

On the complete model with stochastic volatility by Hobson and Rogers

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In the complete model with stochastic volatility by Hobson and Rogers, preference independent options prices are solutions to degenerate partial differential equations obtained by including additional state variables describing the dependence on past prices of the underlying. In this paper, we aim to emphasize the mathematical tractability of the model by presenting analytical and numerical results comparable with the known ones in the classical Black–Scholes environment.

Keywords: Black–Scholes model; stochastic volatility;
path-dependent contingent claim; Kolmogorov equation; hypoelliptic equation

1. Introduction

It is widely believed that the celebrated Black–Scholes model (Black & Scholes 1973) cannot describe option prices dynamics in real markets. The characteristic observed effects of ‘smiles’ and ‘skews’ contrast with the assumption of constant volatility of the underlying asset. Indeed there is strong evidence that volatility depends in some way on the present stock price (*leverage effect*) and on realization of past prices (*volatility persistence*), as well as on the moneyness, time and maturity (*volatility term structure*): we refer, for instance, to Tompkins (2001), Derman & Kani (1998), Ghysels *et al.* (1996) and Bollerslev *et al.* (1992).

For this reason many authors have developed variations of the Black–Scholes model, aiming to capture at least some of the above observed features. Within the framework of continuous Itô processes, these alternative theories divide into two broad classes: *single-factor* models, in which the volatility is a deterministic function of present or past values of the underlying price, and *multi-factor* models, which allow for exogenous stochastic influence (see, for instance, the monograph by Epps (2000) and the recent papers by Barucci *et al.* (2003) and Davis (2004) for comprehensive bibliographic references).

An apparent advantage of single- over multi-factor models is that the Brownian motion which drives the asset price is the only source of randomness. Then the arbitrage argument that underlies the Black–Scholes model is preserved and there are unique preference-independent prices for contingent claims. The simplest single-factor model is the so-called *level-dependent volatility* in which the underlying price

S_t satisfies

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad (1.1)$$

where B is a Brownian motion and $\sigma = \sigma(t, S_t)$ is a deterministic function of the current value S_t and time. Special cases are the CEV model by Cox & Ross (1976), where $\sigma(S_t) = S_t^{\alpha-1}$, $\alpha \in]0, 1[$, and the *implied tree* models by Derman & Kani (1998), Dupire (1997) and Rubinstein (1994).

Empirical analysis has shown that level-dependent volatility still fails to price derivative securities better than the usual Black–Scholes model does. This is the conclusion of the study by Dumas *et al.* (1998), who suggest a volatility related to *past* changes of the underlying prices, as the last candidate model before resorting to fully stochastic volatility.

An interesting class of such models was proposed by Hobson & Rogers. In Hobson & Rogers (1998) a specific version is examined in which the volatility is defined in terms of the difference between the current price and an exponential average of past prices. As in the discrete-time ARCH and GARCH environments, this is designed to reflect the perception that large movements of the asset price in the past tend to forecast higher future volatility. In Hobson & Rogers (1998) it is shown that the model indeed exhibits a wide variety of smiles and skews, and can account for volatility term structures as the average of prices evolves through time (see also the study by Figà-Talamanca & Guerra (2000)). Hobson & Rogers claim to solve numerically the associated partial differential equation (PDE) by a finite-difference method: unfortunately no scheme is explicitly given in Hobson & Rogers (1998), nor are convergence results proved. Actually, it is difficult to obtain these results by the ‘classical’ theory of parabolic PDEs. Indeed, incorporating past prices enters an additional state variable on which the derivative’s arbitrage-free price depends. Then, as in the case of path-dependent contingent claims such as Asian or look-back options, the augmented PDE associated to the model is strongly degenerate and standard analytical and numerical techniques for uniformly parabolic equations do not apply.

On the other hand, we recall that Barucci *et al.* (2001) successfully studied the problem of pricing Asian options in the framework of PDEs of Kolmogorov type. This class of degenerate diffusion PDEs, whose precise definition is given in §2, shares most of the prominent features of classical parabolic equations.

In this paper we present analytical and numerical results on Kolmogorov PDEs with variable coefficients which apply to the Hobson–Rogers model for European and Asian-style derivatives. Consequently, we wish to emphasize the mathematical tractability of this model. Indeed our results are comparable with those known for the classical Black–Scholes environment. In particular, we aim to propose a finite-difference scheme for the Hobson–Rogers model and prove convergence results.

Let us recall that nonlinear Kolmogorov equations have also been recently considered by Peszek (1995) for pricing options with memory feedback and by Antonelli & Pascucci (2002) and Pascucci & Polidoro (2003) for a stochastic differential utility problem.

The paper is organized as follows. In the next section, we present theoretical results about Kolmogorov PDEs which allow us to treat a variety of path-dependent contingent claims. In §3, we prove numerical results by suitably adapting finite-differences schemes to the degenerate setting.

2. Kolmogorov equations

Degenerate PDEs of Kolmogorov type naturally arise in the problem of pricing path-dependent contingent claims. The simplest significant example is given by Asian-style derivatives: if we assume that the stock price S_t is a lognormal stochastic process, i.e. a solution to the stochastic differential equation (1.1) with μ and σ being fixed constants, then the price U of a geometric average Asian option is a solution to the equation

$$\partial_t U + rS\partial_S U + \frac{1}{2}\sigma^2 S^2 \partial_{SS} U + \log(S)\partial_A U = rU, \quad (2.1)$$

where ‘ A ’ denotes the path-dependent variable and r is the risk-free rate (see, for example, Wilmott *et al.* 1993). By an elementary change of variables, (2.1) can be reduced to the following PDE in \mathbb{R}^3 :

$$\partial_{x_1 x_1} u + x_1 \partial_{x_2} u - \partial_t u = 0. \quad (2.2)$$

Although (2.2) is strongly degenerate, in the sense that there is a complete lack of diffusion in the x_2 -direction, Kolmogorov (1934) constructed an *explicit* fundamental solution to (2.2) of Gaussian type, which is a C^∞ function outside the diagonal (cf. (2.12)). Consequently, equation (2.2) has a closed-form solution and is hypoelliptic, that is, every distributional solution to (2.2) is a C^∞ function. As we shall see later, the Kolmogorov method can be applied in a very general framework.

In the Hobson–Rogers model (Hobson & Rogers 1998), the stock price S_t is described by an Itô process of the form

$$dS_t = \mu(D_t)S_t dt + \sigma(D_t)S_t dB_t, \quad (2.3)$$

where D_t represents the deviation from the trend and is defined as the difference between the current value and a geometric, exponentially weighted average of past prices. More precisely, if $Z_t = \log(e^{-rt}S_t)$ denotes the discounted log-price, then

$$D_t = Z_t - \int_0^{+\infty} \lambda e^{-\lambda\tau} Z_\tau - \tau d\tau,$$

where λ is a positive parameter. Hobson & Rogers prove the existence of an equivalent probability measure under which the discounted price $e^{-rt}S_t$ is a martingale. Then usual no-arbitrage arguments can be used to determine the price of the derivative.

In Hobson & Rogers (1998), it is proved that the price U_t of a European-style contingent claim with terminal pay-off $U_T = \varphi(S_T)$ is given by

$$U_t = u(\log(S_t), \log(S_t) - D_t, T - t),$$

where $u = u(x_1, x_2, t)$ is the solution to the following degenerate PDE with variable coefficients in \mathbb{R}^3 :

$$\frac{1}{2}\sigma^2(x_1 - x_2)(\partial_{x_1 x_1} u - \partial_{x_1} u) + \lambda(x_1 - x_2)\partial_{x_2} u - \partial_t u = 0, \quad (2.4)$$

with initial datum $u(x_1, x_2, 0) = \varphi(e^{x_1})$.

The same argument can be used for the more general case of a geometric average Asian option. Let A_t be the path-dependent process describing the geometric average of the underlying:

$$dA_t = \log(S_t) dt.$$

The price U_t of a derivative with terminal pay-off $U_T = \varphi(S_T, A_T)$ is then given by

$$U_t = u(\log(S_t), \log(S_t) - D_t, A_t, T - t),$$

where $u = u(x_1, x_2, x_3, t)$ is the solution to the PDE

$$\frac{1}{2}\sigma^2(x_1 - x_2)(\partial_{x_1 x_1} u - \partial_{x_1} u) + \lambda(x_1 - x_2)\partial_{x_2} u + x_1 \partial_{x_3} u - \partial_t u = 0, \quad (2.5)$$

with initial datum $u(x_1, x_2, x_3, 0) = \varphi(e^{x_1}, x_3)$. We emphasize that (2.5) is a PDE in \mathbb{R}^4 with diffusion only in the x_1 -direction.

The above examples motivate the study of the following class of degenerate PDEs.

Definition 2.1. We call the *Kolmogorov equation* (in short, KE) a PDE of the form

$$Ku \equiv \sum_{i,j=1}^p (a_{ij}(x, t)\partial_{x_i x_j} u + c_i(x, t)\partial_{x_i} u) + \sum_{i,j=1}^N b_{ij} x_i \partial_{x_j} u - \partial_t u = 0, \quad (2.6)$$

where $(x_1, \dots, x_N, t) \equiv (x, t) \equiv z \in \mathbb{R}^{N+1}$, under the following structural conditions:

(i) there exists $\mu > 0$ such that

$$\mu^{-1}|\xi|^2 < \sum_{i,j=1}^p a_{ij}(z)\xi_i \xi_j < \mu|\xi|^2 \quad (2.7)$$

for every $\xi \in \mathbb{R}^p$ and $z \in \mathbb{R}^{N+1}$;

(ii) the matrix $B = (b_{ij})$ is constant and has the following block form:

$$\begin{pmatrix} * & B_1 & 0 & \cdots & 0 \\ * & * & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & B_m \\ * & * & * & \cdots & * \end{pmatrix}, \quad (2.8)$$

where the B_j -valued blocks are $p_{j-1} \times p_j$ matrices of rank p_j , $j = 1, 2, \dots, m$, and the blocks denoted by '*' are arbitrary. Moreover, the p_j values are positive integers such that

$$p = p_0 \geq p_1 \geq \cdots \geq p_m \geq 1, \quad \text{and} \quad p + p_1 + \cdots + p_m = N.$$

Example 2.2. Equation (2.2) is obviously a KE; in this case

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is also clear that every parabolic equation is a KE.

To verify that equations (2.4) and (2.5) of the Hobson–Rogers model are KEs, it is convenient to recall that the structural assumptions of definition 2.1 are *equivalent* to the following classical Hörmander condition (Hörmander 1967) (see, for instance, Lanconelli & Polidoro 1994). The well-known Hörmander condition gives a characterization of hypoelliptic degenerate PDEs with smooth coefficients. In our setting it reads as follows.

Condition 2.3 (Hörmander condition). The first-order differential operators

$$\partial_{x_1}, \dots, \partial_{x_p} \quad \text{and} \quad Y = \sum_{i,j=1}^N b_{ij} x_i \partial_{x_j} - \partial_t, \tag{2.9}$$

together with their commutators, span \mathbb{R}^{N+1} at every point.

Example 2.4. Equation (2.4) is a KE. Indeed, for every $\lambda > 0$, the first-order differential operators

$$X \equiv \partial_{x_1}, \quad Y = \lambda(x_1 - x_2)\partial_{x_2} - \partial_t, \quad [X, Y] \equiv XY - YX = \lambda\partial_{x_2} \tag{2.10}$$

are linearly independent at every point and form a basis of \mathbb{R}^3 . Analogously, equation (2.5) is a KE since

$$\begin{aligned} X &= \partial_{x_1}, & \tilde{Y} &= \lambda(x_1 - x_2)\partial_{x_2} + x_1\partial_{x_3} - \partial_t, \\ [X, \tilde{Y}] &= \lambda\partial_{x_2} + \partial_{x_3}, & [\tilde{Y}, [X, \tilde{Y}]] &= \lambda^2\partial_{x_2} \end{aligned}$$

span \mathbb{R}^4 at every point.

Definition 2.5. A KE in the form (2.6) with constant a_{ij} and $c_i \equiv 0$ for $i, j = 1, \dots, p$ is called a ‘Kolmogorov equation with constant coefficients’.

The denomination ‘KE with constant coefficients’ stems from the theory of parabolic PDEs. Indeed, a constant-coefficients parabolic equation is nothing more than a translation-invariant equation on the Euclidean space. Similarly, a constant-coefficients KE in the form (2.6) has the remarkable property of being invariant with respect to the *non-Euclidean* left translations in the Lie group law

$$(x, t) \circ (y, s) = (y + E(s)x, t + s), \quad (x, t), (y, s) \in \mathbb{R}^{N+1}, \tag{2.11}$$

where $E(t) = \exp(-tB^T)$, and B^T denotes the transpose of the matrix B .

In this case, the explicit expression of $\Gamma(z; \zeta)$, the fundamental solution of (2.6) evaluated in z with a pole at ζ , is given by $\Gamma(z; \zeta) = \Gamma(\zeta^{-1} \circ z; 0)$, where

$$\Gamma(x, t; 0, 0) = \frac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}} \exp(-\frac{1}{4}\langle C^{-1}(t)x, x \rangle - t \operatorname{tr}(B)), \tag{2.12}$$

for $(x, t) \in \mathbb{R}^N \times \mathbb{R}^+$ and $C(t)$ is the positive definite matrix

$$C(t) = \int_0^t E(s)AE^T(s) ds, \quad t > 0. \tag{2.13}$$

As usual, a solution to the Cauchy problem for (2.6) can be expressed in closed form, as the convolution of the initial datum with the fundamental solution Γ . As in classical theory, constant-coefficients KEs serve as an essential class of prototypes and many results can be extended to the general situation of variable coefficients by perturbation arguments (see, for example, the survey by Lanconelli *et al.* (2002)).

Among constant-coefficient KEs, a particularly important role is played by equations for which all the $*$ -blocks in (2.8) are zero matrices. Indeed these KEs have the remarkable property of being *homogeneous* with respect to the dilations $(\delta_\lambda)_{\lambda>0}$ in \mathbb{R}^{N+1} defined by

$$\delta_\lambda = \operatorname{diag}(\lambda I_{p_0}, \lambda^3 I_{p_1}, \dots, \lambda^{2m+1} I_{p_m}, \lambda^2), \tag{2.14}$$

where I_{p_j} denotes the $p_j \times p_j$ identity matrix. More precisely, it holds that

$$K \circ \delta_\lambda = \lambda^2(\delta_\lambda \circ K), \quad \forall \lambda > 0. \quad (2.15)$$

Subelliptic operators on homogeneous Lie groups were first studied by Folland (1975).

Note that equation (2.2) is homogeneous with respect to the dilations

$$\delta_\lambda(x_1, x_2, t) = (\lambda x_1, \lambda^3 x_2, \lambda^2 t).$$

In contrast, in the KEs arising in the Hobson–Rogers model, not all the $*$ -blocks are null.

The main result of this section is the following theorem, proved in Di Francesco & Pascucci (2004) (see also Polidoro (1994) for the case of null $*$ -blocks). In the next statement, $\bar{\partial}_x = (\partial_{x_0}, \dots, \partial_{x_p})$ denotes the gradient in the first p variables, Y_z and Y_ζ , respectively, denote the first-order differential operators in (2.9) acting on the variables $z = (x, t)$ and $\zeta = (y, s)$. Moreover, we say that a function φ is *slowly increasing* if

$$|\varphi(x)| \leq \alpha e^{\beta|x|^2}, \quad \forall x \in \mathbb{R}^N, \quad (2.16)$$

for some positive constants α, β .

Theorem 2.6. *Assume that the coefficients of the KE (2.6) are infinitely differentiable in $\mathbb{R}^N \times]0, T[$ and that all the derivatives are bounded functions. Then (2.6) has a fundamental solution Γ : in particular, for every $\varphi \in C(\mathbb{R}^N)$, slowly increasing with suitably small β in (2.16), the function*

$$u(x, t) = \int_{\mathbb{R}^N} \Gamma(x, t; y, 0) \varphi(y) \, dy \quad (2.17)$$

is the unique slowly increasing, classical solution to the Cauchy problem

$$\left. \begin{aligned} Ku &= 0 && \text{in } \mathbb{R}^N \times]0, T[, \\ u(\cdot, 0) &= \varphi && \text{in } \mathbb{R}^N. \end{aligned} \right\} \quad (2.18)$$

Moreover, the function $\Gamma(z; \zeta)$ is infinitely differentiable for $z \neq \zeta$ and we have the following estimate of the derivatives:

$$|\bar{\partial}_x^h \bar{\partial}_y^k Y_z^m Y_\zeta^n \Gamma(x, t; y, s)| \leq c(t-s)^{-(|h|+|k|)/2-m-n} \Gamma_\mu(x, t; y, s), \quad (2.19)$$

where Γ_μ is the fundamental solution of the constant-coefficients KE with $(a_{ij}) = \mu I_p$ and μ as in (2.7). The explicit expression of Γ_μ is given in (2.12). The constant c depends on the multi-indexes h, k and on the non-negative integers m, n .

Remark 2.7. The hypotheses on the regularity of the coefficients in the previous statement can be considerably relaxed. For the existence of Γ , it is sufficient to assume that the coefficients are bounded and Hölder continuous. Analogously, estimate (2.19) still holds true under the usual hypotheses: the regularity of order $m + \alpha$, $\alpha > 0$, of the coefficients implies an estimate of the derivatives of order $m + 2$ of Γ . Note that estimate (2.19) involves only the first p spatial derivatives $\partial_{x_1}, \dots, \partial_{x_p}$ and the vector field Y . If we consider $\bar{\partial}_x$ as first-order derivatives, then Y weighs as a second-order derivative, in perfect analogy with the classical parabolic case. Estimates for the other spatial derivatives can be obtained from (2.19) by the Hörmander condition: for instance, in the case of equation (2.4), by (2.10) we have $\partial_{x_2} = \partial_{x_1} Y - Y \partial_{x_1}$ so that ∂_{x_2} should be considered as a third-order derivative.

3. A numerical scheme

We are concerned with the numerical solution to the Cauchy problem for the KE (2.4). Here we extend some results by Barucci *et al.* (2001) for the homogeneous KE (2.2) with constant coefficients. As we shall see below, the main new difficulties in treating the Hobson–Rogers equation (2.4) are due to the lack of homogeneity. Our scheme allows us to consider the problem

$$a\partial_{xx}u + b\partial_xu + (\alpha x + \beta y)\partial_yu - \partial_tu = 0 \quad \text{in } S_T \equiv \mathbb{R}^2 \times]0, T[, \quad (3.1)$$

$$u(\cdot, 0) = \varphi \quad \text{in } \mathbb{R}^2, \quad (3.2)$$

where $a = a(x, y, t) \geq a_0 > 0$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$. Preferring straightforwardness to generality, we assume that the coefficients a, b are smooth functions with bounded derivatives. We remark explicitly that the condition $\alpha \neq 0$ ensures that (3.1) is a KE, since it verifies the Hörmander condition.

We first consider the case of the model equation

$$Ku \equiv \partial_{xx}u + (x - y)\partial_yu - \partial_tu = 0, \quad (3.3)$$

corresponding to $a = \alpha = 1 = -\beta$ and $b = 0$ in (3.1). In view of the theory of KEs, it seems natural to consider and approximate the first-order operators related to the equation as in (2.9), rather than the Euclidean derivatives as usual. Thus, as in example 2.4, we consider, for fixed x, y , the first-order part of K as a unique ‘directional’ derivative, which we denote by Y :

$$Y = (x - y)\partial_y - \partial_t. \quad (3.4)$$

We aim to approximate operator K by finite differences on the uniform grid

$$G = \{(j\Delta_x, k\Delta_y, n\Delta_t) \mid j, k, n \in \mathbb{Z}\}.$$

Since K is strongly degenerate, it is known that the classical parabolic boundary-value problem for (3.3) is not well posed. In order to comply with the non-Euclidean geometry naturally associated to K (see, for example, Manfredini 1997), we impose

$$\Delta_y = \Delta_x \Delta_t. \quad (3.5)$$

Since equation (3.3) is *not homogeneous* (in the sense of (2.15)), the discrete operator that one might naively associate to K is not well defined on G because it involves points not belonging to the grid. However, it is possible to approximate K efficiently by the following ‘corrected’ operator:

$$K_G u(x, y, t) = D_h^2 u(x, y + \delta(x - h[y/h]), t - \delta) - \frac{u(x, y, t) - u(x, y + \delta(x - h[y/h]), t - \delta)}{\delta}, \quad (3.6)$$

where $\delta, h > 0$, $[x]$ denotes the integer part of $x \in \mathbb{R}$ and

$$D_h^2 u(x, y, t) = \frac{u(x + h, y, t) - 2u(x, y, t) + u(x - h, y, t)}{h^2}$$

is the usual second-order central difference. Indeed we have the following lemma.

Lemma 3.1. *Operator K_G is well defined on the grid G with $\Delta_x = h$, $\Delta_t = \delta$ and approximates K in the sense that, for every smooth function u , it holds that*

$$\|K_G u - K u\|_{L^\infty} \leq \Delta_x \|\partial_y u\|_{L^\infty} + \Delta_t (\|Y^2 u\|_{L^\infty} + \|Y \partial_x^2 u\|_{L^\infty}) + \Delta_x^2 \|\partial_x^4 u\|_{L^\infty} + \Delta_x \Delta_t \|\partial_{xxy} u\|_{L^\infty}. \quad (3.7)$$

Proof. By (3.5), the first assertion is obvious. To prove (3.7), we first observe that

$$0 \leq y - h \left\lceil \frac{y}{h} \right\rceil < h, \quad h > 0, \quad y \in \mathbb{R}. \quad (3.8)$$

Then we have

$$\begin{aligned} & \left| \frac{u(x, y + \delta(x - h[y/h]), t - \delta) - u(x, y, t)}{\delta} - Y u(x, y, t) \right| \\ & \leq \left| \frac{u(x, y + \delta(x - h[y/h]), t - \delta) - u(x, y + \delta(x - y), t - \delta)}{\delta} \right| \\ & \quad + \left| \frac{u(x, y + \delta(x - y), t - \delta) - u(x, y, t)}{\delta} - Y u(x, y, t) \right| \end{aligned}$$

(by the mean value theorem and (3.8))

$$\leq h \|\partial_y u\|_{L^\infty} + \delta \|Y^2 u\|_{L^\infty}. \quad (3.9)$$

An analogous estimate for the second-order part of K completes the proof. \blacksquare

Lemma 3.2 (maximum principle). *Let v be a solution to the discrete Cauchy problem*

$$K_G v = f \quad \text{in } S_T \cap G, \quad (3.10)$$

$$v(\cdot, 0) = \psi \quad \text{in } G_0, \quad (3.11)$$

where f, ψ are bounded functions and $G_0 = G = \{(j\Delta_x, k\Delta_y) \mid j, k \in \mathbb{Z}\}$. If the stability condition

$$\frac{\Delta_t}{(\Delta_x)^2} \leq \frac{1}{2} \quad (3.12)$$

holds, then

$$\sup_{G \cap S_T} |v| \leq \sup_{G_0} |\psi| + T \sup_{G \cap S_T} |f|. \quad (3.13)$$

Proof. Let us denote

$$v_{j,k}^n = v(j\Delta_x, k\Delta_y, n\Delta_t), \quad j, k, n \in \mathbb{Z}.$$

Once we have observed that (3.10) is equivalent to

$$\begin{aligned} v_{j,k}^{n+1} &= \left(1 - 2 \frac{\Delta_t}{(\Delta_x)^2}\right) v_{j,k+j-[k\Delta_t]}^n \\ & \quad + \frac{\Delta_t}{(\Delta_x)^2} (v_{j,k+j-[k\Delta_t]}^n + v_{j,k-j-[k\Delta_t]}^n) - \Delta_t f_{j,k+j-[k\Delta_t]}^n, \end{aligned}$$

for $j, k, n \in \mathbb{Z}$ with $0 \leq n\Delta_t < T$, the thesis follows by a standard induction argument. \blacksquare

Lemma 3.3. *If φ is Lipschitz continuous in \mathbb{R}^N and u is a solution to problem (2.18), then for every $s \in]0, T[$ we have*

$$|u(x, t + s) - u(x, t)| \leq c(1 + |x|)\sqrt{s}, \quad \forall (x, t) \in S_{T-s}, \quad (3.14)$$

where the constant c only depends on K, T and the Lipschitz constant of φ .

Proof. By the representation formula (2.17) it suffices to consider the case $t = 0$. Then, if $C = C(t)$ denotes the matrix in (2.13) with $A = \mu I_p$, by (2.17) and (2.19) we infer that

$$\begin{aligned} |u(x, s) - \varphi(x)| &\leq \frac{c}{\sqrt{\det C(s)}} \int_{\mathbb{R}^N} e^{-(C^{-1}(s)(x-E(s)y), x-E(s)y)/4} |\varphi(y) - \varphi(x)| dy \\ &= \bar{c} \int_{\mathbb{R}^N} e^{-|\xi|^2/4} |\varphi(E(-s)(x - C^{1/2}(s)\xi)) - \varphi(x)| d\xi \end{aligned}$$

(by the change of variables $\xi = C^{-1/2}(s)(x - E(s)y)$) and the thesis follows by the Lipschitz continuity of φ . ■

We are now in a position to prove the following convergence result.

Theorem 3.4. *Let u be a solution to the Cauchy problem (3.3), (3.2) with initial datum φ bounded and Lipschitz continuous. Let u_G be a solution to the correspondent discrete problem (3.10), (3.11) with $f = 0$ and $\psi = \varphi$. Assume the stability condition (3.12). Then u_G tends to u uniformly on compacts.*

Proof. For a fixed suitably small $s > 0$, we define

$$u^s(x, t) = u(x, t + s), \quad (x, t) \in S_{T-s}$$

and we denote by u_G^s the solution to (3.10), (3.11) with $f = 0$ and $\psi = u^s(\cdot, 0)$. Let M be a compact subset of $\mathbb{R}^2 \times]0, T[$. By lemma 3.3, we have

$$\sup_{z \in M} |u(z) - u^s(z)| \leq c\sqrt{s}. \quad (3.15)$$

Moreover, keeping in mind that $\partial_y = [\partial_x, Y]$ is a third-order derivative, by the representation formula (2.17) and estimate (2.19), we have

$$\left. \begin{aligned} \sup_{S_{T-s}} |\partial_y u^s| &\leq cs^{-3/2} \sup |\varphi|, \\ \sup_{S_{T-s}} (|\partial_x^4 u^s| + |Y \partial_{xx} u^s| + |Y^2 u^s|) &\leq cs^{-2} \sup |\varphi|, \\ \sup_{S_{T-s}} |\partial_{xxy} u^s| &\leq cs^{-5/2} \sup |\varphi|, \end{aligned} \right\} \quad (3.16)$$

with c independent of s . Now, the function $v = u^s - u_G^s$ is a solution to (3.10), (3.11) with $f = K_G u^s - K u^s$ and initial datum $\psi = 0$. Therefore, applying to v the maximum principle and lemma 3.1 combined with the estimates (3.16) and the stability condition (3.12), we get

$$\sup_{G \cap S_{T-s}} |u^s - u_G^s| \leq c(\Delta_x s^{-3/2} + \Delta_x^2 s^{-2} + \Delta_x^3 s^{-5/2}), \quad (3.17)$$

with $c = c(\varphi)$.

Finally, $u_G^s - u_G$ is a solution to (3.10), (3.11) with $f = 0$ and initial datum $\psi = u(\cdot, s) - \varphi$. By (3.15), $|\psi| \leq cs^{1/2}$. Hence, by the maximum principle, we have

$$\sup_{G \cap S_{T-s}} |u_G^s - u_G| \leq c\sqrt{s}. \quad (3.18)$$

Combining (3.15), (3.17) and (3.18), we deduce

$$\sup_{G \cap M} |u - u_G| \leq c(\sqrt{s} + \Delta_x s^{-3/2} + \Delta_x^2 s^{-2} + \Delta_x^3 s^{-5/2}),$$

and, by choosing $s = \sqrt{\Delta_x}$, we get

$$\max_{M \cap G} |u - u_G| \leq c(\Delta_x)^{1/4}, \quad (3.19)$$

where the positive constant c only depends on K , M and φ . ■

Remark 3.5. The rate of convergence of the scheme is given explicitly in (3.19). Clearly, it is not comparable with the classical results for parabolic PDEs and it is consequent to the correction made in the definition of K_G in (3.6). As a refinement of the above scheme, one can use linear interpolation and approximate K by the discrete operator

$$\theta K_G + (1 - \theta)\tilde{K}_G, \quad \theta = \theta(x, y, t) \in [0, 1],$$

where \tilde{K}_G is formally defined as in (3.6) with the term $[y/h]$ replaced by $[y/h] + 1$. The above technique can also be adapted straightforwardly to implicit difference schemes.

Theorem 3.4 can be generalized straightforwardly to problem (3.1), (3.2). In the definition of the grid, (3.5) should be replaced by $\Delta_y = |\alpha|\Delta_x\Delta_t$, and the approximating operator becomes

$$\begin{aligned} K_G u(x, y, t) &= a(x, y, t) D_h^2 u(x, \tilde{y}, t - \delta) \\ &+ b(x, y, t) \frac{u(x + h \operatorname{sgn}(b(x, y, t)), \tilde{y}, t - \delta) - u(x, \tilde{y}, t - \delta)}{h \operatorname{sgn}(b(x, y, t))} \\ &- \frac{u(x, y, t) - u(x, \tilde{y}, t - \delta)}{\delta}, \end{aligned} \quad (3.20)$$

where

$$\tilde{y} = y + \delta \left(\alpha x + |\alpha| h \left[\frac{\beta y}{|a|h} \right] \right).$$

In (3.20), the first-order term bu_x is approximated by the usual upwind difference, that is, by using the forward difference where b is positive and the backward difference where it is negative. We then have the following theorem.

Theorem 3.6. *Let u be a solution to (3.1), (3.2) and u_G a solution to (3.10), (3.11) with $f = 0$ and $\psi = \varphi$. Under the stability condition*

$$\frac{\Delta_t}{(\Delta_x)^2} \leq \frac{1}{2 \sup a + \sup |b|},$$

for every compact subset M of $\mathbb{R}^2 \times]0, T[$ there exists a positive constant c such that

$$\sup_{M \cap G} |u - u_G| \leq c(\Delta_x)^{1/4}.$$

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