1.4) of the dilations δ_λ, it is convenient to denote the components of $x\in\mathbb{R}^N$ by

$$(x^{(0)},x^{(1)},\dots,x^{(r)}), \tag{1.6}$$

where $x^{(k)}\in\mathbb{R}^{m_k}$ for $k=0,\dots,r$. Moreover, the natural number

$$Q=m_0+3m_1+\cdots+(2r+1)m_r$$

is usually called the *homogeneous dimension of* $\mathbb{R}^N$ with respect to δ_λ.

The following Harnack inequality for positive solutions u of (1.1) has been proved by Kupcov [13], Garofalo and Lanconelli [8] and Lanconelli and one of us [14], by using some mean value formulas:

$$u(x_2,t_2)\ge c\;u(x_1,t_1), \tag{1.7}$$

for $(x_1,t_1),(x_2,t_2)\in\mathbb{R}^{N+1}$ with $t_1<t_2$ and for some positive $c=c(x_1,x_2,t_1,t_2)$. In this paper we aim to give a short and intuitive proof of the Harnack inequality by using an original variational argument due to Li and Yau [15]. In Corollary 1.2, we also refine the known results in that we find explicitly the optimal constant in (1.7). Our proof is based on the following gradient estimate for positive solutions to L in a strip $\mathbb{R}^N\times]0,T[$, $T>0$ (cf. Proposition 4.2):

$$-Yu+\frac{Q}{2t}u\ge\frac{\langle ADu,Du\rangle}{u}. \tag{1.8}$$

A Harnack inequality can be directly derived from estimate (1.8). Indeed, let $W=W(x,t)$ denote a vector field in $\mathbb{R}^N$; by adding $2\langle ADu,W\rangle+u\langle AW,W\rangle$ to both side of (1.8), we deduce

$$-Yu+\frac{Q}{2t}u+2\langle ADu,W\rangle+u\langle AW,W\rangle\ge 0. \tag{1.9}$$

Now we make a suitable choice of W. Fix $z_j=(x_j,t_j)\in\mathbb{R}^N\times]0,T[$, $j=1,2$, with $t_1<t_2$, and let us call any integral curve of the vector fields $\partial_{x_1},\dots,\partial_{x_{m_0}},-Y$ connecting z_1 to z_2 an *L-admissible path.* More precisely, we set

$$\mathcal{A}_{z_1,z_2}=\{\gamma\in C^\infty([t_1,t_2];\mathbb{R}^{N+1})\ \mid\ \dot\gamma=\sum_{i=1}^{m_0}\dot\gamma_i\partial_{x_i}-Y(\gamma),\ \gamma(t_j)=z_j,\ i=1,2\} \tag{1.10}$$

as the class of the L-admissible paths. We remark that $\mathcal{A}_{z_1,z_2}$ is not empty by Hörmander's rank condition and Chow's Theorem [6] (see also Nagel, Stein and

Wainger [16] for a general theory of metrics associated to vector fields). Fix $\gamma \in \mathcal{A}_{z_1,z_2}$, and put

$$W = \frac{A_0^{-1}\dot{\gamma}^{(0)}}{2}$$

in (1.9) (recall notation (1.6)) to obtain

$$\frac{d}{dt}u(\gamma) + \frac{Q}{2t}u(\gamma) + u(\gamma)\frac{1}{4}\langle A_0^{-1}\dot{\gamma}^{(0)}, \dot{\gamma}^{(0)}\rangle \geq 0, \qquad \text{in } [t_1, t_2].$$

Dividing by $u(\gamma)$ and integrating in the variable t over the interval $[t_1, t_2]$, we finally prove the first assertion of the following

Theorem 1.1 (Harnack inequality)**.** *Let u be positive solution to $Lu = 0$ in $\mathbb{R}^N \times]0, T[$. Let $x_1, x_2 \in \mathbb{R}^N$ and $0 < t_1 < t_2 < T$. Then*

$$u(x_2, t_2) \geq \left(\frac{t_1}{t_2}\right)^{\frac{Q}{2}} u(x_1, t_1) \exp\left(-\frac{1}{4}\inf \Phi(\gamma)\right), \tag{1.11}$$

where

$$\Phi(\gamma) = \int_{t_1}^{t_2} \langle A_0^{-1}\dot{\gamma}^{(0)}(s), \dot{\gamma}^{(0)}(s)\rangle ds, \qquad \gamma \in \mathcal{A}_{z_1,z_2}, \tag{1.12}$$

and the infimum in (1.11) *is taken over all L-admissible paths connecting (x_1, t_1) to (x_2, t_2). Moreover, there exists a unique polynomial function $\gamma \in \mathcal{A}_{z_1,z_2}$ which is the minimum of Φ.*

The argument used above is quite general and applies to many different problems: parabolic equations on manifolds (Li and Yau [15]), porous medium and p-diffusion equations (Auchmuty and Bao [1]), and sum of squares of vector fields (Cao and Yau [3]). The new difficulty in our problem is due to the fact that Φ is a strongly degenerate functional since it involves only the first m_0 components of γ. This is clearly related to the degeneracy of the differential equation (1.1). On the other hand the Hörmander condition ensures that Φ has the usual coercivity and compactness properties on the family of the L-admissible curves. The last assertion and the existence of the minimum of Φ will be proved in Section 2.

Although the exponent in (1.11) has an implicit expression, it can be written explicitly in the most interesting cases. For instance, in [1] it is proved that

$$u(x_2, t_2) \geq \left(\frac{t_1}{t_2}\right)^{\frac{N}{2}} u(x_1, t_1) \exp\left(-\frac{1}{4}\frac{|x_1 - x_2|^2}{t_2 - t_1}\right),$$

for every non-negative solution to the heat equation. This is a sharp version of the classical parabolic Harnack inequality by Pini [18] and Hadamard [9].

In Section 3, we extend the above inequality to the case of a Lie algebra of step two (considered by Garofalo and Lanconelli in [8]) and three (considered by Sonin [20] and Ivasishen and Voznyak [11]) and we prove the following corollary.

Corollary 1.2. *Let u be a positive solution to $Lu = 0$, where L is as in* (1.1)-(1.2) *with $r = 1, 2$. Then*

$$u(x_2,t_2) \geq \left(\frac{t_1}{t_2}\right)^{\frac{Q}{2}} u(x_1,t_1) \exp\left(-\frac{1}{4}\langle \mathcal{C}^{-1}(t_2-t_1)(x_2 - E(t_2-t_1)x_1), (x_2 - E(t_2-t_1)x_1)\rangle\right) \tag{1.13}$$

holds, where $\mathcal{C}$ and E are defined in equation (1.15) *below.*

We emphasize that estimate (1.13) is sharp since, in the general case $r \geq 0$, the fundamental solution of L in (1.1) is

$$\Gamma(z_2,z_1) = \frac{c_0}{(t_2-t_1)^{\frac{Q}{2}}} \exp\left(-\frac{1}{4}\langle \mathcal{C}^{-1}(t_2-t_1)(x_2 - E(t_2-t_1)x_1), (x_2 - E(t_2-t_1)x_1)\rangle\right), \tag{1.14}$$

for $t_2 > t_1$, and $\Gamma(z_2,z_1) = 0$ for $t_2 \leq t_1$. In (1.14), we denote

$$E(t) = \exp(-tB^T) \quad \text{and} \quad \mathcal{C}(t) = \int_0^t E(s)AE^T(s)ds, \tag{1.15}$$

where B^T is the transpose matrix of B and $c_0 = (4\pi)^{-\frac{N}{2}}(\det \mathcal{C}(1))^{-\frac{1}{2}}$. Note that Hörmander's condition ensures that $\mathcal{C}(t) > 0$ for any $t > 0$ (cf. Proposition A.1 in [14], see also [12]).

The paper is organized as follows. In Sections 2 and 3, we prove respectively Theorem 1.1 and Corollary 1.2. In Section 4, we prove the gradient estimate (1.8) for positive solutions to L.

2. Proof of Theorem 1.1

The purpose of this section is to complete the proof of Theorem 1.1 by showing that the functional Φ in (1.12) has a unique minimum which is a polynomial L-admissible path. To this end, we characterize the minima of Φ as critical points: we set

$$d\Phi(\gamma,\eta) = \int_{t_1}^{t_2} \langle A_0^{-1}\dot{\gamma}^{(0)}(s), \dot{\eta}^{(0)}(s)\rangle ds, \tag{2.1}$$

and we say that γ is a critical point of Φ in $\mathcal{A}_{z_1,z_2}$ if

$$d\Phi(\gamma,\eta) = 0 \quad \text{for every} \quad \eta \in \mathcal{A}_{0,0}. \tag{2.2}$$

We claim: *γ is a minimum of Φ in $\mathcal{A}_{z_1,z_2}$ if and only if it is a critical point of Φ.*

The "only if" part of the claim is standard, while the "if" part is a consequence of the fact that Φ is quadratic. Indeed, let γ be a critical point of Φ in $\mathcal{A}_{z_1,z_2}$. Then, for every $\tilde{\gamma} \in \mathcal{A}_{z_1,z_2}$, we have

$$\Phi(\tilde{\gamma}) = \Phi(\gamma) + 2d\Phi(\gamma, \gamma - \tilde{\gamma}) + \Phi(\gamma - \tilde{\gamma}) \geq \Phi(\gamma),$$

since $(\gamma - \tilde{\gamma}) \in \mathcal{A}_{0,0}$ and $\Phi(\gamma - \tilde{\gamma}) \geq 0$. This proves the claim.

In the sequel we assume for simplicity, since it is not restrictive, that $A_0 = I_{m_0}$. Aiming to further simplify the proof, we recall that the operator L in (1.1) has the remarkable property of being invariant with respect to a Lie product in $\mathbb{R}^{N+1}$. More precisely, we denote by ℓ_ζ, $\zeta \in \mathbb{R}^{N+1}$, the left translation $\ell_\zeta(z) = \zeta \circ z$ in the group law

$$(x,t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau), \qquad (x,t), (\xi,\tau) \in \mathbb{R}^{N+1};$$

then we have

$$L\left(u \circ \ell_\zeta\right) = (Lu) \circ \ell_\zeta.$$

Hence it is sufficient to put $z = z_1^{-1} \circ z_2$ and prove Theorem 1.1 in the case of $z_1 = 0$ and $z_2 = z$. We also recall that the explicit expression of z is

$$z = (x,t) = (x_2 - E(t_2 - t_1)x_1, t_2 - t_1). \tag{2.3}$$

We next introduce some notations in order to rewrite $d\Phi$ in the more convenient form (2.11); then we characterize the critical points in terms of differential equations. Lastly, we show that the Euler-Lagrange equation has a unique polynomial solution in $\mathcal{A}_{0,z}$, which is the minimum of Φ.

We remark that an L-admissible path γ is a solution to the system

$$\dot\gamma^{(k)} = -B_k^T \gamma^{(k-1)}, \qquad k = 1, \dots, r. \tag{2.4}$$

Thus, if we set $M_0 = I_{m_0}$ and

$$M_k = (-1)^k B_k^T \cdots B_1^T, \qquad k = 1, \dots, r, \tag{2.5}$$

we have

$$\frac{d^k}{dt^k}\gamma^{(k)} = M_k \gamma^{(0)}, \qquad k = 0, \dots, r. \tag{2.6}$$

Next, we denote $V_0 = \mathrm{Ker}(M_1)$ and by V_r the orthogonal space of the kernel of M_r in $\mathbb{R}^{m_0}$, that is,

$$V_r = (\mathrm{Ker}(M_r))^\perp.$$

Moreover, we define inductively the linear subspace V_k of $\mathbb{R}^{m_0}$ by the following formula:

$$V_k \oplus V_{k+1} \oplus \cdots \oplus V_r = (\mathrm{Ker}(M_k))^\perp, \qquad k = 1, \dots, r.$$

Then $\mathbb{R}^{m_0}$ is the direct sum of V_k for $k = 0, \dots, r$, and any L-admissible path γ has the following unique representation

$$\gamma^{(0)} = \gamma^{(0,0)} + \cdots + \gamma^{(0,r)}, \qquad \gamma^{(0,k)} \in V_k,\ k = 0, \dots, r. \tag{2.7}$$

We remark that, since the matrix M_k has maximum rank m_k, then $\dim V_r = m_r$ and

$$\dim V_k = m_k - m_{k+1}, \qquad k = 0, \dots, r-1. \tag{2.8}$$

Moreover, by formula (2.6), we have

$$\frac{d^k}{dt^k}\gamma^{(k)} = M_k \sum_{h=k}^{r} \gamma^{(0,h)}, \qquad k = 0, \dots, r. \tag{2.9}$$

If we denote by M_k^{-1} the (unique) right inverse of M_k such that $M_k^{-1}(\mathbb{R}^{m_k}) = (\mathrm{Ker}(M_k))^\perp$, then from (2.9) we infer that

$$\gamma^{(0,k)} = M_k^{-1}\frac{d^k}{dt^k}\gamma^{(k)} - \sum_{h=k+1}^{r} \gamma^{(0,h)}, \qquad k = 0, \dots, r. \tag{2.10}$$

Therefore, it is clear that we can rewrite the functional $d\Phi$ as follows:

$$d\Phi(\gamma,\eta)=\sum_{k=0}^{r}\int_0^t\langle M_k^{-1}\frac{d^{k+1}}{ds^{k+1}}\gamma^{(k)}(s)-\sum_{h=k+1}^{r}\dot\gamma^{(0,h)}(s),\qquad M_k^{-1}\frac{d^{k+1}}{ds^{k+1}}\eta^{(k)}(s)-\sum_{h=k+1}^{r}\dot\eta^{(0,h)}(s)\rangle ds. \tag{2.11}$$

Next, we prove existence and uniqueness of the critical point of Φ in $\mathcal{A}_{0,z}$. We need the following

Lemma 2.1. *Let $\gamma\in\mathcal{A}_{0,z}$ and $\eta\in\mathcal{A}_{0,0}$ be such that $\eta^{(0)}(s)\in V_k$ for some $k=0,\dots,r$ and for any $s\in[0,t]$. Then*

$$d\Phi(\gamma,\eta)=(-1)^{k+1}\int_0^t\langle\frac{d^{k+2}}{ds^{k+2}}\gamma^{(0,k)}(s),M_k^{-1}\eta^{(k)}(s)\rangle ds. \tag{2.12}$$

Proof. Since $\eta^{(0)}\equiv\eta^{(0,k)}$, by (2.10) we clearly have

$$\eta^{(0,h)}=M_h^{-1}\frac{d^h}{dt^h}\eta^{(h)}-\sum_{j=h+1}^{r}\eta^{(0,j)}=\begin{cases}M_k^{-1}\frac{d^k}{dt^k}\eta^{(k)}, & \text{if } h=k,\\ 0, & \text{otherwise.}\end{cases}$$

Then, it holds that

$$\begin{aligned}d\Phi(\gamma,\eta)&=\int_0^t\langle M_k^{-1}\frac{d^{k+1}}{ds^{k+1}}\gamma^{(k)}(s)-\sum_{h=k+1}^{r}\dot\gamma^{(0,h)}(s),M_k^{-1}\frac{d^{k+1}}{ds^{k+1}}\eta^{(k)}(s)\rangle ds\\&=\int_0^t\langle M_k^{-1}\frac{d^{k+1}}{ds^{k+1}}\gamma^{(k)}(s),M_k^{-1}\frac{d^{k+1}}{ds^{k+1}}\eta^{(k)}(s)\rangle ds\end{aligned}$$

(by (2.9) and integrating by parts, since there is no contribution at the boundary due to the fact that $\eta\in\mathcal{A}_{0,0}$)

$$=(-1)^{k+1}\int_0^t\langle\sum_{h=k}^{r}\frac{d^{k+2}}{ds^{k+2}}\gamma^{(0,h)}(s),M_k^{-1}\eta^{(k)}(s)\rangle ds.$$

Finally, the thesis follows since, by hypothesis, $M_k^{-1}\eta^{(k)}\in V_k$. □

Now, by making a suitable choice of the path η in (2.2), we derive some necessary and sufficient conditions for the existence of a critical point γ of Φ in $\mathcal{A}_{0,z}$. Fixed $k\in\{0,1,\dots,r\}$, $v\in V_k$ and $\phi\in C_0^\infty([0,t])$, we consider

$$\eta=\left(\frac{d^k\varphi}{ds^k}v,\frac{d^{k-1}\varphi}{ds^{k-1}}M_1v,...,\varphi M_kv,0,...,0\right).$$

Clearly $\eta\in\mathcal{A}_{0,0}$ and $\eta^{(0)}\in V_k$. Then we can apply Lemma 2.1, that gives

$$d\Phi(\gamma,\eta)=(-1)^{k+1}\int_0^t\langle\frac{d^{k+2}}{ds^{k+2}}\gamma^{(0,k)}(s),v\rangle\varphi(s)ds=0$$

(recall that $M_k^{-1}\eta^{(k)}=\varphi v\in V_k$). Thus, being φ and v arbitrary, we deduce the following *necessary* condition for γ to be a critical point of Φ in $\mathcal{A}_{0,z}$:

(2.13)
$\gamma^{(0,k)}$ *is a polynomial of degree less than or equal to $k+1$, for $k=0,1,\dots,r$.*

Actually, (2.13) is also *sufficient* for γ to be critical. Indeed it is clear that, again by Lemma 2.1, condition (2.13) implies $d\Phi(\gamma,\eta)=0$ at least for η of the special form required by Lemma 2.1, that is, $\eta \in \mathcal{A}_{0,0}$ such that $\eta^{(0)} \in V_k$ for some $k = 0, \dots, r$. On the other hand, every $\eta \in \mathcal{A}_{0,0}$ can be uniquely represented as a sum of paths of the special form above, so that the thesis follows from the linearity of $d\Phi(\gamma, \cdot)$.

In order to prove that there exists a unique path γ in $\mathcal{A}_{0,z}$ satisfying condition (2.13), we introduce the linear subspace W_k of $\mathbb{R}^N$ defined by the following formula:

$$W_k = V_k \oplus B^T V_k \oplus ... \oplus \left(B^T\right)^k V_k, \qquad k = 0, ..., r.$$

Note that, by condition (1.2), $\mathbb{R}^N$ is the direct sum of W_k for $k = 0, \dots, r$. We next show that that there exists a unique path γ in $\mathcal{A}_{0,z}$ satisfying condition (2.13). For $k = 0, ..., r$ we consider the component of γ in W_k, namely

$$\gamma^{(0,k)} + \gamma^{(1,k)} + ... + \gamma^{(k,k)}, \qquad \gamma^{(j,k)} \in \left(B^T\right)^j V_k, \quad j = 1, \dots, k.$$

Since γ is L-admissible, from (2.4) we have

$$\frac{d^h}{ds^h}\gamma^{(k,k)} = M_k M_{k-h}^{-1} \gamma^{(k-h,k)}, \qquad h = 0, \dots, k. \tag{2.14}$$

In view of (2.13), $\gamma^{(k,k)}$ is of the form

$$\gamma^{(k,k)}(s) = \sum_{j=0}^{2k+1} \alpha_j s^j, \qquad s \in [0, t],$$

for some $\alpha_j \in \mathbb{R}^{m_k - m_{k+1}}$. Requiring $\gamma(0) = 0$, we get from (2.14)

$$0 = \gamma^{(k,k)}(0) = \dot{\gamma}^{(k,k)}(0) = \cdots = \frac{d^k}{ds^k}\gamma^{(k,k)}(0),$$

that is,

$$\alpha_0 = \alpha_1 = \cdots = \alpha_k = 0.$$

Hence $\gamma^{(k,k)}(s) = s^{k+1} g(s)$, for some polynomial function

$$g(s) = \sum_{j=0}^{k} \frac{\beta_j}{j!}(s - t)^j, \qquad s \in [0, t].$$

The coefficients β_j can be uniquely determined by imposing the condition $\gamma(t) = x$. Indeed, by (2.14), we obtain the system of linear equations

$$\begin{aligned} x^{(k,k)} &= \gamma^{(k,k)}(t) = t^{k+1}\beta_0, \\ -B_k^T x^{(k-1,k)} &= \dot{\gamma}^{(k,k)}(t) = (k+1)t^k \beta_0 + t^{k+1}\beta_1, \\ &\cdots \\ (-1)^k M_k x^{(0,k)} &= \frac{d^k}{ds^k}\gamma^{(k,k)}(t) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k+1)!}{(j+1)!} t^{j+1}\beta_j, \end{aligned}$$

which is clearly uniquely solvable. Then there is a unique polynomial path γ satisfying (2.13), thus the proof of Theorem 1.1 is accomplished.

3. Proof of Corollary 1.2

In this section we prove Corollary 1.2 by computing explicitly the minimum γ of Φ. As in the previous section, it is sufficient to put $z = z_1^{-1} \circ z_2$ and prove the claim in $\mathcal{A}_{0,z}$. A natural candidate seems to be the path

$$\eta(s) = \mathcal{C}(s)\mathcal{C}^{-1}(t)x$$

which connect the origin to x and satisfies condition (2.13). Moreover, as we shall see later,

$$\int_0^t |\dot\eta^{(0)}(s)|^2 ds = \langle \mathcal{C}^{-1}(t)x, x\rangle,$$

which is the exponent appearing in (1.13) (see also (2.3)). Unfortunately, η is not L-admissible. However, we look for the minimizing path γ by a suitable perturbation of η. More precisely, we set

$$\begin{aligned}\gamma(s) =&\Big(\int_0^s (p(\sigma)+q(\sigma))d\sigma, M_1\int_0^s\int_0^{s_1}(p(\sigma)+q(\sigma))d\sigma ds_1,\dots,\\ &M_r\int_0^s\int_0^{s_1}\cdots\int_0^{s_r}(p(\sigma)+q(\sigma))d\sigma\cdots ds_2 ds_1, s\Big), \qquad s\in[0,t],\end{aligned} \tag{3.1}$$

where

$$p(s) = \frac{d}{ds}\left(\mathcal{C}(s)\mathcal{C}^{-1}(t)x\right)^{(0)}, \qquad s\in[0,t],$$

and q is a suitable polynomial function. By imposing $\gamma\in\mathcal{A}_{0,z}$, we determine q and it turns out that it verifies the condition

$$\Phi(\gamma) = \int_0^t |p(s)+q(s)|^2 ds = \int_0^t |p(s)|^2 ds. \tag{3.2}$$

This proves the corollary, since at the end of this section we shall prove the following identity:

$$\int_0^t |p(s)|^2 ds = \langle \mathcal{C}^{-1}(t)x, x\rangle. \tag{3.3}$$

In the sequel, we shall use the following non-standard notation: given an $n\times m$ matrix M and $1\le i\le j\le n$, we denote by $[M]_{i,j}$ the $(j-i+1)\times m$ matrix obtained from M eliminating the rows from 1 to $i-1$ and from $j+1$ to n.

We need the following preliminary

Lemma 3.1. *For $k=1,\dots,r$,*

$$\int_0^t\int_0^{s_1}\cdots\int_0^{s_k} p(s)ds\cdots ds_2 ds_1 = \sum_{h=0}^k \frac{t^{k-h}}{(k-h)!}(-1)^h M_h^{-1}x^{(h)}. \tag{3.4}$$

Proof. We first recall (1.15) and remark that

$$p(s) = \frac{d}{ds}\int_0^s [E(\sigma)I_{m_0}E^T(\sigma)]_{1,m_0} d\sigma\ \mathcal{C}^{-1}(t)x = [E^T(s)]_{1,m_0}\mathcal{C}^{-1}(t)x. \tag{3.5}$$

Moreover we note that, for $h=1,\dots,r$ and $s\in\mathbb{R}$,

$$\frac{s^h}{h!}M_h[E^T(s)]_{1,m_0} = [E(s)I_{m_0}E^T(s)]_{m_{h-1}+1,m_h}. \tag{3.6}$$

Then, we have

$$M_h \int_0^t \frac{s^h}{h!} p(s) ds$$

(by (3.5))

$$= M_h \int_0^t \frac{s^h}{h!} [E^T(s)]_{1,m_0} ds \; \mathcal{C}^{-1}(t) x$$

(by (3.6))

$$= [\mathcal{C}(t)]_{m_{h-1}+1,m_h} \mathcal{C}^{-1}(t) x = x^{(h)}. \tag{3.7}$$

We are now in a position to prove the thesis. We have

$$\begin{aligned}
\int_0^t \int_0^{s_1} \cdots \int_0^{s_k} p(s) ds \cdots ds_2 ds_1 &= \int_0^t p(s) \int_s^t \int_{s_k}^t \cdots \int_{s_2}^t ds_1 \cdots ds_{k-1} ds_k ds \\
&= \int_0^t \frac{(t-s)^k}{k!} p(s) ds \\
&= \sum_{h=0}^k (-1)^h \frac{t^{k-h}}{(k-h)!} \int_0^t \frac{s^h}{h!} p(s) ds
\end{aligned}$$

and, by (3.7), we obtain (3.4). □

Next we determine q by imposing that γ in (3.1) belongs to $\mathcal{A}_{0,z}$. In particular, since γ is L-admissible and $\gamma(0) = 0$ by construction, we determine q by requiring that $\gamma^{(k)}(t) = x^{(k)}$ for $k = 0, \dots, r$; this leads to the conditions (3.8)-(3.9) below. Indeed, since obviously

$$\int_0^t p(s) ds = x^{(0)},$$

we infer that $\gamma^{(0)}(t) = x^{(0)}$ is equivalent to

$$\int_0^t q(s) ds = 0. \tag{3.8}$$

Furthermore, by Lemma 3.1, $\gamma^{(k)}(t) = x^{(k)}$ for $k = 1, \dots, r$, is equivalent to

$$\int_0^t \frac{(t-s)^k}{k!} q(s) ds = M_k^{-1} x^{(k)} - \sum_{h=0}^k \frac{t^{k-h}}{(k-h)!} (-1)^h M_h^{-1} x^{(h)}, \qquad s \in [0,t]. \tag{3.9}$$

The proof of (3.3) is straightforward:

$$\int_0^t |p(s)|^2 ds = \langle \int_0^t E(s) I_{m_0} E^T(s) ds \; \mathcal{C}^{-1}(t) x, \mathcal{C}^{-1}(t) x \rangle = \langle \mathcal{C}^{-1}(t) x, x \rangle.$$

We now conclude the proof of Corollary 1.2, and we apply the above results to the cases $r = 1$ and $r = 2$. In both cases it is easy to verify that the conditions (3.8)-(3.9) determine the polynomial

$$q(s) = \frac{6(2s-t)}{t^3} (t x^{(0)} + 2(B_1^T)^{-1} x^{(1)}).$$

Then, a direct computation shows that (3.2) holds and this completes the proof.

4. Gradient estimates

In this section we prove that the positive solutions to L verify the gradient estimate (1.8). Here we use the explicit expression (1.14) of the fundamental solution Γ of (1.1). As a first result of this section, we show that Γ verifies the equation

$$-Y\Gamma + \frac{Q}{2t}\Gamma = \frac{\langle AD\Gamma, D\Gamma\rangle}{\Gamma} \tag{4.1}$$

in $\mathbb{R}^N\times]0,+\infty[$. Then we prove the gradient estimate (1.8) by means of a representation formula for positive solutions.

Proposition 4.1. *The fundamental solution Γ of L satisfies* (4.1) *in $\mathbb{R}^N\times]0,+\infty[$.*

Proof. We first recall that (see [14])

$$\mathcal{C}(t) = \delta_{\sqrt{t}}\mathcal{C}(1)\delta_{\sqrt{t}}, \qquad \forall t > 0.$$

Therefore, if we set

$$\mathcal{C}^{-1}(1) = \tilde{C} = (\tilde{c}_{ij}),$$

then the fundamental solution takes the form

$$\Gamma(x,t) = \omega_N t^{-\frac{Q}{2}} \exp\left(-\frac{1}{4}\langle \tilde{C}\delta_{\frac{1}{\sqrt{t}}}x, \delta_{\frac{1}{\sqrt{t}}}x\rangle\right), \qquad t>0,$$

for some dimensional constant ω_N. Therefore, if $1\le i,j\le m_0$, we have

$$\partial_{x_i}\Gamma(x,t) = -\frac{\Gamma(x,t)}{2\sqrt{t}}(\tilde{C}\delta_{\frac{1}{\sqrt{t}}}x)_i,$$

$$\partial_{x_ix_j}\Gamma(x,t) = \frac{\Gamma(x,t)}{4t}(\tilde{C}\delta_{\frac{1}{\sqrt{t}}}x)_i(\tilde{C}\delta_{\frac{1}{\sqrt{t}}}x)_j - \frac{\tilde{c}_{ij}}{2t}\Gamma(x,t);$$

consequently

$$\operatorname{div}(AD\Gamma) = \frac{\langle AD\Gamma, D\Gamma\rangle}{\Gamma} - \frac{\Gamma}{2t}\sum_{i,j=1}^{m_0} a_{ij}\tilde{c}_{ij}. \tag{4.2}$$

On the other hand, we have

$$\begin{aligned} Y\Gamma(x,t) &= \Gamma(x,t)\left(\frac{Q}{2t} - \frac{1}{4}Y\langle \mathcal{C}^{-1}(t)x,x\rangle\right)\\ &= \Gamma(x,t)\left(\frac{Q}{2t} - \frac{1}{2}\langle x, B\mathcal{C}^{-1}(t)x\rangle + \frac{1}{4}\langle \frac{d}{dt}\mathcal{C}^{-1}(t)x,x\rangle\right). \end{aligned} \tag{4.3}$$

Evaluating equation

$$\operatorname{div}(AD\Gamma(x,t)) = -Y\Gamma(x,t) \tag{4.4}$$

at $x=0$, we get from (4.2) and (4.3)

$$\sum_{i,j=1}^{m_0} a_{ij}\tilde{c}_{ij} = Q. \tag{4.5}$$

Hence, (4.1) follows from (4.2), (4.4) and (4.5). □

Proposition 4.2. *Let u be a positive solution of $Lu=0$ in a strip $\mathbb{R}^N\times]0,T[$, $T>0$. Then u satisfies the gradient estimate* (1.8)*, that is,*

$$-Yu + \frac{Q}{2t}u \ge \frac{\langle ADu, Du\rangle}{u}$$

in $\mathbb{R}^N\times]0,T[$.

Proof. It is known (see [19]) that, fixed $t_0 \in]0,T[$, $u>0$ has the representation

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t,y,t_0)u(y,t_0)dy, \qquad (x,t) \in \mathbb{R}^N \times]t_0,T[. \tag{4.6}$$

Then, by Proposition 4.1, we have

$$\begin{aligned} -Yu + \frac{Q}{2t}u &= \int_{\mathbb{R}^N} \left(-Y\Gamma(\cdot,y,t_0) + \frac{Q}{2t}\Gamma(\cdot,y,t_0) \right) u(y,t_0)dy \\ &= \int_{\mathbb{R}^N} \left(\frac{\langle AD\Gamma(\cdot,y,t_0), D\Gamma(\cdot,y,t_0)\rangle}{\Gamma(\cdot,y,t_0)} \right) u(y,t_0)dy \end{aligned}$$

(by the Hölder inequality, since u is positive)

$$\geq \langle A \int_{\mathbb{R}^N} D\Gamma(\cdot,y,t_0)u(y,t_0)dy, \int_{\mathbb{R}^N} D\Gamma(\cdot,y,t_0)u(y,t_0)dy\rangle \left(\int_{\mathbb{R}^N} \Gamma(\cdot,y,t_0)u(y,t_0)dy \right)^{-1}$$

and this concludes the proof. □

Remark 4.3. The previous result can be straightforwardly generalized to the case of positive super-solutions u of L, i.e. positive solutions to $Lu \leq 0$. Indeed it suffices to represent u by

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t,y,t_0)u(y,t_0)dy - \int_{t_0}^{t} \int_{\mathbb{R}^N} \Gamma(x,t,y,s)Lu(y,s)dyds,$$

and proceed as above.

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