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# Calibration of a path-dependent volatility model: empirical tests.

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## Abstract

The Hobson and Rogers model for option pricing is considered. This stochastic volatility model preserves the completeness of the market and can potentially reproduce the observed smile and term structure patterns of implied volatility. A calibration procedure based on ad-hoc numerical schemes for hypoelliptic PDEs is proposed and used to quantitatively investigate the pricing performance of the model. Numerical results based on S&P500 option prices are discussed.

*Key words:* Option pricing, Hobson & Rogers model, Stochastic volatility, Calibration, Inverse problem

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## 1 Introduction

Among non-constant volatility models in complete markets, the Hobson and Rogers (HR) model [24] seems to be one of the more appealing. In this model the volatility is a function of the trend of the underlying asset, defined as the difference between the spot price and a weighted average of past prices of the underlying asset. Since no exogenous source of risk is added, market completeness is preserved and the standard arbitrage pricing theory applies. Moreover, preliminary tests performed in [24] show that this model is potentially capable to reproduce the observed smiles and volatility term structure patterns.

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A simplified version of the HR model is defined as follows: in a Wiener space with one-dimensional Brownian motion  $W$ , we denote by  $S$  the stock price and by  $D$  the deviation of prices from the trend, defined by

$$D_\tau = Z_\tau - \int_0^{+\infty} \lambda e^{-\lambda s} Z_{\tau-s} ds, \quad \lambda > 0, \quad (1)$$

where  $Z_\tau = \log(e^{-r\tau} S_\tau)$  is the discounted log-price and the parameter  $\lambda$  amounts to the rate at which past prices are weighted. We assume that  $S$  is the solution to the stochastic differential equation

$$dS_\tau = \mu(D_\tau) S_\tau d\tau + \sigma(D_\tau) S_\tau dW_\tau. \quad (2)$$

The coefficients  $\mu$  and  $\sigma$  in (2) are positive functions satisfying usual hypotheses in order to guarantee that the system of SDEs (1)-(2) is uniquely solvable and the solution  $(S_\tau, D_\tau)$  is a Markovian process.

We denote by  $U = U(S, D, \tau)$  the price at time  $\tau$  of an European option with maturity  $T$ , expressed as a function of the state variables: time  $\tau$ , asset price  $S$  and deviation  $D$ . Then usual no arbitrage arguments lead to the pricing PDE (cf. formula (4.4) in [24])

$$U_\tau - rU + rSU_S - \left( \lambda D + \frac{\sigma^2}{2} \right) U_D + \frac{\sigma^2}{2} (S^2 U_{SS} + 2SU_{SD} + U_{DD}) = 0, \quad (3)$$

for  $(S, D, \tau) \in \mathbb{R}_+ \times \mathbb{R} \times ]0, T[$ . As in the Black&Scholes framework, the drift term in (2) does not enter in the valuation PDE while a key role is played by the volatility function  $\sigma$  which is an input of the model and has to be estimated in order to fit market observations.

Compared with the assumption  $\sigma = \sigma(\tau, S_\tau)$  of the widespread local volatility models, the path dependence property seems to be very realistic and natural. For instance it is well known that volatility increases after market reversals: this is difficultly captured by a model which only takes into account of the present price of the underlying and cannot relate past to present prices. Moreover a path-dependent model incorporates information on the preceding behavior of investors. Then, in some sense, the model “knows” how investors behave in different market circumstances and can also keep into account of the (positive or negative) trend of the asset. For this reason it seems that the HR model does not need to be frequently recalibrated, which is a known deficiency of local volatility models.

It should be recalled that the main advantages of the HR model against other stochastic volatility models consists on the fact that only one source of uncertainty drives both the price and the latent process. For this reason the model is able to incorporate path-dependency without losing the market completeness property [24]. Recent works addressing the applications of stochastic volatility

models to the pricing, hedging and calibration of options includes [3,27,37]. Additional references on estimation, selection and forecasting using stochastic volatility model can be found in [2].

Another fine feature of the HR model is that, due to some invariance property of (3), a simple change of variables allows to evaluate all plain vanilla option prices corresponding to different strikes and different time-to-maturities in a single run (cf. Remark 1). This considerably speeds up the calibration procedure by PDEs' techniques. Actually that approach also has the natural advantage of allowing to compute the derivatives with respect to the parameters of the solution which will be useful in the procedure.

For these reasons, the HR model raised some interest among researchers. We recall that parameters calibration ( $\lambda$  in the average weight and the volatility function  $\sigma$ ) was studied by Platania and Rogers [30], Figà-Talamanca and Guerra [17], Hahn, Putschögl and Sass [22]. Sekine [31] obtained a closed-form expression of the Laplace transform  $E[e^{-z \log S_T}]$  for a special parametrization of  $\sigma$ . More general path dependent volatility models were proposed by Hubalek, Teichmann and Tompkins [25]; Foschi and Pascucci [19] proposed the use of a more general weight function, possibly with compact support, in (1) and analyzed the out-of-sample performance of the path dependent volatility in comparison with standard stochastic volatility models. An extension to the framework of term-structure modeling was given by Chiarella and Kwon [7]. Hahn, Putschögl and Sass [23] considered the HR dynamics in a portfolio optimization problem. The robustness of the HR model with respect to the data and parameters was studied by Hallulli and Vargiolu [5]. Trifi [34] showed that the HR model is the continuous time limit of an ARCH-type model. Recently the free boundary and optimal stopping problems for American options in a path dependent volatility model were studied in [29] and [14].

As a first goal of this paper, in Section 2 we provide accurate numerical schemes ad hoc designed for the degenerate parabolic equation (3). These schemes exploit the intrinsic regularity properties of the PDE derived by the Hörmander condition and allow for an efficient implementation.

In Section 3 the inverse problem arising in the calibration is stated as a simple nonlinear least squares problem. Aiming to motivate the model, Hobson and Rogers consider in [24] a volatility function of the form

$$\sigma(D) = \min \left\{ \eta \sqrt{1 + \varepsilon D^2}, N \right\}, \quad (4)$$

for some large constant  $N$  and positive parameters  $\varepsilon, \eta$ : then they show that the model can indeed exhibit smiles and skews of different directions. In this note we aim to select  $\sigma$  without imposing a priori assumptions on its shape but simply calibrating it to market prices of plain vanilla options. More precisely,

in order to maintain the approach as much flexible as possible, we assume that  $\sigma$  is approximated in a space of piecewise cubic Hermite polynomials.

In Section 3, we also introduce what we call the *Hobson-Rogers recalibrated model* which is a refinement of the standard HR model: in this alternative, as suggested by Hobson and Rogers, the level of the offset  $D_t$  used to evaluate new derivatives is inferred from the current cross section of option prices, instead of using the observed trajectory of  $S_t$  for its estimation. This new approach seemingly overcomes possible observation or model errors and gives a better fit to market prices.

Finally, in the last part of the paper, the results of the calibration are tested on a set of S&P500 index options prices and experimental results regarding the fitting of the model to observed prices are presented.

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## 2 Kolgorov schemes

For an European call option with strike  $K$ , equation (3) is coupled with the initial condition

$$U(S, D, T) = (S - K)^+. \quad (5)$$

We simplify the pricing PDE by a simple change of variables. We rewrite equation (3) as

$$\mathcal{L}u := a(\partial_{xx}u - \partial_x u) + (x - y)\partial_y u - \partial_t u = 0, \quad (6)$$

where  $u = u(x, y, t)$  is determined by the transformation

$$U(S, D, \tau) = Ke^{-r(T-\tau)}u(r(T-\tau)+\log(S/K), r(T-\tau)+\log(S/K)-D, \lambda(T-\tau)), \quad (7)$$

and

$$a(x, y) = \frac{\sigma^2(x - y)}{2\lambda}. \quad (8)$$

By this change of variables, problem (3), (5) is equivalent to the Cauchy problem for (6) in the strip  $\mathbb{R}^2 \times [0, \lambda T]$  with initial condition

$$u(x, y, 0) = (e^x - 1)^+, \quad (x, y) \in \mathbb{R}^2. \quad (9)$$

**Remark 1** Problem (6)-(9) is independent of  $K$ . Therefore formula (7) gives option prices corresponding to different strikes by solving a unique PDE.

Due to the additional state variable  $D$  on which the option price depends, equation (6) is of degenerate type since the quadratic form associated to the second order part of  $\mathcal{L}$  is singular. However (6) belongs to the noteworthy subclass of Hörmander PDEs today called of Kolmogorov or Ornstein-Uhlenbeck type. For this class a very satisfactory theory has been developed and many sharp analytical results are available even under weak regularity assumptions: see, for instance, [26] for an exhaustive survey on this topic. In particular, in [12] conditions are given for the existence and uniqueness of a classical solution to the Cauchy problem (6)-(9); moreover sharp estimates for  $u$  and its derivatives are provided.

### 2.1 Finite difference schemes of Kolmogorov type for the pricing PDE

Since the natural framework for the study of the properties of equation (6) is the theory of subelliptic PDEs, also for the numerical approximation the best results are obtained in a non-Euclidean setting. Indeed it is known that the differential operators

$$Xu := \partial_x u, \quad Yu := (x - y)\partial_y u - \partial_t u \quad (10)$$

are the main (in some intrinsic sense) directional derivatives and we can rewrite equation (6) in the form

$$a(X^2 - X)u + Yu = 0.$$

Therefore, in the numerical solution of the pricing equation by finite difference methods, it is natural and more efficient to approximate the directional derivatives  $X$  and  $Y$  rather than the usual Euclidean ones. Some “intrinsic” schemes for the HR model were first introduced in [11]. Here we refine those results and introduce some schemes, hereafter called Kolmogorov schemes, whose main features are the following:

- i) in the discretization of the PDE in a finite region, *no boundary conditions on the  $y$ -variable are required*. This is a significant advantage since there are no obvious financial motivations for imposing conditions on the option price  $U(S, D, t)$  for some fixed  $D$ ;
- ii) solving the scheme only involves the inversion of a *tri-diagonal* matrix which leads to a fast and easy implementation.

To be more specific, we consider the uniform grid

$$G = \{(i\Delta_x, j\Delta_y, n\Delta_t) \mid i, j, n \in \mathbb{Z}, n \geq 0\}, \quad (11)$$

and approximate as usual the derivatives  $\partial_x u$  and  $\partial_{xx} u$  by the centered differences and the three-point schemes, respectively:

$$\partial_x u(x, y, t) \sim D_{\Delta_x} u(x, y, t) = \frac{u(x + \Delta_x, y, t) - u(x - \Delta_x, y, t)}{2\Delta_x}, \quad (12)$$

and

$$\partial_{xx} u(x, y, t) \sim D_{\Delta_x}^2 u(x, y, t) = \frac{u(x + \Delta_x, y, t) - 2u(x, y, t) + u(x - \Delta_x, y, t)}{\Delta_x^2}, \quad (13)$$

Thus, the approximation

$$\begin{aligned} \partial_{xx} u(x, y, t) - \partial_x u(x, y, t) &\sim D_{\Delta_x}^2 u(x, y, t) - D_{\Delta_x} u(x, y, t) \\ &= d_1 u(x - \Delta_x, y, t) + d_2 u(x, y, t) + d_3 u(x + \Delta_x, y, t), \end{aligned} \quad (14)$$

with  $d_1 = 1/\Delta_x^2 + 1/(2\Delta_x)$ ,  $d_2 = -2/\Delta_x^2$  and  $d_3 = 1/\Delta_x^2 - 1/(2\Delta_x)$ , is of order  $\Delta_x^2$ .

The second main derivative  $Y$  is approximated either by

$$Yu(x, y, t) \sim Y_{\Delta_t}^+ u(x, y, t) = \frac{\tilde{u}(x, y, t) - \tilde{u}(x, y - (x - y)\Delta_t, t + \Delta_t)}{\Delta_t}, \quad (15)$$

or by

$$Yu(x, y, t) \sim Y_{\Delta_t}^- u(x, y, t) = \frac{\tilde{u}(x, y + (x - y)\Delta_t, t - \Delta_t) - \tilde{u}(x, y, t)}{\Delta_t}, \quad (16)$$

where  $\tilde{u}(x, y, t)$  denotes the linear interpolation of  $u$  at the point  $(x, y, t)$  based on the two nearest grid points. Specifically,

$$\tilde{u}(x, y, t) = (1 - \gamma)u(x, \tilde{y}, t) + \gamma u(x, \tilde{y} + \Delta_y, t), \quad (17)$$

where  $\gamma = (y - \tilde{y})/\Delta_y$  and  $\tilde{y} = [y/\Delta_y]\Delta_y$  denoting by  $[\cdot]$  the integer part. Since  $\tilde{u}(x, y, t)$  approximates  $u(x, y, t)$  with an error of the order of  $\Delta_y$ , then the approximations (15) and (16) are of the order of  $\Delta_t + \Delta_y$ . We remark that interpolation (17) is necessary because  $(x, y, t)$  and  $(x, y - (x - y)\Delta_t, t - \Delta_t)$  cannot both belong to the same uniform grid. In [10] a different change of variables has been proposed in place of (7). That approach allowed for both the points to belong to the grid, but at the cost of imposing the grid size condition  $\Delta_y = \Delta_x \Delta_t$ .

The discrete operators  $\mathcal{L}_G^+$  and  $\mathcal{L}_G^-$  are defined by

$$\mathcal{L}_G^\pm u = a(D_{\Delta_x}^2 u - D_{\Delta_x} u) + Y_{\Delta_t}^\pm u, \quad (18)$$

and approximate  $\mathcal{L}$  in the sense that

$$\|\mathcal{L}u - \mathcal{L}_G^\pm u\|_{L^\infty} \leq C \left( \Delta_x^2 + \Delta_t + \frac{\Delta_y^2}{\Delta_t} \right), \quad (19)$$

for some positive constant  $C$  depending on the  $L^\infty$ -norms of  $a$ ,  $\partial_{xxx}u$ ,  $\partial_y u$ ,  $\partial_x^4 u$ ,  $Y^2 u$ ,  $\partial_{xx} Y u$ , and  $\partial_{xy} u$ .

Hereafter, we refer to  $\mathcal{L}_G^+$  and  $\mathcal{L}_G^-$  respectively as explicit and implicit schemes for the discretization of  $\mathcal{L}$ . The implicit scheme is unconditionally stable, while the stability condition for the explicit method is given by  $\Delta_t \leq \frac{\Delta_x^2}{2 \sup a}$  and  $\Delta_x < 2$ .

In the Appendix we formulate the discretization of the PDE (6) by means of (18), as the block bi-diagonal linear system (A.4). It is remarkable that the solution of such a system only requires the inversion of a tri-diagonal matrix which can be computed very efficiently.

## 2.2 Calibration and continuous dependence results

In the calibration procedure we consider the volatility  $\sigma$  as a smooth function of a parameter vector  $\alpha \in \mathbb{R}_+^p$  for some  $p \in \mathbb{N}$ ; more precisely, we assume that  $\sigma = \sigma(d; \alpha) \in C^1(\mathbb{R} \times \mathbb{R}_+^p)$  and that there exist two positive constants  $C_1, C_2$  such that

$$C_1 \leq \sigma(d, \alpha) \leq C_2, \quad |\partial_{\alpha_k} \sigma(d, \alpha)| \leq C_2 \quad (d, \alpha) \in \mathbb{R} \times \mathbb{R}_+^p, \quad k = 1, \dots, p.$$

Thus we rewrite the dynamics (2) of the price as

$$dS_t = \mu(D_t) S_t dt + \sigma(D_t; \alpha) S_t dW_t, \quad (20)$$

and denote by  $u(\cdot; \alpha)$  the solution to (6)-(9) with  $a(x, y; \alpha) = \frac{\sigma^2(x-y; \alpha)}{2\lambda}$ . For what follows, it will be useful to compute the derivatives of  $u$  with respect to the parameters  $\alpha$ . A linear system, similar to (A.4), derives from the discretization of the corresponding PDE. Indeed the following result about continuous dependence with respect to the parameters holds.

**Theorem 2** *Let  $\mathcal{L}^{(\alpha)}$  be the operator defined by (6) with diffusion coefficient  $a = a(\cdot; \alpha)$  and consider the solution  $u$  to the Cauchy problem*

$$\begin{cases} \mathcal{L}^{(\alpha)} u(\cdot; \alpha) = 0, & \text{in } \mathbb{R}^2 \times ]0, \lambda T[, \\ u(x, y, 0; \alpha) = u_0(x, y), & \text{in } \mathbb{R}^2, \end{cases} \quad (21)$$

where the initial datum  $u_0$  is Hölder continuous and such that  $|u_0(x, y)| \leq C_3 e^{C_4(x^2+y^2)}$  with  $C_4$  suitably small. Then,  $u(z; \cdot) \in C^1(\mathbb{R}_+^p)$  for every  $z =$



$(x, y, t) \in \mathbb{R}^2 \times [0, \lambda T]$ . Moreover for  $k = 1, \dots, p$ , the derivative  $v = \partial_{\alpha_k} u(\cdot; \alpha)$  satisfies the PDE

$$\mathcal{L}^{(\alpha)} v = -(\partial_{\alpha_k} a(\cdot; \alpha))(\partial_{xx} u(\cdot; \alpha) - \partial_x u(\cdot; \alpha)), \quad (22)$$

in  $\mathbb{R}^2 \times ]0, \lambda T[$ , with initial condition

$$v(x, y, 0) = 0, \quad \text{for } (x, y) \in \mathbb{R}^2. \quad (23)$$

The proof of the theorem is based on the following

**Lemma 3** *Under the above assumptions,  $\partial_x u, \partial_{xx} u$  are continuous functions w.r.t. the variables  $(x, y, t, \alpha)$  and there exist some positive constants  $C_5, C_6$  and  $\delta > 0$ , only dependent on  $C_1, \dots, C_4$  and the Hölder constant of  $u_0$ , such that*

$$|\partial_x u(x, y, t; \alpha)| + |\partial_{xx} u(x, y, t; \alpha)| \leq C_5 \frac{e^{C_6(x^2+y^2)}}{t^{1-\delta}}, \quad (24)$$

for every  $(x, y, t) \in \mathbb{R}^2 \times ]0, \lambda T[$  and  $\alpha \in \mathbb{R}_+^p$ .

A detailed proof of Lemma 3 is given in [13]. The proof is rather delicate since it involves the study of some singular integrals related to degenerate parabolic PDEs. We only mention that the continuity of  $\partial_x u$  and  $\partial_{xx} u$  can be proved by using the techniques in [12] (see, in particular, Chap. 5 regarding some potential estimates); estimate (24) can be proved as Theorem 8.2, Chap. V in [15], by using the pointwise estimates for the fundamental solution and its derivatives provided in [12], Theorem 1.4.

*Proof of Theorem 2.* For simplicity, we only consider the case  $p = 1$ . We set  $z = (x, y, t) \in \mathbb{R}^2 \times ]0, \lambda T[$  and denote by  $\Gamma^{(\alpha)}$  the fundamental solution of  $\mathcal{L}^{(\alpha)}$ . Since  $v(z) := u(z; \alpha) - u(z; \alpha_0)$  is solution to the problem

$$\begin{cases} \mathcal{L}^{(\alpha_0)} v = -(a(\cdot; \alpha) - a(\cdot; \alpha_0))(\partial_{xx} u(\cdot; \alpha) - \partial_x u(\cdot; \alpha)), & \text{in } \mathbb{R}^2 \times ]0, \lambda T[, \\ v(x, y, 0) = 0, & \text{in } \mathbb{R}^2, \end{cases}$$

we have

$$\begin{aligned} \frac{u(z; \alpha) - u(z; \alpha_0)}{\alpha - \alpha_0} &= \int_0^t \int_{\mathbb{R}^2} \Gamma^{(\alpha_0)}(z; \xi, \eta, s) \cdot \\ &\quad \frac{a(\xi, \eta; \alpha) - a(\xi, \eta; \alpha_0)}{\alpha - \alpha_0} (\partial_{\xi\xi} - \partial_\xi) u(\xi, \eta, s; \alpha) d\xi d\eta ds. \end{aligned}$$

Therefore, as  $\alpha$  goes to  $\alpha_0$ , by the dominated convergence theorem combined with Lemma 3, we infer that  $u$  is differentiable w.r.t.  $\alpha$  and it holds

$$\partial_\alpha u(z; \alpha_0) = \int_0^t \int_{\mathbb{R}^2} \Gamma^{(\alpha_0)}(z; \xi, \eta, s) \partial_\alpha a(\xi, \eta; \alpha_0) (\partial_{\xi\xi} - \partial_\xi) u(\xi, \eta, s; \alpha_0) d\xi d\eta ds.$$

This concludes the proof.  $\square$

### 2.3 Boundary conditions

The numerical solution of (6) by finite-difference methods requires the discretization of the equation in a bounded region and the specification of some initial-boundary conditions. More precisely, we approximate the Cauchy problem (6)-(9) in the cylinder

$$Q = \{(x, y, t) \mid |x| < \mu, |y| < \nu \text{ and } 0 < t < \lambda T\}, \quad (25)$$

for some suitably large  $\mu, \nu$ . By transformation (7), this corresponds to the initial-boundary value problem for (3) in the domain of the points  $(S, D, t)$  such that

$$Ke^{-\mu-r(T-t)} < S < Ke^{\mu-r(T-t)}, |D - \log(e^{r(T-t)}S/K)| < \nu \text{ and } 0 < t < T.$$

The conditions on the parabolic boundary of  $Q$ , defined by

$$\partial_P Q = \partial Q \cap \{(x, y, t) \mid t < \lambda T\},$$

are set as follows:

$$u(x, y, 0) = (e^x - 1)^+, \quad \text{for } x \in [-\mu, \mu], y \in [-\nu, \nu]; \quad (26)$$

moreover, we set

$$(\partial_{xx}u - \partial_x u)(\pm\mu, y, t) = 0, \quad \text{for } y \in ]-\nu, \nu[, t \in ]0, \lambda T[. \quad (27)$$

We note explicitly that (27) corresponds to condition  $\partial_{SS}U = 0$  in the original variables, that imposes linearity on  $U$  at the border. This condition is standard in the Black&Scholes framework [33,35] and it has been analyzed in [28,36].

It is remarkable that the Kolmogorov schemes avoid imposing conditions on the lateral boundary  $\{y = \pm\nu\}$ , provided that  $\nu$  is suitably large. To be more specific, let us first introduce some notation. Fixed  $i_0, j_{0,n} \in \mathbb{N}$  for  $n \in \mathbb{N} \cup \{0\}$ , we denote

$$u_{i,j}^n = u(i\Delta_x, j\Delta_y, n\Delta_t), \quad i, j \in \mathbb{Z}, \quad |i| \leq i_0, \quad |j| \leq j_{0,n}. \quad (28)$$

Applying the discrete operator in (14) to  $u_{i,j}^n$  for  $|i| \leq i_0 - 1$  gives

$$D_{\Delta_x^2} u_{i,j}^n - D_{\Delta_x} u_{i,j}^n = (d_1 u_{i-1,j}^n + d_2 u_{i,j}^n + d_3 u_{i+1,j}^n). \quad (29)$$

Consider now the discretization (16) and assume that  $(x, y, t) = (i\Delta_x, j\Delta_y, n\Delta_t)$  belongs to the grid. Then we have

$$\tilde{u}(x, y + (x - y)\Delta_t, t - \Delta_t) = (1 - \gamma) u_{i,j+k}^{n-1} + \gamma u_{i,j+k+1}^{n-1}$$

and

$$Y_{\Delta_t}^- u_{i,j}^n = \frac{1}{\Delta_t} \left( u_{i,j}^n - (1 - \gamma) u_{i,j+k}^{n-1} - \gamma u_{i,j+k+1}^{n-1} \right), \quad (30)$$

where  $k$  and  $\gamma$  are, respectively, the integer and fractional part of  $(x-y)\Delta_t/\Delta_y$ , that is

$$k = \left\lfloor \frac{(x-y)\Delta_t}{\Delta_y} \right\rfloor = \left\lfloor \left( i \frac{\Delta_x}{\Delta_y} - j \right) \Delta_t \right\rfloor \quad \text{and} \quad \gamma = \left| \frac{(x-y)\Delta_t}{\Delta_y} - k \right|. \quad (31)$$

Applying the discrete operator  $\mathcal{L}_G^-$  to  $u_{i,j}^n$  we have

$$a_{ij} \left( D_{\Delta_x^2} u_{i,j}^n - D_{\Delta_x} u_{i,j}^n \right) + Y_{\Delta_t}^- u_{i,j}^n = 0, \quad |i| \leq i_0 - 1, \quad |j| \leq j_{0,n}, \quad (32)$$

where  $a_{ij} = a(i\Delta_x, j\Delta_y)$ . On the other hand, assuming  $\mu = i_0\Delta_x$ , condition (27) is equivalent to

$$Y_{\Delta_t}^- u_{i,j}^n = 0, \quad i = \pm i_0, \quad |j| \leq j_{0,n}. \quad (33)$$

Next we fix  $j_{0,N} \in \mathbb{N}$  and examine the domain of dependence of the set of values

$$U_N = \{ u_{i,j}^N \mid |i| \leq i_0, \quad |j| \leq j_0^N \};$$

more precisely, for  $0 \leq n \leq N-1$ , we specify  $j_{0,n}$  as the maximum of the set of the indexes  $j$ 's such that  $U_N$  depends on  $u_{i,j}^n$  through conditions (32)-(33). Moreover we set  $\nu_n = j_{0,n}\Delta_y$ . Since, by (30) and (31), it holds

$$j_{0,n-1} = j_{0,n} + \left\lfloor j_{0,n}\Delta_t + i_0 \frac{\Delta_x\Delta_t}{\Delta_y} \right\rfloor + 1 \leq j_{0,n}(1 + \Delta_t) + i_0 \frac{\Delta_x\Delta_t}{\Delta_y} + 1,$$

it follows that

$$\nu_{n-1} \leq \nu_n(1 + \Delta_t) + \mu\Delta_t + \Delta_y,$$

and thus  $\nu_{N-n} \leq z_n$ , where  $z_n$  is defined by the difference equation

$$z_{n+1} = (1 + \Delta_t)z_n + \mu\Delta_t + \Delta_y, \quad z_0 = \nu_N,$$

which has solution  $z_n = (1 + \Delta_t)^n (y_0 + \mu + \Delta_y/\Delta_t) - \mu - \Delta_y/\Delta_t$ : indeed, recall that the solution of the difference equation  $z_{n+1} = \alpha z_n + \beta$ , with initial value  $z_0$  and  $\alpha \neq 1$ , is given by  $z_n = \alpha^n (z_0 - z_*) + z_*$ , where  $z_* = \beta/(1 - \alpha)$  is the equilibrium value. Finally, we deduce

$$\nu_n = (1 + \Delta_t)^{(N-n)} (\nu_N + \mu + \Delta_y/\Delta_t) - \mu - \Delta_y/\Delta_t$$

and

$$\begin{aligned}\nu_0 &= (1 + \Delta_t)^{\lambda T / \Delta_t} (\nu_N + \mu + \Delta_y / \Delta_t) - \mu - \Delta_y / \Delta_t \\ &\leq e^{\lambda T} \nu_N + (e^{\lambda T} - 1)(\mu + \Delta_y / \Delta_t).\end{aligned}$$

Thus we have proved the following result.

**Theorem 4** *Assume that*

$$\frac{\Delta_y}{\Delta_t} = C_0, \quad (34)$$

*for some constant  $C_0$ . Then, in order to approximate the solution  $u(x, y, \lambda T)$  for  $|x| \leq \mu$  and  $|y| \leq \nu_N$ , conditions on the lateral boundary  $\{y = \pm \tilde{\nu}\}$ , where  $\tilde{\nu} = e^{\lambda T} \nu_N + (e^{\lambda T} - 1)(\mu + C_0)$ , are superfluous.*

**Remark 5** *Notice that, under condition (34), the approximation error of  $\mathcal{L}_G^\pm$  in (19) reduces to an order of  $\Delta_x^2 + \Delta_t$ .*

*Moreover, condition (34) ensures that the width of the initial region can be chosen independently of the refinement of the grid. Alternatively, one can solve (6) in the prism*

$$\{(x, y, t) \mid |x| < \mu, |y| < e^{\lambda T - t} \nu_N + (e^{\lambda T - t} - 1)(\mu + C_0), 0 < t < \lambda T\},$$

*rather than in the whole cylinder  $Q$ .*

### 3 Calibration

In this section we consider the calibration of the HR model, that is the estimation of the volatility function  $\sigma$  from observed market prices of European options. As  $u$  is observed only at a finite number of points, specific restrictions should be imposed on  $\sigma$  in order to obtain a well posed problem. For this reason, as in Subsection 2.2, we assume that  $\sigma = \sigma(\cdot; \alpha)$  depends on a vector  $\alpha = (\alpha_1, \dots, \alpha_p)$  of real positive parameters and denote by  $u(x, y, t; \alpha)$  the solution to the Cauchy problem (6)-(9) corresponding to

$$a = \frac{\sigma^2(x - y; \alpha)}{2\lambda}.$$

This defines a mapping from  $\mathbb{R}_+^p$  to  $C^\infty(\mathbb{R}^2 \times ]0, \lambda T[)$ . The aim of this section is to develop and test a numerical procedure to “invert” that function.

Let  $\hat{f}_i$  be the observed option value at the point  $z_i \equiv (x_i, y_i, t_i)$ , for  $i = 1, 2, \dots, M$ , and let  $f_i(\alpha)$  be the price given by (7) in terms of the solution  $u(x_i, y_i, t_i; \alpha)$  of the PDE (6) for a given  $\alpha$  at the observation point  $z_i$ . Since the

point  $z_i$  may not belong to the grid  $G$ , the value of  $f_i(\alpha)$  is approximated by using a linear interpolation of the nearest points of the grid. The error made in fitting the  $i$ -th observed value for a given  $\alpha$  is denoted by

$$\varepsilon_i(\alpha) = f_i(\alpha) - \hat{f}_i.$$

We aim to find  $\alpha$  that best fits the data by solving the nonlinear least squares (NLLS) problem

$$\min_{\alpha \in \mathbb{R}_+^p} \sum_{i=1}^M \varepsilon_i^2(\alpha)/w_i + \rho(\alpha), \quad (35)$$

where  $w_i$  is some weight given to the  $i$ -th observation which will be specified later in Subsection 3.2 and  $\rho$  is positive definite regularization function introduced to penalize large estimates of  $\sigma$  and to render the objective function free of non-optimal stationary points. The NLLS problem (35) is solved using the Matlab routine `lsqnonlin` which is a trust-region method based on the interior-point method described in [8]. The algorithm needs the first order derivatives  $\partial_{\alpha_k} u$  at the points  $z_i$ , for  $i = 1, \dots, M$ , which are computed by solving a set of  $p + 1$  PDEs (6) and (22): thus the computational cost linearly increases with the number  $p$  of parameters involved. In the experiments performed, the non-convexity of the objective function affected only the speed of convergence of the method, but not its robustness.

### 3.1 The dataset

The calibration procedure here described is applied to a set of European options quotations on the S&P 500 index from the Chicago Board Options Exchange (CBOE). Only calls with time-to-maturity from two weeks to six months are considered, moreover the average of bid and ask prices is used as reference. Observations have been taken each 15 minutes from 11:00 to 14:00 of each trading day in the period from Nov-15-2002 to May-23-2003. The empirical distribution of this dataset, which contains 190397 observations, w.r.t. absolute time, deviation from the trend, time-to-maturity and moneyness is shown in Figure 1.

Following Ait-Sahalia and Lo [1] we do not use quotations on the underlying  $S_t$  and realized dividends  $\delta_t$ , because the first ones are affected by synchronization errors and the second ones are not necessarily equal to the expected rates at the time the option is priced. In order to avoid pricing anomalies that can arise from these problems the following procedure is used: at each time  $t$  and for each maturity  $T$  and strike  $K$ , in order to avoid arbitrage opportunities,

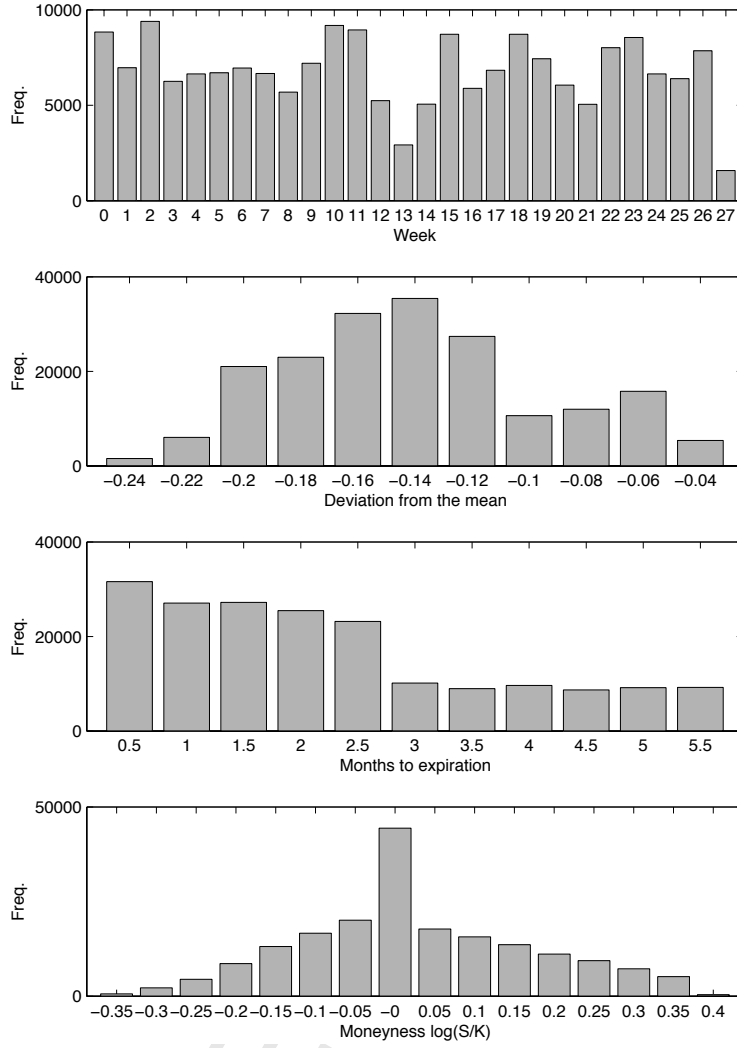


Figure 1. Distribution of observations. The four panels show the number of observations plotted against time (measured in weeks), deviation from the mean, time to maturity (measured in months) and moneyness.

the following parity relations hold:

$$F_{T-t} = S_t e^{(r_{t,T} - \delta_{t,T})(T-t)},$$

and

$$U_{T-t}^{\text{call}}(S_t, K, r_{t,T}, \delta_{t,T}) + K e^{-r_{t,T}(T-t)} = U_{T-t}^{\text{put}}(S_t, K, r_{t,T}, \delta_{t,T}) + F_{T-t} e^{-r_{t,T}(T-t)},$$

where  $F_{T-t}$ ,  $U_{T-t}^{\text{call}}$  and  $U_{T-t}^{\text{put}}$  denote, respectively, the future, call and put prices with expiration at  $T$ ,  $r_{t,T}$  and  $\delta_{t,T}$  the corresponding interest and dividend rates. Thus, future prices can be inferred from synchronous quotations of put

and call options with the same strike and maturity. In order to obtain reliable values,  $F_{T-t}$  is computed as the weighted average of the implied futures over all the available strikes. Log of volumes plus one are used as weights, even if a simple average produces similar results. Next, in order to reduce the number of input parameters of the calibration procedure, we use the homogeneity relation

$$e^{r_{t,T}(T-t)}U_{T-t}(S_t, K, r_{t,T}, \delta_{t,T}) = U_{T-t}(F_{T-t}, K, 0, 0) = f(F_{T-t}, M_t, T-t),$$

to replace option prices  $U(S_t, K, r_{t,T}, \delta_{t,T})$  with ex-rates prices  $U(F_{T-t}, K, 0, 0)$ .

Notice that many authors modify each cross section before the calibration. For example in [6] implied volatilities curves are computed by smoothing splines from market observations and in [4] constrained cubic splines are used to smooth market prices and to enforce non-arbitrage conditions. In both cases the reported results refer to the modified values. Here, we only infer the underlying future price and *use raw data for the fitting*. Furthermore, differently from other investigations, where transactions or daily closing prices are used, we use a dataset built from intra-day observation of bid-ask prices which do not necessarily correspond to some transaction.

To compute the exponential trend  $M_t$  and the corresponding deviation  $D_t$  we have used closing day prices from October 1982 to September 2002 and then intra-day prices until May 2003. Index and trend computed with  $\lambda = 1$  are shown in Figure 2. We have chosen  $\lambda = 1$  as it gives the best reproduction of the term structure of implied volatilities. The period considered in the dataset has an initial decreasing phase followed by sharp rise of the index level.

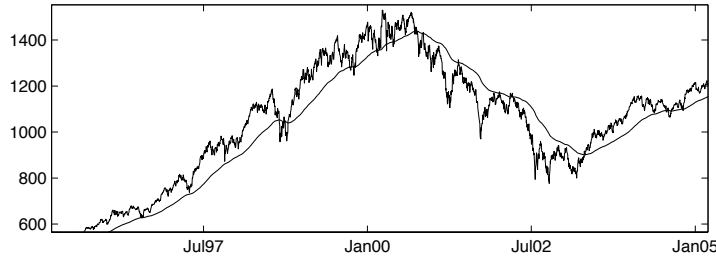


Figure 2. Price of the S&P 500 index  $S_t$  and the corresponding exponential trend computed with  $\lambda = 1$  for the years 1997-2004.

The dependency of the option prices on  $D_t$  is evident from Figure 3. The four panels plot the implied volatilities against the adjusted log-moneyness  $\log(F/K)/\sqrt{T-t}$ : as it was noted by Foque et al. [18] the implied volatility smile of a cross section of index options is a function of  $\log(F/K)/\sqrt{T-t}$  only. This relation captures the term structure of option implied volatilities. Each panel refer to a different range of  $D_t$ . The base of the smile shifts from

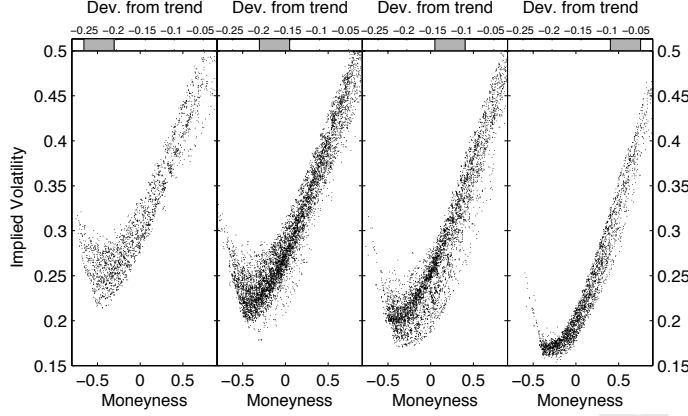


Figure 3. Effects of the deviation from the trend on marked implied volatilities. The implied volatilities are plotted against adjusted log-moneyness  $\log(F/K)/\sqrt{T-t}$  and grouped by different ranges of  $D$  as shown by the bar in the top of each panel.

an implied volatility of 0.22 to one of 0.15 and from a log-moneyness of -0.5 to one of -0.25. Notice that, the observations of  $D_t$ , like  $S_t$ , are affected by errors, due to synchronization or to market inefficiencies.

### 3.2 Calibration results

Since the observations are not homogeneously distributed with respect to time-to-maturity, moneyness and deviation from the trend, we adopt the following strategy to choose the weights  $w_i$  in (35). We divide option transactions in 18 groups based on maturity ( $[0, 3]$  or  $]3, 6]$  months), log-moneyness ( $] - \infty, -0.1]$ ,  $] -0.1, 0.1]$  and  $]0.1, \infty[$ ) and deviation from trend ( $] - \infty, -0.2]$ ,  $] -0.2, -0.15]$ ,  $] -0.15, -0.1]$ ,  $] -0.1, 0]$ ). The weight  $w_i$  of the  $i$ th observation has been chosen equal to the number of elements in the corresponding group.

The HR model has been calibrated to the data for two different choices of the volatility function  $\sigma^2(D)$ . Following [20], the first choice consists in a piecewise cubic Hermite polynomial  $\sigma_{\text{Spline}}^2(D)$  interpolating the abscissae  $\sigma_{\text{Spline}}^2(D_i) = \alpha_i$ , for  $1 \leq i \leq 7$  at the knots  $D_i \in \{-1, -2/3, -1/3, 0, 1/3, 2/3, 1\}$ . In Figure 4 the interpolating polynomial and the knots are represented by a dashed line and by circles, respectively. The function  $\sigma_{\text{Spline}}^2(D)$  is chosen linear outside the interval  $[-1, 1]$  continuous up to the first derivative in  $D = 1$  and  $D = -1$ . Further, its positiveness is ensured by the constraints  $\alpha_i \geq 0$ ,  $1 \leq i \leq 7$ , in the NLLS problem (35). Hereafter, we refer to this function as the *Spline volatility function* and to the corresponding model as the *Spline model*.



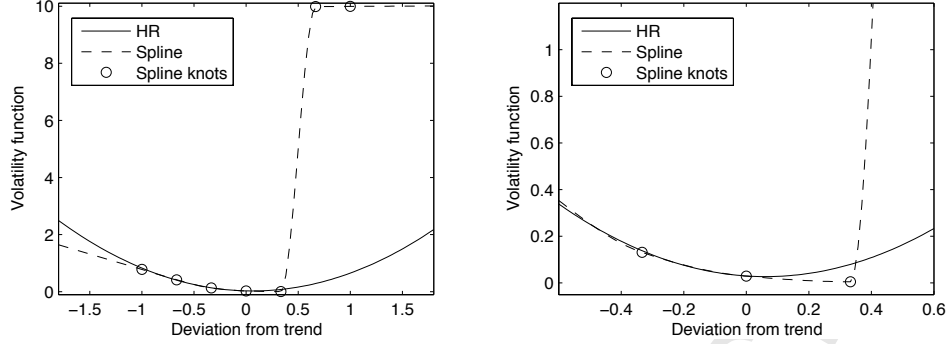


Figure 4. Calibrated Volatility functions  $\sigma_{HR}^2$  and  $\sigma_{Spline}^2$ .

The second choice consists in

$$\sigma_{HR}^2(D) = \min \left\{ \alpha_1 + \alpha_2(D - \alpha_3)^2, \sqrt{5} \right\},$$

with the constraints  $\alpha_i \geq 0$ , for  $i = 1, 2$  and  $\alpha_3$  unconstrained. We refer to this function as the *HR volatility function* and to the corresponding model as the *HR model*. The calibrated parameters of the two functions and their standard deviations are reported in Table 2. Figure 4 shows the graphs of the two volatility functions.

In order to overcome possible observation or model errors in  $D_t$ , we have re-calibrated the offset parameter  $\alpha_3$  of the HR model for each day of the dataset. More precisely, the parameters  $\alpha_1$  and  $\alpha_2$  have been kept fixed to the previously estimated values, that is to the values reported in Table 2 and the parameter  $\alpha_3$  is calibrated using the observations of each given day: this, results in a series of estimates  $\alpha_t^{(3)}$  which allows to recover the option implied deviations defined by  $\widetilde{D}_t = D_t + \alpha_3 - \alpha_t^{(3)}$ . The two time-series  $D_t$  and  $\widetilde{D}_t$  are shown in Figure 5, where it can be noticed that the difference is often substantial. We refer to this calibration as the *HR re-calibrated model*.

To analyze the quality of the fit, four kinds of errors are considered:

- absolute or dollar (valuation) error:  $\varepsilon_i = f_i - \hat{f}_i$ ;
- percentage or relative error:  $\hat{\varepsilon}_i = \varepsilon_i / \hat{f}_i$ ;
- error outside the bid-ask spread, or simply outside error:

$$\varepsilon_i^{OE} = \text{sign}(\varepsilon_i) \max(f_i - \hat{f}_i^{ask}, \hat{f}_i^{bid} - f_i, 0);$$

- percentage or relative outside error:  $\hat{\varepsilon}_i^{OE} = \varepsilon_i^{OE} / \hat{f}_i$ ;

where  $\hat{f}_i^{ask}$  and  $\hat{f}_i^{bid}$  are the bid and ask prices of the  $i$ th observation (cf. [16]). In order to partially eliminate the bias in percentage errors, only calls with price larger than 10\$ are considered when computing statistics for relative errors (note that the value of the underlying is 887\$ in mean).

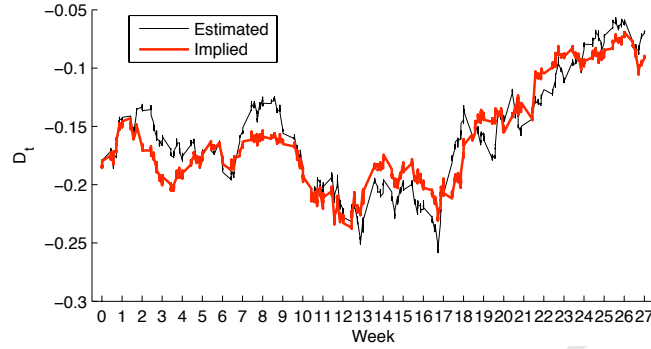


Figure 5. Comparison of two estimators for  $D_t$ . The thin line represents the time series computed from historical data by discretizing (1). The thick line corresponds to the value for  $D_t$  implied from each daily cross sections of options.

Table 1 contains the following resuming statistics for the three models and the four types of errors: mean error, standard deviation, extreme values, Root Mean Square Error (RMSE)

$$\text{RMSE}(\varepsilon_i) = \left( M^{-1} \sum_{i=1}^M \varepsilon_i^2 \right)^{1/2},$$

and Mean Absolute Error (MAE),

$$\text{MAE}(\varepsilon_i) = M^{-1} \sum_{i=1}^M |\varepsilon_i|.$$

The four panels in Figures 6 and 7, respectively, plot the RMSEs of absolute and relative outside errors for different ranges of absolute time, deviation from the trend, time-to-expiration and moneyness.

We first compare the fits of the HR and Spline specifications. The overall performances of both models are good, also considering the length of the period and that the volatility function, and thus the pricing kernel, is kept constant. The good results are confirmed by the histograms of absolute and percentage errors for the HR and Spline calibrations shown in Figure 8. From the third panel of Figures 6 and 7 it should be noticed that both the models well explain the term-to-maturity structure of option prices. This has been pointed as a deficit of stochastic volatility models in [18,21,32].

The Spline model obtains slightly better results than those of the HR model, but the difference is so small that the HR model should be preferred for its parsimony. It can be seen from Figure 4 that the two volatility functions differ only on their tails and especially on the right ones. However, as indicated by the standard deviations reported in Table 2,  $\sigma_{\text{Spline}}^2$  is completely imprecise in that region. In fact, the last two parameters of  $\sigma_{\text{Spline}}^2$  are not identifiable

Absolute Errors						
Model	RMSE	MAE	Mean	Std. Dev.	Minimum	Maximum
Spline	1.853	1.458	-0.757	1.692	-5.944	7.049
HR	1.857	1.463	-0.789	1.681	-6.113	7.131
HR Recal.	1.532	1.229	-0.613	1.403	-4.948	4.81
Absolute Outside errors						
Model	RMSE	MAE	Mean	Std. Dev.	Minimum	Maximum
Spline	1.153	0.740	-0.366	1.094	-5.532	6.043
HR	1.154	0.741	-0.387	1.087	-5.846	6.125
HR Recal	0.852	0.533	-0.271	0.808	-4.131	3.976
Percentage Errors						
Model	RMSE	MAE	Mean	Std. Dev.	Minimum	Maximum
Spline	5.16%	3.29%	-0.71%	5.11%	-34.40%	33.08%
HR	5.19%	3.34%	-1.05%	5.09%	-36.56%	32.05%
HR Recal.	4.33%	2.73%	-0.47%	4.30%	-26.00%	34.06%
Percentage Outside Errors						
Model	RMSE	MAE	Mean	Std. Dev.	Minimum	Maximum
Spline	3.37%	1.70%	-0.34%	3.35%	-29.57%	29.69%
HR	3.37%	1.73%	-0.56%	3.33%	-31.72%	28.66%
HR Recal.	2.62%	1.19%	-0.26%	2.60%	-23.01%	31.62%

Table 1

Statistics for absolute (relative) errors and outside errors for the calibrations with the Spline, HR and HR re-calibrated models.

because they do not significantly affect the solution  $u(x, y, t)$  of (6) at the observation points.

Next we discuss the performance of the HR re-calibrated model. This approach allows for a further improvement in the fit: both the absolute and relative outside errors are reduced of one third (cf. Table 1). Furthermore, implying  $D_t$  from option prices allows to better capture the term structure of option prices (see the third panel in Figures 6 and 7). While the distribution of the residuals of the HR and Spline models, shown in Figure 8, is clearly not normal, that of the HR re-calibrated model seems to be a normal one. This should point out a misspecification of  $D_t$  in the first two models that almost disappears in

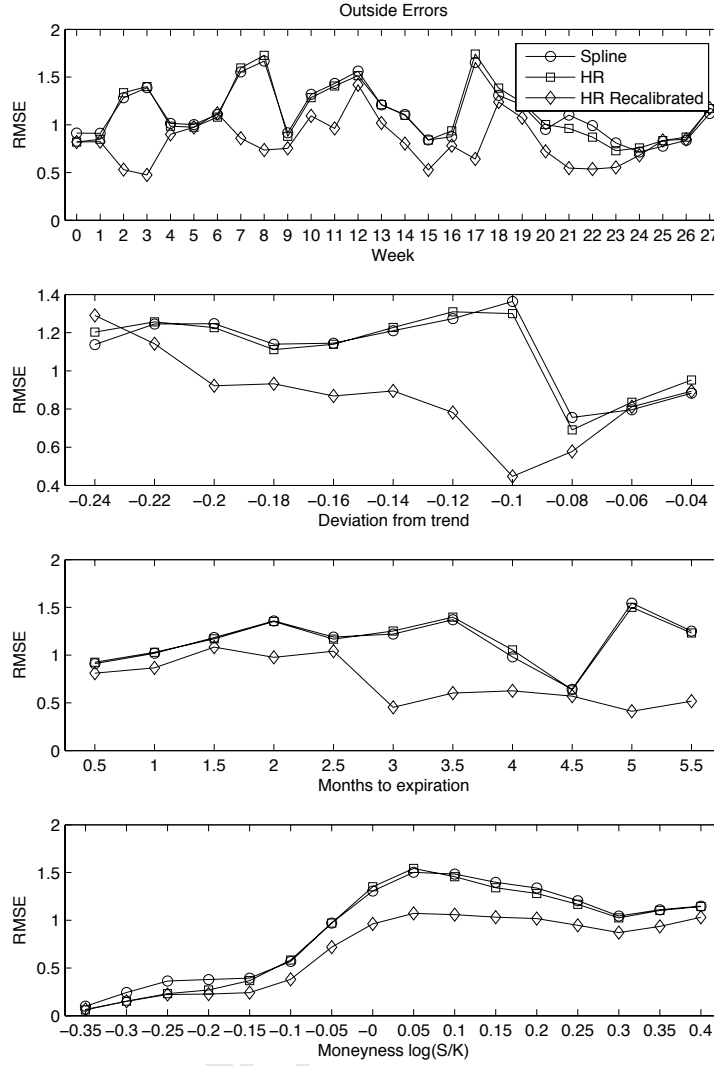


Figure 6. Plot of RMSEs of absolute outside errors against observation time (measured in weeks), deviation from the mean, time to maturity (measured in months) and moneyness. Results for the Spline, HR and HR re-calibrated models are represented by circles, squares and diamond, respectively.

the HR re-calibrated model. This also suggests that further investigation in the model with respect to the specification of the state variable  $D_t$  is needed. With this respect, we mention the extensions considered in [19] and [25] and the use of an additional deviation  $D_{\lambda_2,t}$  with a faster decay (a shorter memory) or a second order offset function

$$D_t^{(2)} = \int_0^\infty \lambda e^{-\lambda\tau} (Z_t - Z_{t-\tau})^2 d\tau,$$

as already considered in [24], to better explain the market behavior.

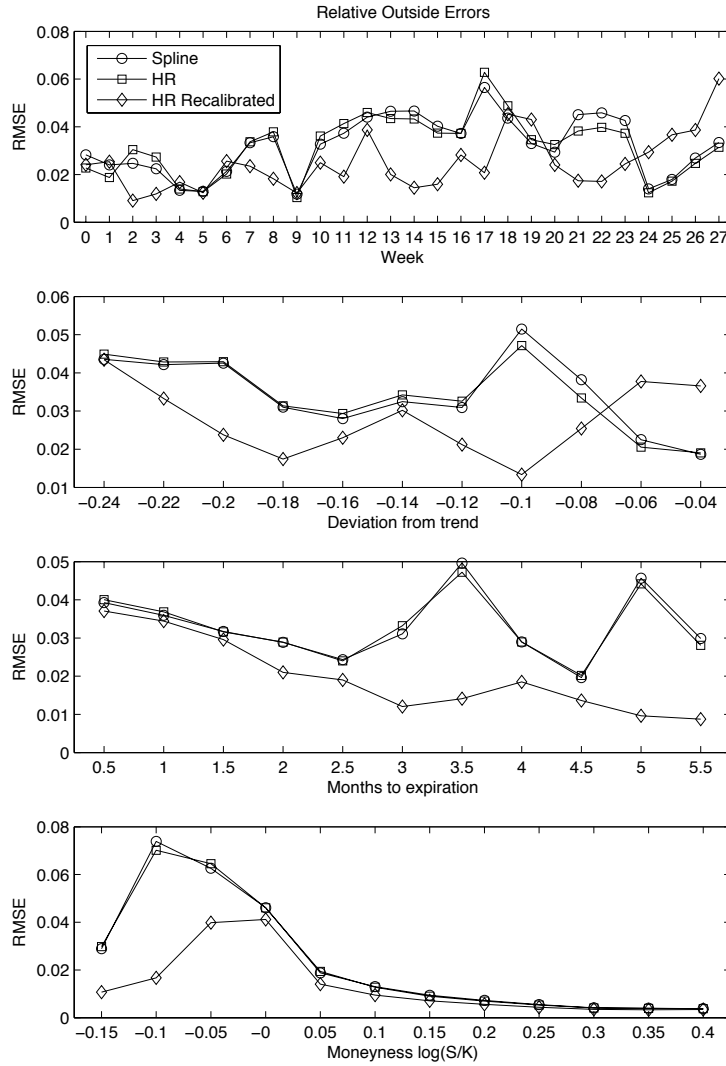


Figure 7. Plot of RMSEs of relative outside errors against observation time (measured in weeks), deviation from the mean, time to maturity (measured in months) and moneyness. Results for the Spline, HR and HR re-calibrated models are represented by circles, squares and diamond, respectively.

Figure 9 shows implied volatilities for the HR model using the daily calibrated deviations  $\tilde{D}_t$ . As can be seen by comparing Figures 3 and 9, the model is not able to fully reproduce the smiles for deep in- or out-of-the money options. Furthermore, smiles implied by the market are more significant especially for shorter maturities.

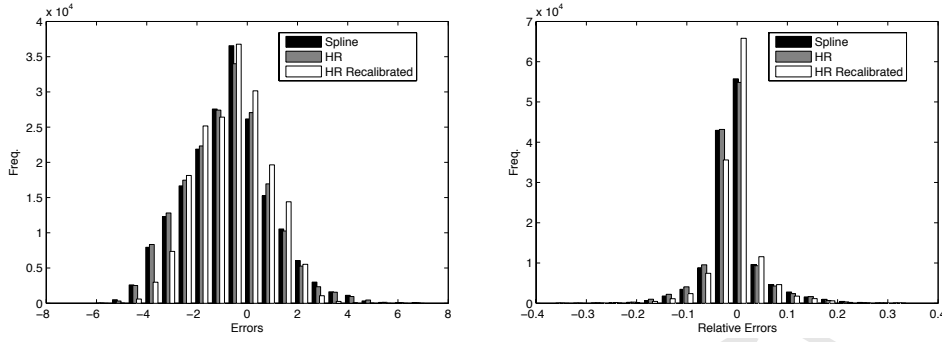


Figure 8. Histograms of the errors in the calibration with the HR and Spline volatility models.

	Spline		HR	
	$\alpha_i$	Std. dev.	$\alpha_i$	Std. dev.
$\alpha_1$	0.7791	(3.59e-2)	0.0272	(8.51e-5)
$\alpha_2$	0.4195	(2.01e-3)	0.7114	(2.11e-3)
$\alpha_3$	0.1308	(1.24e-4)	0.0616	(6.51e-4)
$\alpha_4$	0.0289	(4.17e-5)		
$\alpha_5$	0.0050	(3.54e-4)		
$\alpha_6$	9.9862	(2.46e+5)		
$\alpha_7$	9.9927	(6.58e+5)		

Table 2

Calibrated parameters for the volatility functions  $\sigma_{spline}$  and  $\sigma_{HR}$ .

### 3.3 Out-of-sample tests

In order to test out-of-sample performances, the HR model has been calibrated on the observations ranging from Nov-15-2002 to Jan-14-2003 and tested on the successive week, month and three months periods. The calibration produces the volatility function

$$\sigma_{HR}(D)^2 = 0.0297 + 0.9360(D - 0.0352)^2,$$

and examples of the in- and out-of-sample fits are shown in Figure 10, where the calibrated payoffs for different maturities are represented by the curves and market prices  $\hat{f}_i$  by the crosses.

Statistics for in-sample and out-of-sample tests are given in Table 3 and RMSEs are plotted against time, deviation, maturity and log-moneyness in Figure 11. As expected, out-of-sample performances get worse as the horizon considered increases. Moreover, since the calibration is performed mainly on short

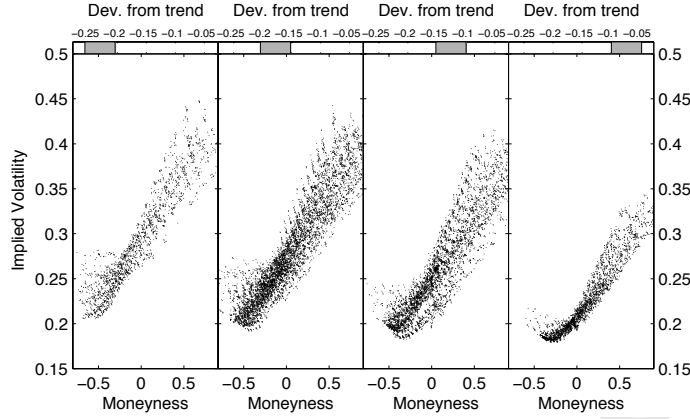


Figure 9. Effects of the deviation from the trend on marked implied volatilities. The implied volatilities are plotted against adjusted log-moneyness  $\log(F/K)/\sqrt{T-t}$  and grouped by different ranges of the deviation as shown by the bar at the top of each panel.

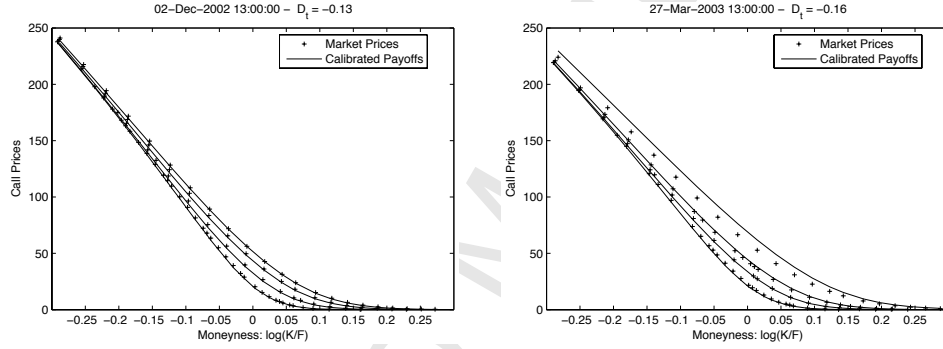


Figure 10. Market prices and call price curves on December 2nd (in sample) and March 27th (out of sample). The calibration has been performed over the period November 15th 2002 - January 14th 2003.

maturity options it does not give a very good prediction for longer maturities. Nevertheless the RMSEs in the first panel follow the same pattern of the RMSEs in Figure 6, with an exception at week 17. Table 3 also confirms the validity of the model which, contrary to local volatility models, seems to have a good performance even if it is not continuously re-calibrated. In particular, the quality of the fit seems to be quite acceptable even one month after the calibration period, confirming the theoretical arguments.

Absolute errors						
	RMSE	MAE	Mean	Std. Dev.	Minimum	Maximum
in-sample	1.71	1.34	-0.46	1.65	-4.28	7.58
out-of-sample						
1 week	1.58	1.18	-0.02	1.58	-3.17	5.46
1 month	2.23	1.49	0.18	2.22	-3.97	10.51
3 months	2.69	1.75	0.60	2.63	-5.16	13.29
Absolute outside errors						
	RMSE	MAE	Mean	Std. Dev.	Minimum	Maximum
in-sample	0.99	0.58	-0.07	0.99	-3.29	6.52
out-of-sample						
1 week	0.97	0.54	0.08	0.97	-2.18	4.45
1 month	1.67	0.84	0.35	1.63	-3.03	10.01
3 months	2.13	1.07	0.65	2.02	-5.06	12.28
Percentage errors						
	RMSE	MAE	Mean	Std. Dev.	Minimum	Maximum
in-sample	4.06%	2.56%	-0.17%	4.06%	-20.17%	33.14%
out-of-sample						
1 week	4.04%	2.64%	1.23%	3.85%	- 5.33%	19.20%
1 month	5.84%	3.78%	1.15%	5.72%	-17.36%	39.71%
3 months	8.19%	4.86%	2.23%	7.88%	-33.17%	48.38%
Percentage outside errors						
	RMSE	MAE	Mean	Std. Dev.	Minimum	Maximum
in-sample	2.50%	1.14%	0.13%	2.50%	-15.27%	30.35%
out-of-sample						
1 week	2.31%	1.16%	0.71%	2.20%	- 2.16%	14.44%
1 month	4.12%	2.08%	1.16%	3.95%	-14.75%	36.77%
3 months	6.45%	3.16%	2.01%	6.13%	-28.16%	45.26%

Table 3

In-sample and out-of-sample statistics for percentage errors  $\varepsilon_i$  and absolute errors  $\hat{\varepsilon}_i$  of the calibration over the period from Jan-17 to Feb-17-2003. Out-of-sample statistics refers to the one week, one month and three months periods starting the 18-Feb-2003.



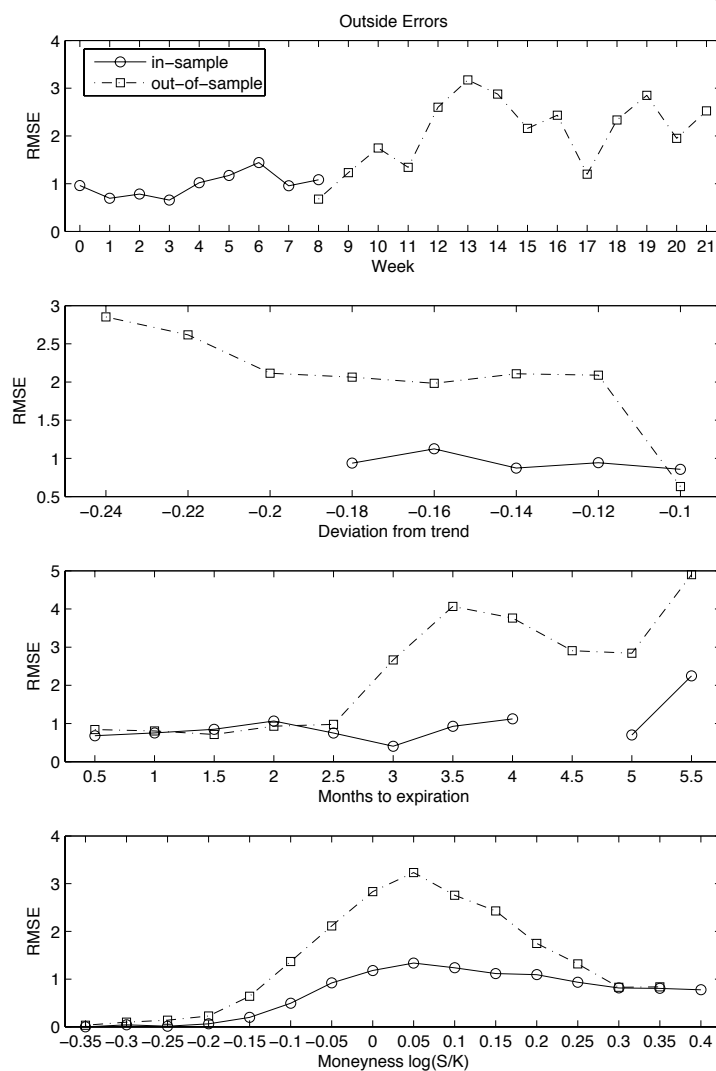


Figure 11. Plot of RMSEs versus observation time (measured in weeks), deviation from the mean, time to maturity (measured in months) and moneyness. In-sample (from January 17th to February 16th) and out-of-sample (from February 17th to May 23th) RMSEs are represented by circles and squares, respectively.

## 4 Conclusions

In this paper we consider the path dependent volatility model proposed by Hobson and Rogers. In this model option prices can be expressed in terms of solution to a hypoelliptic parabolic PDE of Kolmogorov type. In order to exploit the intrinsic regularity properties of the degenerate PDE, we introduce a non-Euclidean finite differences scheme which allows for an accurate approximation of option prices.

Aiming to test the effectiveness of the model we study its calibration as a nonlinear least squares problem. In this approach we estimate the volatility function both in a parametric and a semi-parametric framework. We perform an empirical experiment on a large dataset of intraday S&P 500 option quotations ranging from November 2002 to May 2003. The results show good performances of the model even when not continuously recalibrated. For this reason the model seems to be a valid alternative to the widely used local volatility models.

## A Appendix: discretization and linear systems

Throughout this Appendix we use the notations of Subsection 2.3. The aim is to formulate the discretization of the partial differential equation (6) as a block bidiagonal linear system. We define  $I = 2i_0 + 1$ ,  $J_n = 2j_{0,n} + 1$  and denote by  $u^n \in \mathbb{R}^{IJ_n}$  the vector containing the values  $u_{i,j}^n$  for  $|i| \leq i_0$  and  $|j| \leq j_{0,n}$ : those values are sorted by the couple of indices  $(j, i)$  in lexicographic order.

Let us now consider the application of the discrete operator  $Y_{\Delta_t}^-$  in (30) to the vector  $u^n$ . The generic element  $Y_{\Delta_t}^- u_{i,j}^n$  is the linear combination of the corresponding element in  $u^n$  and two elements,  $u_{i,j+k}^{n-1}$  and  $u_{i,j+k+1}^{n-1}$  of  $u^{n-1}$ . Thus, applying  $Y_{\Delta_t}^-$  to  $u_n$  is equivalent to the difference of two linear operators,  $\Delta_t^{-1} \mathcal{I}_n$  and  $\Delta_t^{-1} Z_n$ , applied respectively to  $u^n$  and  $u^{n-1}$ , where  $\mathcal{I}_n$  denotes the identity operator in  $\mathbb{R}^{IJ_n}$ . Specifically, the vector with elements  $Y_{\Delta_t}^- u_{i,j}^n$  is given by

$$\frac{1}{\Delta_t}(u^n - Z_n u^{n-1}),$$

where  $Z_n \in \mathbb{R}^{IJ_n \times IJ_{n-1}}$  is the matrix such that the entry corresponding to the index  $i, j$  of  $Z u^{n-1}$  is given by

$$(1 - \gamma)u_{i,j+k}^{n-1} + \gamma u_{i,j+k+1}^{n-1}.$$

Then it turns out that the linear system (32) can be rewritten in matrix form

$$(\mathcal{I}_n + \Delta_t A_n D_n) u^n - Z_n u^{n-1} = 0, \quad (\text{A.1})$$

for  $1 \leq n \leq N$ , where  $A \in \mathbb{R}^{IJ_n \times IJ_n}$  is the diagonal matrix with elements  $a_{ij}$  and

$$D_n = \begin{pmatrix} \check{D} & 0 & \cdots & 0 \\ 0 & \check{D} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \check{D} \end{pmatrix} \quad \text{and} \quad \check{D} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ d_1 & d_2 & d_3 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & d_1 & d_2 & d_3 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.2})$$

are tridiagonal matrices of order  $IJ_n$  and  $I$ , respectively.

Similarly, combining the forward and backward schemes allows to derive the  $\theta$ -method:

$$\theta \Delta_t A_n D_n u^n + (1 - \theta) \Delta_t Z_n A_{n-1} D_{n-1} u^{n-1} - Z_n u^{n-1} + u^n = 0,$$

or

$$\bar{A}_1^n u^n = \bar{A}_2^n u^{n-1}, \quad 1 \leq n \leq N, \quad (\text{A.3})$$

with  $\bar{A}_1^n = (\mathcal{I}_n + \theta \Delta_t A_n D_n)$  and  $\bar{A}_2^n = Z_n (\mathcal{I}_n - (1 - \theta) \Delta_t A_{n-1} D_{n-1})$ . As usual, the  $\theta$ -method reduces to the explicit, implicit or Crank-Nicholson schemes when  $\theta = 0, 1$  or  $0.5$ , respectively. Notice that the  $\theta$ -method is unconditionally stable for  $0.5 \leq \theta \leq 1$ . The matrices  $\bar{A}_1^n$  and  $D_n$  have an identical structure, specifically they are block diagonal with tridiagonal blocks. Thus, the computational cost required to solve (A.3) is of the order of  $IJ_n$ . Furthermore, the structure of the matrices can be exploited to design computationally efficient and/or parallel algorithms for the solution of the PDE (6).

Finally, by setting  $N = \lambda T / \Delta_t$  it turns out that considering (A.3) for  $n = 1, \dots, N$  and imposing the initial conditions  $u_0 = v_0$  is equivalent to the linear system

$$\begin{pmatrix} I & 0 & 0 & \cdots & 0 \\ -\bar{A}_2^1 & \bar{A}_1^1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\bar{A}_2^{N-1} & \bar{A}_1^{N-1} & 0 \\ 0 & \cdots & 0 & -\bar{A}_2^N & \bar{A}_1^N \end{pmatrix} \begin{pmatrix} u^0 \\ u^1 \\ \vdots \\ u^{N-1} \\ u^N \end{pmatrix} = \begin{pmatrix} v^0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

or, with the appropriate substitution,

$$\bar{A}\bar{u} = \bar{v}, \quad (\text{A.4})$$

where  $\bar{A}$  is block bidiagonal and  $\bar{u} \in \mathbb{R}^q$  where

$$q = \sum_{n=0}^N IJ_n. \quad (\text{A.5})$$

Existence and uniqueness of the solution are related to the non-singularity of  $\bar{A}_1^n$  and, in view of the expressions of  $d_1, d_2, d_3$ , are clearly guaranteed if  $\frac{\Delta_t}{\Delta_{x^2}}$  is suitably small. Stability and numerical stability are driven by the properties of the matrices  $\bar{A}_1^n$  and  $\bar{A}_2^n$ .

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