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Parametrix Approximation of Diffusion Transition Densities*

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Abstract. A new analytical approximation tool, derived from the classical PDE theory, is introduced in order to build approximate transition densities of diffusions. The tool is useful for approximate pricing and hedging of financial derivatives and for maximum likelihood and method of moments estimates of diffusion parameters. The approximation is uniform with respect to time and space variables. Moreover, easily computable error bounds are available in any dimension.

Key words. diffusions, transition densities, option pricing, analytic approximations, parabolic equations, parametrix method

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1. Introduction and motivation.

The ability of approximating, with some explicit measure of error, the fundamental solution of a parabolic partial differential equation (PDE) has become of paramount relevance for economical and financial applications. The origin of this can be traced back to the 1960s with the widespread introduction of diffusion-based modeling. In financial applications, transition densities play a central role in several instances from pricing and hedging of financial derivatives to parameter estimation and calibration (by likelihood or moment method based techniques) and solution of optimal control problems.

The early examples of diffusion models considered only very simple specifications, usually in the class of linear diffusions, for which explicit transition densities are known. However, the requirement of more statistical realism and the need for a multidimensional framework stimulated more complex models whose explicit solutions are unavailable. This favored the implementation of standard (in other fields) numerical procedures, e.g., finite differences or finite elements methods, and the development of a number of very useful Monte Carlo techniques.

An alternative or, more precisely, a complement to the use of numerical methods is given by the powerful machinery of analytical approximations. These, in fields like physics and engineering, are the tools of choice for the study of the qualitative properties of a model and for comparison between models. Moreover, these techniques provide the basis for efficient numerical algorithms specific to each model. Today the literature concerning asymptotic expansions and singular perturbation theory applied to finance is vast. We quote, among

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others, the papers by Whalley and Wilmott [34] in the study of transaction costs; Hagan and Woodward [19] and Hagan et al. [18] for the implied volatility in CEV and SABR models; Fouque, Papanicolaou, and Sircar [15], Fouque et al. [16], Howison [21], Widdicks et al. [35], and Svoboda-Greenwood [32] for other local and stochastic volatility models; Barone-Adesi and Elliott [3], Broadie and Detemple [5], and Kuske and Keller [27] in the study of American options; Widdicks et al. [36] in multiasset option pricing; Turnbull and Wakeman [33], Zhang [37], and Dewynne and Shaw [10] concerning average (Asian) options; and Broadie, Glasserman, and Kou [6] in the study of discrete barrier options. For other analytical or partially analytical approximation methods we refer the reader to Aït-Sahalia [1] and [2].

In this paper we propose a new technique, the *parametrix method*, for the analytical approximation of the fundamental solution of a general parabolic PDE. As far as we know, contrary to perturbation techniques, the parametrix has not been previously examined as a numerical method, nor has it been already employed for approximation purposes in finance or other fields. Indeed, the classical parametrix technique was introduced by Levi [28] as a *theoretical method* to prove the existence of the fundamental solution of a parabolic PDE:[1] as such, it is not optimized for the purpose of numerical computation of solutions.

The first main contribution of this paper is the conversion and adjustment of this classical tool in PDE analysis as a numerical method. In particular, we modify the parametrix technique to yield an approximation useful for computational purposes. Moreover, we introduce the new concept of the *backward parametrix* and we show how it is both more appropriate for computations and more useful for model interpretation purposes than the standard parametrix. As we will see, the backward parametrix lends itself to a direct probabilistic interpretation which is not readily available for the standard parametrix. This interpretation is very useful when the backward parametrix is applied to the problem of pricing financial derivatives as it allows a financial interpretation of the leading and the correction terms in the expansion.

The second main contribution is the derivation, in the new context, of easy-to-compute, a priori (i.e., independent of the solution), *uniform bounds* on the approximation error both for the PDE solution and for its derivatives. The hypotheses under which the series approximation uniformly converges and under which the evaluation of the error term for a truncated series holds are very general. They can be checked by the sole knowledge of some general property of the diffusion model and are substantially the same hypotheses commonly required for a diffusion problem and a parabolic problem to be equivalent. While perhaps the most easy to check, these hypotheses are not the most general. Indeed the proposed method also works in cases where these hypotheses are not true: we show an example of this in section 3. The results of the paper are then applied to a number of relevant particular cases and compared with other numerical and seminumerical evaluation procedures known in the literature.

The rest of the paper is organized as follows. In section 2, we derive and slightly generalize the classical parametrix expansion in the one-dimensional setting; moreover, we introduce the

[1] A comprehensive presentation of the classic parametrix method for uniformly parabolic PDEs can be found, for instance, in [17]. We also quote the papers [11] and [26], where the parametrix method is applied to a wider class of (possibly degenerate) equations that includes the pricing PDEs for Asian options. While well known in the classical theory of parabolic PDEs, the parametrix series is, as far as we know, rather unknown in the field of mathematical finance with the exception of [4] and of a quotation in [1].

new backward parametrix. We also give a financial interpretation of the derivation of the parametrix. In section 3 we find closed form approximate solutions in a general local volatility model. In section 4 we perform some numerical tests and compare the performance of the parametrix with other known approximate methods. In section 5, we derive a priori bounds on the approximation error for both the forward parametrix and the backward parametrix. The appendix contains a number of lemmas used in the proofs.

2. Forward and backward parametrix approximations. The aim of this section is to give the main ideas by presenting our results in the simple case of a one-dimensional model. In section 5 the parametrix is derived in its full generality and complete proofs are given. Our contribution is twofold: first, we compute explicit error bounds for the classical parametrix (hereafter called the *forward parametrix*) approximation introduced by Levi [28]; second, we introduce the so-called *backward parametrix*, an alternative expansion that is more significant for the financial interpretation and from the computational point of view.

In what follows we denote by $z = (x, t)$, $\zeta = (\xi, \tau)$, and $w = (y, s)$ the points in $\mathbb{R} \times \mathbb{R}$ and consider the parabolic PDE

$$Lu(z) := a(z)\partial_{xx}u(z) + b(z)\partial_x u(z) + c(z)u(z) - \partial_t u(z) = 0. \tag{2.1}$$

The fundamental solution $\Gamma = \Gamma(z; \zeta)$ of L is a function such that the following hold:

(i) $L\Gamma(\cdot; \zeta) = 0$ in $\mathbb{R}^2 \setminus \{\zeta\}$ for any ζ;

(ii) for every bounded and continuous function $\varphi = \varphi(x)$ and $\tau \in \mathbb{R}$, a classical solution to the Cauchy problem

$$\begin{cases} Lu(x, t) = 0, & x \in \mathbb{R},\ t > \tau, \\ u(x, \tau) = \varphi(x), & x \in \mathbb{R}, \end{cases} \tag{2.2}$$

is given by

$$u(x, t) = \int_{\mathbb{R}} \Gamma(x, t; \xi, \tau)\varphi(\xi)d\xi. \tag{2.3}$$

Under the assumption that L is uniformly parabolic (i.e., the coefficient a is greater than a constant $a_0 > 0$) and has bounded and Hölder continuous coefficients, it is well known that a fundamental solution for L exists: this *theoretical result* can be proved by the parametrix method. We now aim to investigate the potentiality of the parametrix as a *numerical method*.

Coming back to the financial interpretation, formula (2.3) gives the forward price at time to maturity $t - \tau$ of a European option with payoff φ. Suppose now that problem (2.2) cannot be solved explicitly. It is then inviting to find an approximation formula for (2.3) *whose principal term is given by (or is at least similar to) the Black–Scholes formula*. This is what the parametrix method allows us to do.

2.1. Forward parametrix. The classical forward parametrix method is based on two ideas. The first is to approximate $\Gamma(z; \zeta)$ by the so-called parametrix defined by

$$Z(z; \zeta) = \Gamma_\zeta(z; \zeta),$$

where, for fixed $w \in \mathbb{R}^2$ and for $\bar{b}, \bar{c}$ arbitrarily fixed real constants, Γ_w is the fundamental solution to the constant coefficient operator

$$L_w u(z) := a(w)\partial_{xx}u(z) + \bar{b}\partial_x u(z) + \bar{c}u(z) - \partial_t u(z). \tag{2.4}$$

Note that L_w is a heat operator and the explicit expression of Γ_w is known:

(2.5)
$$\Gamma_w(z;\zeta) = \frac{1}{\sqrt{4\pi a(w)(t-\tau)}} \exp\left(-\frac{(x-\xi)^2}{4a(w)(t-\tau)} - \frac{\bar{b}}{2a(w)}(x-\xi) - \left(\frac{\bar{b}^2}{4a(w)} - \bar{c}\right)(t-\tau)\right)$$

for $t > \tau$.

In the standard parametrix method (cf., for instance, [17]) the constants $\bar{b}$ and $\bar{c}$ are chosen to be null. However, the context of this paper suggests using as a parametrix the fundamental solution of the heat equation which is "most similar" to the equation under analysis. This flexibility in the choice of the operator may result in considerably sharp approximations; see, for instance, section 4, where the some local volatility models are examined.

The second idea is that of supposing that the fundamental solution Γ of L is in the form

$$\Gamma(z;\zeta) = Z(z;\zeta) + \int_\tau^t \int_{\mathbb{R}} Z(z;w)\Phi(w;\zeta)dw. \tag{2.6}$$

In view of the financial applications, in what follows we assume $\zeta = (\xi, 0)$. In order to identify Φ in (2.6), we notice that, since

$$L\Gamma(\cdot;\zeta) = 0 \qquad \text{in } \mathbb{R} \times]0, +\infty[$$

for any ζ, we get

$$0 = LZ(z;\zeta) + L\int_0^t \int_{\mathbb{R}} Z(z;w)\Phi(w;\zeta)dw. \tag{2.7}$$

But formally we have

$$\begin{aligned} L\int_0^t \int_{\mathbb{R}} Z(z;w)\Phi(w;\zeta)dw &= \int_0^t \int_{\mathbb{R}} LZ(z;w)\Phi(w;\zeta)dw - \partial_t \int_0^t \int_{\mathbb{R}} Z(z;w)\Phi(w;\zeta)dw \\ &= \int_0^t \int_{\mathbb{R}} LZ(z;w)\Phi(w;\zeta)dw - \Phi(z;\zeta) \end{aligned} \tag{2.8}$$

so that

$$\Phi(z;\zeta) = LZ(z;\zeta) + \int_0^t \int_{\mathbb{R}} LZ(z;w)\Phi(w;\zeta)dw. \tag{2.9}$$

Notation 2.1. *To avoid confusion, when necessary, we write $L^{(z)}$ instead of L in order to indicate that the operator L is acting in the variable z.*

Formula (2.9) can be solved iteratively and yields

$$\Gamma(z;\zeta) = \sum_{n=0}^{+\infty} Z_n(z;\zeta), \tag{2.10}$$

with $Z_0(z;\zeta) = Z(z;\zeta)$ and

$$Z_n(z;\zeta) = \int_0^t \int_{\mathbb{R}} Z(z;w)(LZ)_n(w;\zeta)dw, \qquad n \in \mathbb{N},$$

where, recalling the notation $w = (y, s)$,

$$\begin{aligned}(LZ)_1(w;\zeta) &= L^{(w)} Z(w;\zeta),\\ (LZ)_{n+1}(w;\zeta) &= \int_0^s \int_{\mathbb{R}} L^{(w)} Z(w;z_0)(LZ)_n(z_0;\zeta)dz_0, \qquad n \in \mathbb{N}.\end{aligned}$$

As we foretold, our first main result consists in the computation of explicit global error bounds of the parametrix approximation. These bounds, provided in Theorem 5.2, are of the following form: for any $T > 0$ there exist two positive constants C, M such that

$$\left| \Gamma(z;\zeta) - \sum_{k=0}^{n} Z_k(z;\zeta) \right| \leq C \frac{(t-\tau)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} \Gamma_M(z;\zeta), \qquad n \geq 0, \tag{2.11}$$

for any $z, \zeta \in \mathbb{R}^2$ such that $0 < t - \tau < T$, where Γ_M is the fundamental solution of the heat operator

$$M\partial_{xx} - \partial_t.$$

Moreover, the constants C and M can be explicitly estimated.

2.2. Backward parametrix. Before examining the financial interpretation, we introduce what we call the *backward parametrix*, which is based on the use of the adjoint operator of L. As we shall see shortly, the backward parametrix is more convenient than the forward parametrix from several points of view: first, it allows us to derive an approximating expansion whose *first term is given exactly by the Black–Scholes formula*, while the subsequent terms can be expressed as solutions to suitable Cauchy problems related to constant coefficient operators that have a clear financial interpretation as well. Second, the approximating terms generated in this way are convolutions of a Gaussian function, and this is convenient from a numerical point of view since we may rely upon several known efficient numerical techniques.

Remark 2.2. *The backward parametrix method does not simply consist in the standard parametrix method applied to the backward PDE: indeed, that would give the same problems as in the forward case. On the contrary, the idea is to use the backward parametrix as an approximation for the forward PDE.*

The formal adjoint operator of L in (2.1), acting in the variable ζ, is defined as

$$\widetilde{L}^{(\zeta)} = a(\zeta)\partial_{\xi\xi} + \widetilde{b}(\zeta)\partial_\xi + \widetilde{c}(\zeta) + \partial_\tau, \tag{2.12}$$

where

$$\widetilde{b}(\zeta) = -b(\zeta) + 2\partial_\xi a(\zeta), \qquad \widetilde{c}(\zeta) = c(\zeta) + \partial_{\xi\xi} a(\zeta) - \partial_\xi b(\zeta).$$

It is known that, under suitable assumptions, $\widetilde{L}$ has a fundamental solution $\widetilde{\Gamma}$ and the following duality formula holds:

$$\widetilde{\Gamma}(\zeta; z) = \Gamma(z; \zeta);$$

in particular, $\widetilde{L}^{(\zeta)}\Gamma(z;\zeta) = 0$ for $z \neq \zeta$. We define the *backward parametrix* as the fundamental solution of a constant coefficient dual operator: more precisely, for $w \in \mathbb{R}^2$ we set

$$\widetilde{L}_w^{(\zeta)} = a(w)\partial_{\xi\xi} + \bar{b}\partial_\xi + \bar{c} + \partial_\tau, \tag{2.13}$$

where $\bar{b}$ and $\bar{c}$ are arbitrarily fixed constants, and consider its fundamental solution

(2.14)
$$\widetilde{\Gamma}_w(\zeta; z) = \frac{1}{\sqrt{4\pi a(w)(t-\tau)}} \exp\left(-\frac{(x-\xi)^2}{4a(w)(t-\tau)} + \frac{\bar{b}}{2a(w)}(x-\xi) - \left(\frac{\bar{b}^2}{4a(w)} - \bar{c}\right)(t-\tau)\right)$$

for $t > \tau$. Then we define the backward parametrix as

$$P(z;\zeta) = \widetilde{\Gamma}_z(\zeta; z),$$

so that in particular we have

$$\widetilde{L}_z^{(\zeta)} P(z;\zeta) = 0. \tag{2.15}$$

As before, we set $\tau = 0$ so that $\zeta = (\xi, 0)$, and, proceeding as in the forward case, we have

$$\Gamma(z;\zeta) = \widetilde{\Gamma}(\zeta; z) = P(z;\zeta) + \sum_{n=1}^{+\infty} \int_0^t \int_{\mathbb{R}} P(w;\zeta)(\widetilde{L}P)_n(z;w)dw, \tag{2.16}$$

where, recalling the notation $w = (y, s)$,

$$(\widetilde{L}P)_1(z;w) = \widetilde{L}^{(w)}P(z;w),$$
$$(\widetilde{L}P)_{n+1}(z;w) = \int_s^t \int_{\mathbb{R}} \widetilde{L}^{(w)}P(z_0;w)(LZ)_n(z;z_0)dz_0, \qquad n \geq 1.$$

In Theorem 5.7 we give explicit global error estimates, completely analogous to (2.11), for the backward approximation truncated at the nth term.

2.3. Parametrix expansions. In section 5, Theorems 5.4 and 5.7, we prove that the solution to the Cauchy problem (2.2) has "forward and backward" expansions of the following form.

- The expansion obtained using the forward parametrix is given by

$$u(z) = \sum_{n=0}^{\infty} u_n(z), \tag{2.17}$$

where

$$u_0(z) = \int_{\mathbb{R}} Z(z;\xi,0)\varphi(\xi)d\xi, \tag{2.18}$$

and, in general, for $n \in \mathbb{N}$,

$$u_n(z) = \int_0^t \int_{\mathbb{R}} Z(z;\zeta) L U_{n-1}(\zeta) d\zeta, \qquad U_{n-1}(z) := \sum_{k=0}^{n-1} u_k(z). \tag{2.19}$$

- Similarly the expansion obtained using the backward parametrix is of the form (2.17), where now

$$\widetilde{u}_0(z) = \int_{\mathbb{R}} P(z;\xi,0)\varphi(\xi)d\xi, \tag{2.20}$$

and, in general, for $n \in \mathbb{N}$,

$$\widetilde{u}_n(z) = \int_0^t \int_{\mathbb{R}} P(z;\zeta) L \widetilde{U}_{n-1}(\zeta) d\zeta, \qquad \widetilde{U}_{n-1}(z) := \sum_{k=0}^{n-1} u_k(z), \qquad n \in \mathbb{N}. \tag{2.21}$$

The main differences between the two parametrix expansions hinted at before are now clear. Each term in the expansion is an "expected value" with respect to the distributions with density $Z(z;\zeta)$ or $P(z;\zeta)$. But, while $P(z;\zeta)$ is the same Gaussian density for each value of the integration variable (z is frozen and the integration is performed varying ζ) and so is a true PDF, $Z(z;\zeta)$ is a different Gaussian (different variance) for each value of the integration variable ζ.

More precisely, let us examine the first term of the expansion for the forward parametrix Z:

$$u_0(z) = \int_{\mathbb{R}} Z(z;\xi,0)\varphi(\xi)d\xi. \tag{2.22}$$

Since the explicit expression of $Z(z;\xi,0) = \Gamma_{(\xi,0)}(z;\xi,0)$ is known, we see that u_0 in (2.22) is very similar to the solution of a Cauchy problem for a constant coefficient operator. On the other hand, the integration in (2.22) is performed with respect to the variable ξ which also appears in $L_{(\xi,0)}$ as the point where the operator L is frozen. Hence, roughly speaking, the first term of the expansion is an "expected value" of the terminal payoff which uses as density a Gaussian with a different volatility (corresponding to the "true" diffusion coefficient) for each point in the integration range. This seems a quite sensible starting point and can obviously be compared with standard "implied volatility" approximations where a different Gaussian distribution (for $\log S$) for each strike is used. Here the suggestion is to use the same distribution but with a different volatility for each terminal value of the stock.

Let us now pass to the backward parametrix expansion zero order term:

$$\widetilde{u}_0(z) = \int_{\mathbb{R}^N} P(z;\xi,0)\varphi(\xi)d\xi.$$

Here the interpretation is straightforward: the zero order term is simply a Black–Scholes option price. Indeed, since

$$P(z;\zeta) = \widetilde{\Gamma}_z(\zeta;z),$$

the parametrix $P(z;\zeta)$ is the terminal log-price density corresponding to starting point (x,t). Notice, however, that a different "volatility" value is used for each initial pair (x,t). Accordingly, if we compute the derivative of the option with respect to the price $S=e^x$, we have that the Delta for the zero order approximation is given, with the obvious notation, by

$$\Delta = \Delta_{\rm BS} + vega * \frac{\partial\sigma}{\partial S}.$$

This Delta computation derived from the parametrix expansion is interesting because it is a direct reinterpretation of one of the ad hoc modifications of the Black–Scholes model used by practitioners in order to get the "correct" answer from the "wrong" model. In fact, this is an example of a "skew correction" for the computation of Delta.

This correction, which can take various shapes (see, e.g., [9]), tries to account for the change of implied volatility which may accompany the change in moneyness of a given option. In a particular specification, often termed "sticky delta," if $\Delta_{\rm BS}$ is the Black–Scholes *delta*, *vega* is the standard Black–Scholes *vega*, and $\sigma(K/S)_S$ is the derivative, with respect to the price, of the volatility used for evaluating the option with strike K with moneyness K/S, then we have a skew corrected *delta* computed as

$$\Delta = \Delta_{\rm BS} + vega * \sigma(K/S)_S.$$

Strictly speaking, this correction is inconsistent as it implies different risk neutral distributions for different strikes at the same date. However, if we read this correction in the framework of the parametrix, we can justify it consistently as an approximation of the "true" delta by a truncated expansion.

Next we consider the subsequent terms in the expansions. Both expansions are similar in that each new term can be interpreted approximately as an expected value. The difference that makes the backward parametrix more readable is that each term in the backward expansion is a true expected value (with respect to the same Gaussian function $P(z;\zeta)$), while in the case of the standard parametrix, $Z(z;\xi,0)$ does not correspond to a real density.

Since each new term can be read as the value (exact or approximate) of a new option in a Black–Scholes world, it is interesting to understand the meaning of such options. To this end, it suffices to recall (2.19) and (2.21) and note that the operator L in the term of order n acts on the "option approximation" derived up to order $n-1$. Therefore, each action of the L operator can be interpreted as a check of the fact that the approximation of order $n-1$ satisfies

$$L\widetilde{U}_{n-1} = 0. \tag{2.23}$$

In other words $L\widetilde{U}_{n-1}$ is a measure of the error implied in supposing that $\widetilde{U}_{n-1}$ satisfies (2.23). This error term is known (as it depends on the $n-1$ approximation) and if added to the original PDE as an inhomogeneous term makes the $n-1$ approximation exact. In the classic literature concerning parabolic equations and, in particular, the heat equation, such terms model the existence of additional heat sources (or sinks). In financial theory similar terms may arise as the result of *transaction costs*: we refer, for instance, to the asymptotic expansion setting for optimal hedging under transaction costs in [34].

We see how the parametrix expansion partitions the value of a given option computed in a non–Black–Scholes world into a series of option values each computed in the Black–Scholes world. This is exact in the case of the backward parametrix and approximately exact, if we recall that Z is not a density, in the classical forward parametrix case. In section 5 we prove how it is possible to bound the overall error derived by truncating the series at the nth term with explicit and easily computable bounds *uniformly decreasing as n goes to infinity and as time to maturity decreases.*

2.4. Computing the second term in the backward parametrix expansion. Formula (2.14) gives the first term of the backward approximation of the fundamental solution of L in (2.1). We now illustrate a method for approximating the second term of the expansion (2.16). Recalling that $\zeta = (\xi, 0)$, we have

$$P_1(z;\zeta) := \int_0^t \int_{\mathbb{R}} P(w;\zeta)\widetilde{L}^{(w)} P(z;w)dw.$$

It turns out that a convenient choice of the coefficients $\bar{b}, \bar{c}$ in (2.13) is

$$\bar{b} = -b(z), \qquad \bar{c} = c(z), \tag{2.24}$$

and therefore we set

$$\widetilde{L}_z^{(\zeta)} = a(z)\partial_{\xi\xi} - b(z)\partial_\xi + c(z) + \partial_\tau. \tag{2.25}$$

Note that, by (2.24), $\widetilde{L}_z^{(\zeta)}$ is the adjoint of the frozen (forward) operator

$$L_z^{(\zeta)} = a(z)\partial_{\xi\xi} + b(z)\partial_\xi + c(z) - \partial_\tau. \tag{2.26}$$

This fact will be used in section 3. We also performed numerical tests that show that, for a local volatility model, the choice (2.24) is indeed optimal in the sense that it minimizes the pricing error among all other possible choices of $\bar{b}, \bar{c}$.

Recalling the notation $w = (y, s)$ and setting

$$I(s) = \int_{\mathbb{R}} P(y,s;\zeta)\widetilde{L}^{(y,s)} P(z;y,s)dy, \tag{2.27}$$

the idea is to use the trapezoidal method to approximate

$$P_1(z;\zeta) = \int_0^t I(s)ds \simeq \frac{t}{2}\left(I(0) + I(t)\right).$$

This allows us to exploit the fact that

$$I(0) = \widetilde{L}^{(\zeta)} P(z;\zeta)$$

since $P(y, 0; \zeta)$ is a Dirac delta centered at ζ. Note that *this approximation avoids the computation of the spatial integral $I(s)$ in (2.27) for any s*: this results in a significant simplification especially for high dimensional models.

By (2.15) we have

$$\widetilde{L}^{(y,s)}P(z;y,s) = \left(\widetilde{L}^{(y,s)} - \widetilde{L}_z^{(y,s)}\right)P(z;y,s)$$

for $t > s$; then

$$I(s) = \int_{\mathbb{R}} P(w;\zeta)\left(\widetilde{L}^{(w)} - \widetilde{L}_z^{(w)}\right)P(z;w)dy$$

(by parts, for $L_z^{(w)}$ as in (2.26))

$$= \int_{\mathbb{R}} P(z;w)\left(L^{(w)} - L_z^{(w)}\right)P(w;\zeta)dy,$$

and, passing to the limit as s goes to t, thanks to the choice (2.24), we obtain

$$I(t) = 0.$$

In conclusion, we have the following *explicit formula* for the backward parametrix approximation with two terms:

$$\Gamma(z;\zeta) \simeq P(z;\zeta) + P_1(z;\zeta) \simeq P(z;\zeta) + \frac{t-\tau}{2}\widetilde{L}^{(\zeta)}P(z;\zeta). \tag{2.28}$$

By using this technique, it is not difficult to determine explicit expressions for higher order approximations. However, we do not report them here since the preliminary experiments we performed in local volatility models (cf. section 4) show a negligible contribution of the terms of order higher than two.

3. Analytic formulae in local volatility models. By using the parametrix method, in this section we derive analytic (closed form) approximation formulae for one-dimensional local volatility models. This result seems significant on its own; however, we would like to emphasize that analogous results are valid for general local or stochastic volatility models even in high dimensions and possibly in a degenerate setting (i.e., for instance, for Asian options): we refer the reader to the forthcoming paper [12] for more results in this direction.

Let us consider a local volatility model where the dynamic of the underlying asset is given by the SDE

$$dS_t = \mu(S_t,t)S_t dt + \sigma(S_t,t)S_t dW_t, \tag{3.1}$$

where W is a one-dimensional Brownian motion and μ, σ are sufficiently regular coefficients. Assuming a constant riskless interest rate r, the price $V(S,t)$ of a European call option with strike K and maturity T is the solution to the Cauchy problem

$$\begin{cases} \frac{\sigma^2(S,t)S^2}{2}\partial_{SS}V(S,t) + rS\partial_S V(S,t) - rV(S,t) + \partial_t V(S,t) = 0, & S > 0,\ t \in]0,T[, \\ V(S,T) = (S-K)^+, & S > 0. \end{cases} \tag{3.2}$$

By the standard change of variables

$$V(S,t) = e^{-r(T-t)}u(r(T-t) + \log S, T-t), \tag{3.3}$$

we have that V solves (3.2) if and only if u solves

$$\begin{cases} a(x,t)(\partial_{xx}u(x,t) - \partial_x u(x,t)) - \partial_t u(x,t) = 0, & x \in \mathbb{R},\ t \in]0,T[, \\ u(x,0) = (e^x - K)^+, & x \in \mathbb{R}, \end{cases} \tag{3.4}$$

where

$$a(x,t) = \frac{1}{2}\sigma^2\left(e^{x-rt}, T-t\right). \tag{3.5}$$

In particular, the option price at $t=0$ is given by

$$V(S,0) = e^{-rT}u(rT + \log S, T), \tag{3.6}$$

where

$$u(x,T) = \int_{\mathbb{R}} \left(e^{\xi} - K\right)^+ \Gamma(x,T;\xi,0)\, d\xi \tag{3.7}$$

and Γ is the fundamental solution of the PDE in (3.4).

Proceeding as in subsection 2.4, we consider the frozen operator corresponding to (2.25)

$$\widetilde{L}_z^{(\zeta)} = a(z)(\partial_{\xi\xi} + \partial_\xi) + \partial_\tau. \tag{3.8}$$

Then the related backward parametrix is given by

$$P(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi a(x,t)(t-\tau)}} \exp\left(-\frac{(x-\xi)^2}{4a(x,t)(t-\tau)} + \frac{1}{2}(x-\xi) - \frac{1}{4}a(x,t)(t-\tau)\right) \tag{3.9}$$

for $t > \tau$.

With this choice the *first term* in the backward parametrix expansion (2.20)–(2.21) corresponds exactly to the Black–Scholes price computed with (constant) volatility:

$$\widetilde{u}_0(x,t) = \int_{\mathbb{R}} \left(e^{\xi} - K\right)^+ P(x,t;\xi,0)\, d\xi = e^x\Phi(d_+) - K\Phi(d_-), \tag{3.10}$$

where

$$d_\pm = \frac{x - \log K \pm a(x,t)t}{\sqrt{4a(x,t)t}}$$

and

$$\Phi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-y^2}\, dy. \tag{3.11}$$

Next we derive the explicit expression of the backward expansion *with two terms*.

Theorem 3.1. *The second order parametrix approximation of the call price in* (3.6) *in the local volatility model* (3.1) *is given by*

$$u(x,t) \simeq \widetilde{u}_0(x,t) + \frac{tK}{2}\big(a(\log K, 0) - a(x,t)\big)P(x,t;\log K, 0), \tag{3.12}$$

where a is as in (3.5) *and $\widetilde{u}_0$ is as defined in* (3.10).

Proof. By (2.28) we have

$$\Gamma(x,t;\xi,\tau) \simeq P(x,t;\xi,\tau) + \frac{t-\tau}{2}\widetilde{L}^{(\zeta)}P(x,t;\xi,\tau),$$

where $\widetilde{L}^{(\zeta)}$ is the adjoint operator of L, acting in the variable $\zeta = (\xi,\tau)$, and P is the backward parametrix in (3.9). We first remark that

$$\left(L^{(\zeta)} - L_z^{(\zeta)}\right)\left(e^{\xi} - K\right) = \left(a(\zeta) - a(z)\right)\left(\partial_{\xi\xi} - \partial_{\xi}\right)\left(e^{\xi} - K\right) = 0. \tag{3.13}$$

Then, recalling the notation $z = (x,t)$, the second term in the parametrix expansion (2.21) is

$$u_1(x,t) = \frac{t}{2}\int_{\mathbb{R}}\left(e^{\xi} - K\right)^{+}\widetilde{L}^{(\zeta)}P(x,t;\xi,0)d\xi$$

(by (2.15))

$$= \frac{t}{2}\int_{\log K}^{+\infty}\left(e^{\xi} - K\right)\left(\widetilde{L}^{(\zeta)} - \widetilde{L}_z^{(\zeta)}\right)P(x,t;\xi,0)d\xi$$

(integrating by parts)

$$= \frac{t}{2}\int_{\log K}^{+\infty}\left(L^{(\zeta)} - L_z^{(\zeta)}\right)\left(e^{\xi} - K\right)P(x,t;\xi,0)d\xi$$
$$- \frac{t}{2}\left[P(z,\zeta)\partial_{\xi}\left(\left(a(\zeta) - a(z)\right)\left(e^{\xi} - K\right)\right)\right]_{\xi=\log K}^{\xi=+\infty}$$

(by (3.13))

$$= -\frac{t}{2}\left[P(z,\zeta)\left(e^{\xi}\left(a(\zeta) - a(z)\right) + \partial_{\xi}a(\zeta)\left(e^{\xi} - K\right)\right)\right]_{\xi=\log K}^{\xi=+\infty}$$
$$= \frac{tK}{2}\big(a(\log K,0) - a(x,t)\big)P(x,t;\log K,0). \qquad ■$$

4. Numerical experiments.

Although the results of this paper, i.e., the introduction of the backward parametrix and the estimation of error bounds, are mainly theoretical, in this section we aim to present some numerical experiment which should convince the reader of the effectiveness of the parametrix method. While more complicated models could have been considered, here we aim only to present some preliminary tests and refer the reader to a forthcoming paper for a more detailed and extensive analysis of the numerical efficiency of the parametrix method for computing option prices and the related sensitivities or Greeks.

In this section we consider the parametrix approximation *with only two terms* and evaluate its performance by comparing it with several other numerical and analytical approximations in some local volatility models. Specifically we consider three classes of models:

- the constant elasticity of variance (CEV) model by Cox and Ross [8];
- local volatility (LV) models of quadratic and hyperbolic form (cf., for instance, Iacus [23], Jäckel [24], and Kahl and Jäckel [25]); and
- the path-dependent (two-dimensional) Hobson–Rogers model [20].

4.1. CEV model. We first consider a particular one-dimensional local volatility model, the well-known CEV model, where the dynamics of the underlying asset, under the risk neutral measure, is given by

$$dS_t = rS_t dt + \sigma S_t^{1-\alpha} dW_t. \tag{4.1}$$

Here r and σ are constants and $\alpha \in [0, 1]$. Then (4.1) corresponds to (3.1) with $\mu(S, t) \equiv r$ and $\sigma(S, t) = \sigma S^{-\alpha}$: in this particular case we have

$$a(x, t) = \frac{\sigma^2 e^{-2\alpha(x-rt)}}{2} \tag{4.2}$$

in the approximation formula (3.12).

We opted for this model since there is a vast literature concerning the approximation of CEV option prices so that we may compare the parametrix performance with several other techniques. Moreover, very accurate formulae are available, and therefore we have reference numbers for an (almost) exact comparison with our approximation. Let us remark explicitly that the CEV pricing PDE is not uniformly parabolic, and, in particular, it does not satisfy the nondegeneracy condition (5.3); nevertheless, at least formally, the parametrix method applies, and we shall see that, as a matter of fact, for a wide range of values of the parameters, it provides very accurate approximations.

We compare the parametrix with six different approximation techniques: note that some of these techniques were introduced *specifically* for the CEV model, while the parametrix is a quite general method.

We distinguish *numerical* from *closed form* approximations. In the first group we consider

- Cox [7] (see also Hsu, Lin, and Lee [22]; Cox and Ross [8]; and Schroder [30]);
- Shaw [31] (cf. Chapter 28); and
- Monte Carlo (MC).

In the second group, that of analytic approximations, we consider

- Hagan and Woodward [19] (see also Obloj [29]);
- Howison [21]; and
- Svoboda-Greenwood [32].

We aim to compare the performance of the parametrix with respect to the above methods in the pricing of European call options for different values of the parameters α, σ, T, and K: typically we consider α ranging from $\frac{1}{4}$ to $\frac{3}{4}$ and the maturity T from one week to one year. We also consider different values of the strike price K, from 1 to 100.

Remark 4.1. *In the CEV model, the strike K and the volatility coefficient σ are inversely correlated. Indeed, by the transformation $Y_t = \frac{S_t}{K}$, from (4.1) we get*

$$dY_t = rY_t dt + \frac{\sigma}{K^\alpha} Y_t^{1-\alpha} dW_t,$$

which shows that K and σ are inversely proportional quantities: increasing the value of the strike corresponds to decreasing the value of the volatility.

Regarding the *numerical* approximations, we recall that the formulae by Cox [7] express the price of a call option as the sum of a series of Gamma cumulative distribution functions.

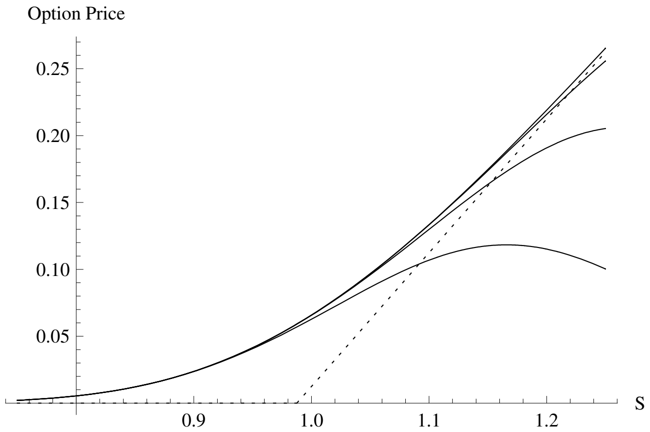

Figure 1. *CEV-expansion option price by Cox [7] in the case $\alpha = \frac{1}{4}$ and $T = \frac{1}{3}$ with a number n of terms in the series expansion equal to $n = 400, 420, 440, 460$.*

It is known that these formulae give a good *local* (at-the-money) approximation of the option price. For instance, Figure 1 shows the Cox option prices in the case $\alpha = \frac{1}{4}$ and $T = \frac{1}{3}$ with a number n of terms in the series expansion equal to $n = 400, 420, 440, 460$: it is evident that for far-from-the-money options this approximation gives wrong prices unless we consider a high number of terms in the series expansion. This is particularly sensible for short times to maturity.

On the other hand, the approximation by Shaw [31] expresses the payoff random variable in terms of Bessel functions and then uses numerical integration to provide the option price. Since it is an adaptive method, the representation of prices is valid globally even if the method may become computationally expensive when we have to compute deep out-of- or in-the-money option prices. We implemented both methods and found that in most cases they are essentially equivalent and provide reliable reference prices. However, as we shall see shortly, the approximation by Hagan and Woodward [19] also seems to be very accurate for all the values of the parameters: since this last approximation gives *closed form* solutions, we decided to use it to produce the reference values for the computation of the errors.

Concerning Monte Carlo (MC), it seems that it is not competitive with any of the other approximations. In particular, the parametrix and the other analytic approximations we considered, when suitably used, are generally much more accurate than MC solutions, and, what is more, they give explicit algebraic formulae for option prices. For instance, in Figure 2 we represent the Hagan–Woodward price, obtained by (4.4), minus the MC price (continuous line) and the parametrix price minus the MC price (dashed line) scaled to the at-the-money price in the CEV model with $\alpha = \frac{1}{4}$ (left) and $\alpha = \frac{1}{2}$ (right). In the experiments, 500.000 MC simulations, combined with an Euler discretization of the SDE with 100 steps, are performed. The values of the other parameters are $T = 0.25$, $\sigma = 30\%$, $r = 5\%$, and $K = 1$. We marked by green (respectively, red) the confidence regions where prices differ no more than 2 (respectively, 2.57) standard deviations from the simulated MC prices: this means that with probability 95% (respectively, 99%) the true price (scaled to the at-the-money price) belongs to the green (respectively, red) region. According to the standard interpretation of a

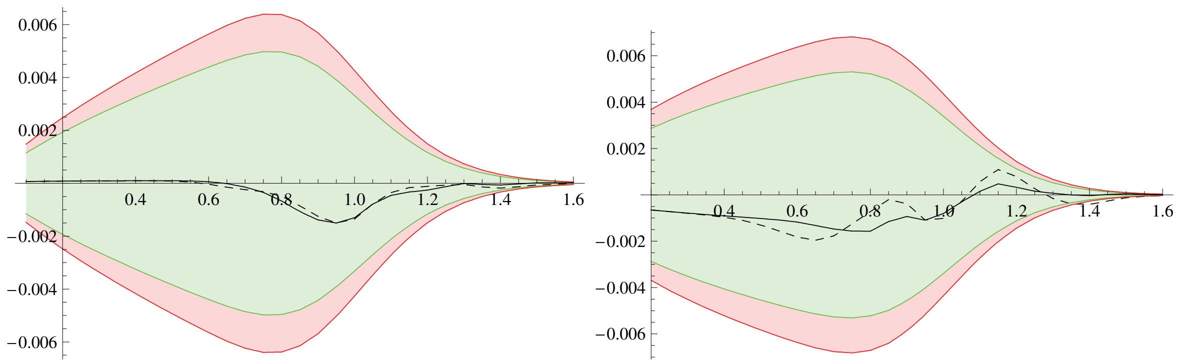

Figure 2. *Hagan–Woodward minus MC price (continuous line) and parametrix minus MC price (dashed line) scaled to the at-the-money price in the CEV model with* $\alpha = \frac{1}{4}$ *(left) and* $\alpha = \frac{1}{2}$ *(right). A* 100*-step Euler discretization of the SDE and an MC with* $N = 500.000$ *simulations have been used. Moreover,* $T = 0.25$, $\sigma = 20\%$, $r = 5\%$, *and* $K = 1$. *The* 95% *and* 99% *confidence regions for MC estimates are marked, respectively, by green and red.*

confidence interval, any price inside the bands can be the *true* price and is compatible (at the given level of confidence) with the MC estimate.

Next we consider the *analytic* approximations. We recall that Hagan and Woodward in [19], by using singular perturbation techniques, obtain explicit formulae for the approximated implied volatility σ_B in a local volatility model where the forward price F_t of the asset obeys an SDE of the form

$$dF_t = \gamma(t) A(F_t)\, dW_t \tag{4.3}$$

for some deterministic and suitably regular functions γ and A. Equation (4.1) can be reduced to (4.1) through the transformation $F_t = e^{r(T-t)} S_t$: in this case we also have

$$\gamma(t) = \sigma e^{r\alpha(T-t)} \quad \text{and} \quad A(F) = F^{1-\alpha}.$$

Therefore, the Hagan–Woodward approximation formula reads

$$\sigma_B = \frac{a}{f^\alpha}\left(1 + \frac{1}{24}\alpha(3-\alpha)\left(\frac{e^{rT}S_0 - K}{f}\right)^2 + \frac{1}{24}\frac{\alpha^2 a^2 T}{f^{2\alpha}}\right), \tag{4.4}$$

where

$$a = \sqrt{\frac{1}{T}\int_0^T \gamma(t)^2 dt} = \sigma\sqrt{\frac{e^{2r\alpha T}-1}{2r\alpha T}} \quad \text{and} \quad f = \frac{e^{rT}S_0 + K}{2}.$$

It is then sufficient to insert the implied volatility σ_B into the Black–Scholes formula to get the option prices. Figure 3 shows the Hagan–Woodward call value minus the Cox call value, scaled with the at-the-money value, as a function of S for different times to maturity and values of α. Since the Hagan–Woodward approximation seems to be quite accurate, hereafter, as already mentioned, we shall use it for computing the "true" (or reference) values in the experiments.

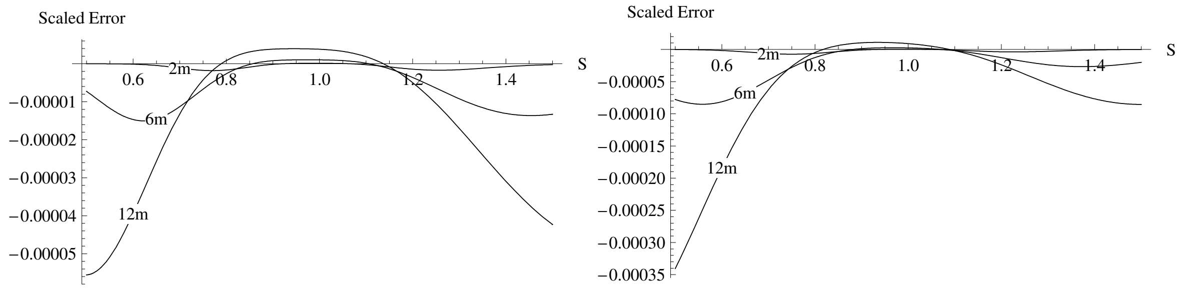

Figure 3. *Hagan–Woodward approximate value minus Cox approximate value (with $n = 1000$ terms in the series expansion), scaled with the at-the-money value, as a function of S, for $T = 2, 6, 12$ months and $\alpha = \frac{1}{4}$ (left), $\alpha = \frac{3}{4}$ (right), $\sigma = 30\%$, $r = 5\%$, and $K = 1$.*

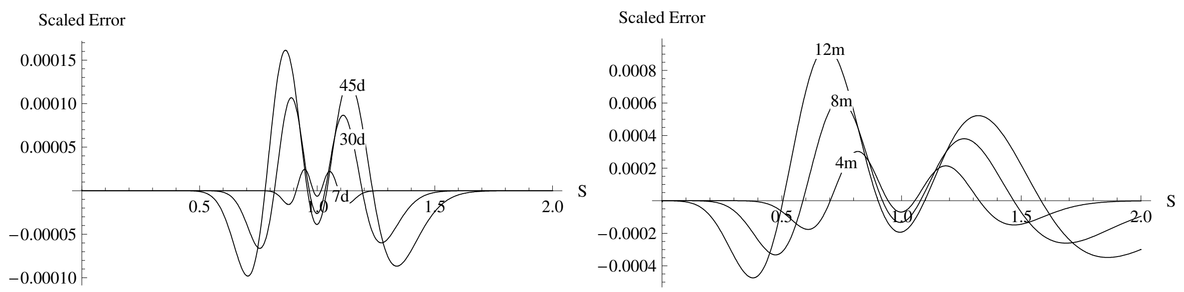

Figure 4. *Scaled errors of the parametrix approximation, as a function of $S \in [0, 2]$, for $T = 7, 15, 45$ days (left) and $T = 4, 8, 12$ months (right). $\alpha = \frac{1}{4}$, $\sigma = 30\%$, $r = 5\%$, and $K = 1$.*

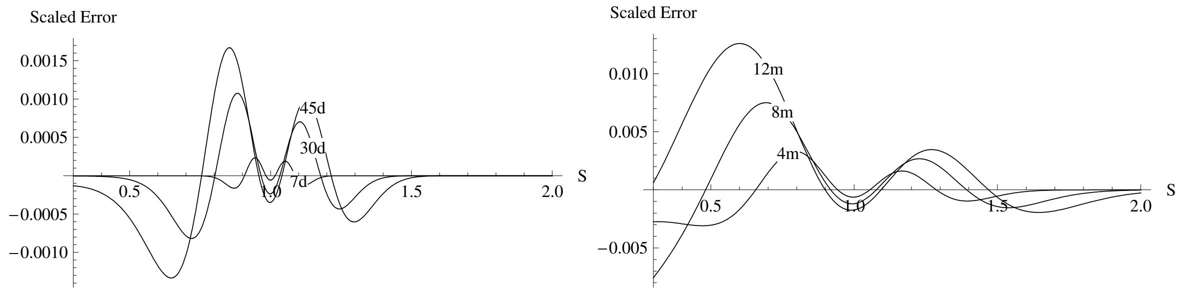

Figure 5. *Scaled errors of the parametrix approximation as a function of $S \in [0.2, 2]$ for $T = 7, 15, 45$ days (left) and $T = 4, 8, 12$ months (right). $\alpha = \frac{3}{4}$, $\sigma = 30\%$, $r = 5\%$, and $K = 1$.*

We next test the computational performance of the parametrix approximation. Figure 4 compares the error relative to the at-the-money option value (in short, the *scaled error*) in the case $\alpha = \frac{1}{4}$ for different times to maturity: $T = 7, 15, 45$ days in the left panel and $T = 4, 8, 12$ months in the right panel.

Figure 5 exhibits analogous results for $\alpha = \frac{3}{4}$: in this case, we have bigger errors than in the case $\alpha = \frac{1}{4}$ which is "closer" to the standard Black–Scholes model. For $\alpha = \frac{3}{4}$, we also separate the case $S \in \left[\frac{2}{10}, 2\right]$ from the case $S \in \left[0, \frac{2}{10}\right]$, which we represent in Figure 6. The reason is that, as previously remarked, the CEV diffusion operator is not uniformly parabolic and

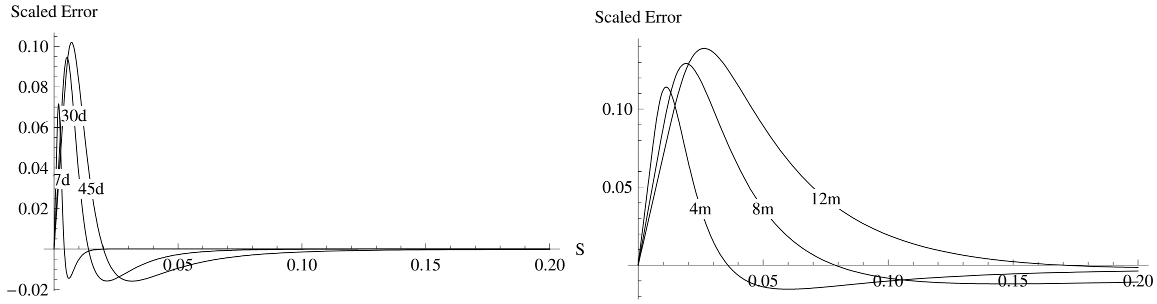

Figure 6. *Scaled errors of the parametrix approximation as a function of* $S \in [0, 0.2]$ *for* $T = 7, 15, 45$ *days (left) and* $T = 4, 8, 12$ *months (right).* $\alpha = \frac{3}{4}$, $\sigma = 30\%$, $r = 5\%$, *and* $K = 1$.

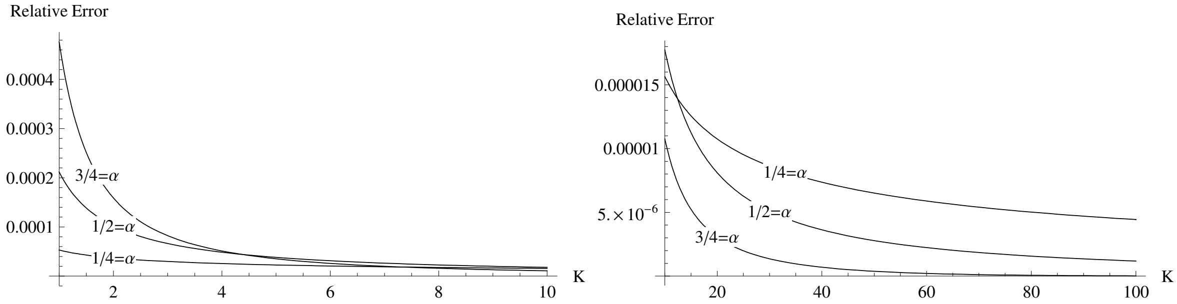

Figure 7. *Relative errors as a function of* $K \in [1, 10]$ *(left) and* $K \in [10, 100]$ *(right) of the at-the-money prices in the parametrix approximation. Here* $T = \frac{1}{2}$, $\sigma = 30\%$, $r = 5\%$, *and* $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

degenerates at the boundary. In particular, in this case, the price process reaches the origin with positive probability and this can be difficultly captured by a nondegenerate diffusion. However, as reported in Figure 6, this fact seems to produce relevant errors only for extremely out-of-the-money options; that is, $S \sim \frac{1}{100}$ when $K = 1$.

Finally, we examine the accuracy of the parametrix approximation as the strike K varies or, equivalently, by Remark 4.1, the volatility σ varies. Figure 7 shows the *relative errors*, defined as

$$\frac{C^{HW} - C^{P}}{C^{HW}},$$

where C^P, C^{HW} are, respectively, the call prices given by the parametrix expansion and by the Hagan–Woodward formula. Here we consider at-the-money options prices as functions of $K \in [10, 100]$. Moreover, $T = \frac{1}{2}$, $\sigma = 30\%$, $r = 5\%$, and the values of $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ are considered. Experiments with other choices of the parameters give essentially the same results and show that the parametrix prices basically coincide with the Hagan–Woodward prices for K large (or σ small): in this case the approximation for $\alpha = \frac{3}{4}$ is even better than it is for $\alpha = \frac{1}{4}$.

Next we consider the matched asymptotic expansion (MAE) proposed by Howison [21]. In this case we can directly employ the approximation formula in [21, page 392] to compute option prices. We also consider the recent paper by Svoboda-Greenwood [32], where another approximated formula for CEV prices is obtained by performing a small time expansion of

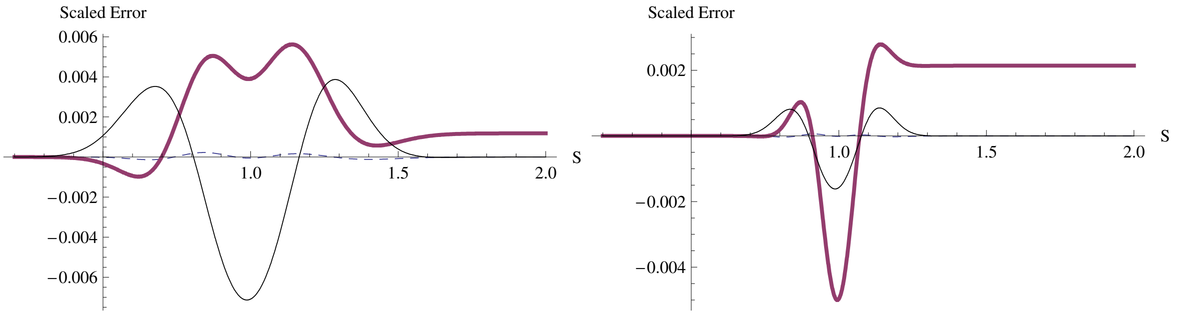

Figure 8. *Scaled errors of parametrix (dashed line), MAE (thick line), and Svoboda-Greenwood (continuous line) as a function of* S*, with* $\alpha = \frac{1}{4}$*; at-the-money volatility is* $\sigma = 30\%$ *(left) and* $\sigma = 15\%$ *(right). Moreover,* $T = 0.25$*,* $r = 5\%$*, and* $K = 1$*.*

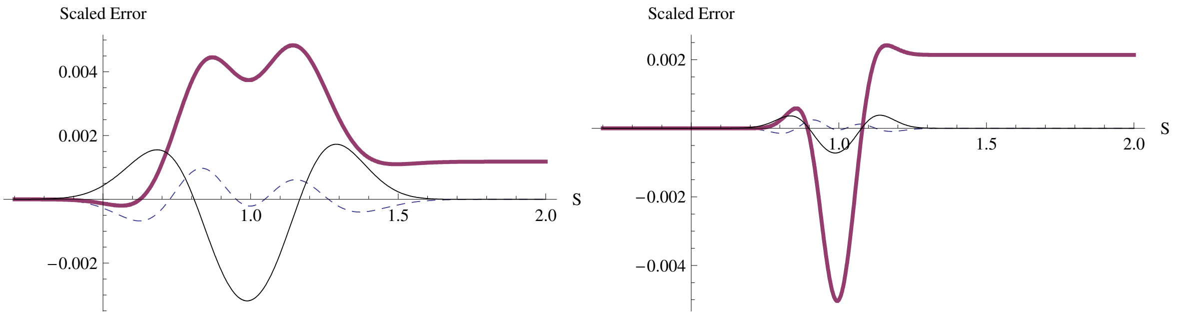

Figure 9. *Scaled errors of parametrix (dashed line), MAE (thick line), and Svoboda-Greenwood (continuous line) as a function of* S*, with* $\alpha = \frac{1}{2}$*; at-the-money volatility is* $\sigma = 30\%$ *(left) and* $\sigma = 15\%$ *(right). Moreover,* $T = 0.25$*,* $r = 5\%$*, and* $K = 1$*.*

the option prices around the forward-at-the-money value of the underlying. More precisely, we consider the SDE

$$dY_t = \gamma(t) f(Y_t) dW_t,$$

which corresponds to (4.1) after the transformation $Y_t = e^{-rt} S_t$, with $\gamma(t) = \sigma e^{-r\alpha t}$ and $f(Y) = Y^{1-\alpha}$. Actually, in [32], only the case of *time independent* $\gamma = \gamma(t)$ is considered. However, by using the same technique we generalize the result in [32, section 2.2] and obtain the following formula for the approximated call option price with strike K and maturity T:

$$C(S,T) \sim (S - y_0)\, \Phi\left(\frac{S - y_0}{y_0^{1-\alpha}\sigma_0}\right) + \left(\left(1 + \frac{(S - y_0)^{1-\alpha}}{2y_0}\right) y_0^{1-\alpha}\sigma_0 + \frac{(1-\alpha)^2}{4}\frac{(S - y_0)^4}{\sigma_0 y_0^{3-\alpha}}\right.$$
$$\left. + \frac{1}{6}(S - y_0)^2 \sigma_0 y_0^{-1-\alpha}(1 + \alpha(2\alpha - 3)) + \frac{1}{12}\sigma_0^3 y_0^{1-3\alpha}(1-\alpha)(9\alpha - 8)\right) \Phi'\left(\frac{S - y_0}{y_0^{1-\alpha}\sigma_0}\right),$$

where $y_0 = Ke^{-rt}$, $\sigma_0 = \sigma\sqrt{\frac{1-e^{-2r\alpha T}}{2r\alpha}}$, and Φ is the standard normal cumulative density function in (3.11).

In Figures 8, 9, and 10 we compare the scaled errors of the parametrix (dashed line), MAE (thick line), and Svoboda-Greenwood (continuous line) approximations. It turns out

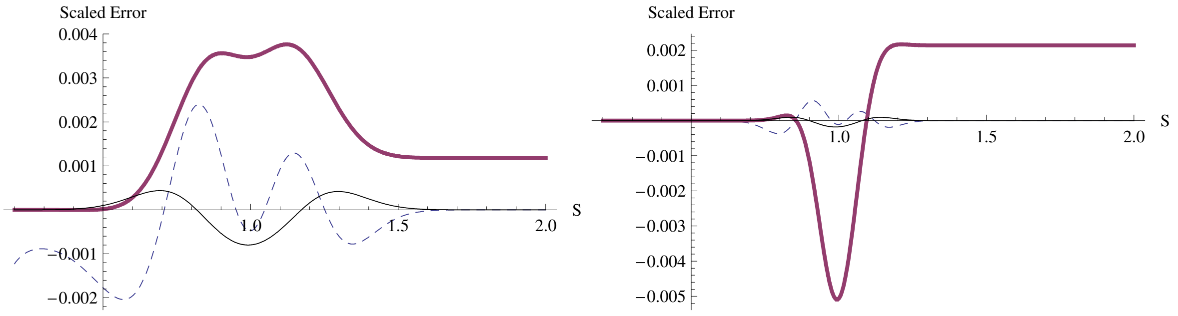

Figure 10. *Scaled errors of parametrix (dashed line), MAE (thick line), and Svoboda-Greenwood (continuous line) as a function of S with $\alpha = \frac{3}{4}$; at-the-money volatility is $\sigma = 30\%$ (left) and $\sigma = 15\%$ (right). Moreover, $T = 0.25$, $r = 5\%$, and $K = 1$.*

that, at least for $\alpha \le \frac{2}{3}$, the parametrix gives the best results: this is confirmed also by other experiments that we do not report here. For $\alpha = \frac{3}{4}$, parametrix and Svoboda-Greenwood errors are of the same order even if the latter seems slightly better. In general the MAE is not competitive with the other two approximations; this is particularly evident when σ is small or K is large. We also remark that the MAE is not correct asymptotically for large S (see also Figure 3 in [21]).

4.2. Parabolic and hyperbolic LV models. We consider two specifications of the volatility function in the general local volatility (LV) model

$$dS_t = rS_t dt + \sigma(S_t, t) S_t dW_t,$$

namely, the *quadratic LV*

$$\sigma(S,t) = \sigma_0 \min\left\{2, \sqrt{1 + (S-\beta)^2}\right\}$$

and the *hyperbolic LV*, as defined in [24],

$$\sigma(S,t) = \sigma_0 \left(\frac{1-\beta+\beta^2}{\beta} S + \frac{\beta-1}{\beta} \left(\sqrt{S^2 + \beta^2(1-S)^2} - \beta \right) \right),$$

where σ_0 and β are suitable parameters.

We remark that, if r is strictly positive, then *both models cannot be reduced in the form* (4.3) where the coefficient A is independent of time. Consequently in this case the Hagan–Woodward and Svoboda-Greenwood approximations do not apply. Therefore, we compare the parametrix with an accurate MC method.

In Figure 11 the black line represents the parametrix call price minus an MC call price, scaled to the at-the-money price, in the quadratic LV model (left) and hyperbolic LV model (right). We set $T = 0.25$, $r = 5\%$, and $K = 1$. Moreover, in the quadratic LV, we put $\beta = 1$ and $\sigma_0 = 20\%$ so that in this model the volatility varies from the minimum 20% at-the-money to the maximum 40%. In the hyperbolic LV, we consider the typical values (cf. [24]) $\sigma = 20\%$ and $\beta = \frac{1}{2}$.

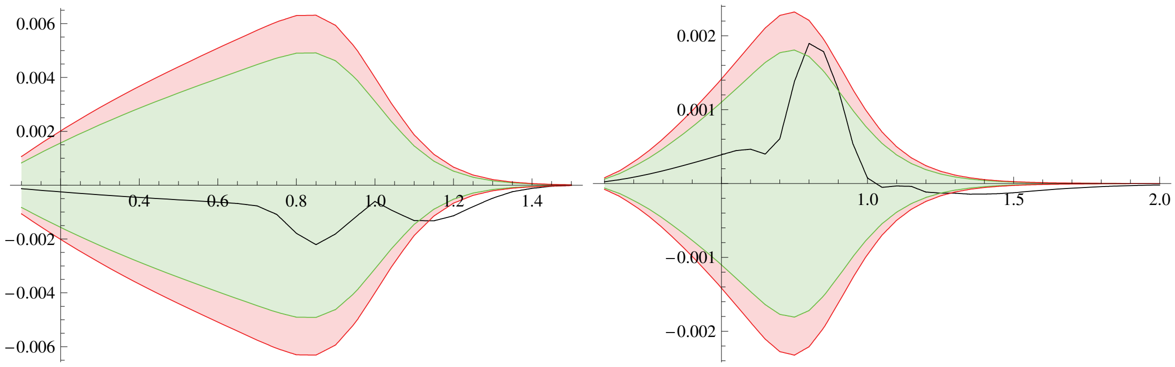

Figure 11. *The black line represents the difference, scaled to the at-the-money price, between the parametrix call price and an MC call price in the quadratic (left) and hyperbolic (right) local volatility models. The* 95% *and* 99% *MC confidence regions are marked, respectively, by green and red. A* 100*-step Euler discretization of the SDEs and an MC with* 500.000 *simulations have been used. Moreover,* $T = 0.25$, $r = 5\%$, *and* $K = 1$.

In the experiments, a 100-step Euler discretization of the SDEs and an MC with 500.000 simulations have been used. As before, we marked by green (respectively, red) the confidence regions where prices differ no more than 2 (respectively, 2.57) standard deviations from the simulated MC prices; in other words, the true price (scaled to at-the-money price) belongs to the green (red) region with probability 95% (99%).

Since in Figure 11 some errors appear corresponding to in-the-money options, for a more comprehensive comparison, we report in Table 1 the MC and parametrix prices (not scaled) for $S \in \{1.2; 1.3; 1.4; 1.5; 1.6\}$.

Table 1

MC and parametrix call prices in quadratic and hyperbolic LV models with $\sigma_0 = 20\%$, $T = 0.25$, $r = 5\%$, *and* $K = 1$. *Moreover,* $\beta = 1$ *in the quadratic LV and* $\beta = \frac{1}{2}$ *in the hyperbolic LV.*

	Call prices			
	Quadratic LV		Hyperbolic LV	
	MC	Parametrix	MC	Parametrix
$S = 1.2$	0.21352	0.213547	0.318564	0.318545
$S = 1.3$	0.312533	0.312535	0.417773	0.417748
$S = 1.4$	0.412432	0.412433	0.517571	0.517548
$S = 1.5$	0.512423	0.512423	0.61752	0.617503
$S = 1.6$	0.612422	0.612422	0.717507	0.717495

Clearly, in view of an extensive use of the parametrix approximation, a deeper analysis of the performance for a wide range of parameters is in order. However, these preliminary results seem promising.

4.3. Hobson–Rogers model. The Hobson–Rogers model [20] was introduced as an extension of the local volatility. In this model the volatility is defined as a function of the whole trajectory of the underlying asset and not only in terms of the spot price. The model was further generalized to a more flexible path dependent volatility model by two of the authors [13]. The main feature is that it generally leads to a complete market. We refer the reader to [14] for an empirical analysis which shows the effectiveness of the model and compares the hedging performance with respect to standard stochastic volatility models.

We consider an average weight ψ that is a nonnegative, piecewise continuous, and integrable function on $]-\infty, T]$. We assume that ψ is strictly positive in $[0, T]$, and we set

$$\Psi(t) = \int_{-\infty}^{t} \psi(s)ds.$$

Then we define the average process as

$$A_t = \frac{1}{\Psi(t)} \int_{-\infty}^{t} \psi(s) Z_s ds, \qquad t \in]0, T],$$

where $Z_t = \log(e^{-rt} S_t)$ denotes the log-discounted price process. The standard Hobson–Rogers model corresponds to the specification $\psi(t) = e^{\lambda t}$ for some positive parameter λ. Then by the Itô formula we have

$$dA_t = \frac{\varphi(t)}{\Phi(t)} (Z_t - A_t)\, dt.$$

In a path dependent volatility model the log-price $Z_t = \log S_t$ has the dynamics

$$dZ_t = \mu(Z_t - A_t)dt + \nu(Z_t - A_t)dW_t,$$

where $\mu = \mu(\cdot)$ and $\nu = \nu(\cdot)$ are suitable functions. Then, by usual no-arbitrage arguments, we obtain the pricing operator

$$Lu = \frac{\nu^2(z - a)}{2} (\partial_{zz} u - \partial_z u) + \frac{\varphi(t)}{\Phi(t)} (z - a)\partial_a u + \partial_t u, \qquad (t, z, a) \in]0, T[\times \mathbb{R}^2. \tag{4.5}$$

We remark that L *is not uniformly parabolic* (i.e., does not satisfy condition (5.3) in $\mathbb{R}^2$). However, if we assume that ν is smooth and bounded from above and below by positive constants, then L is a *hypoelliptic* operator belonging to the general class of Kolmogorov operators for which the parametrix method has been successfully employed in [11] to construct a fundamental solution. Therefore, it is possible to extend all the theoretical results of this paper to include L in (4.5).

As in the previous examples, we tested the parametrix against the MC method, and in the experiments a 100-step Euler discretization of the SDEs and 500.000 simulations have been used. In Figure 12 we compare the parametrix and MC approximations for typical values of the parameters, assuming different values for the initial average A_0—specifically $A_0 = Z_0$ in the left panel and $A_0 = Z_0 - \frac{1}{5}$ in the right panel. In Table 2 we also report some of the corresponding MC and parametrix prices.

The last experiment in Figure 13 is similar, but we consider different maturities, namely, 1 month in the left panel and 6 months in the right panel.

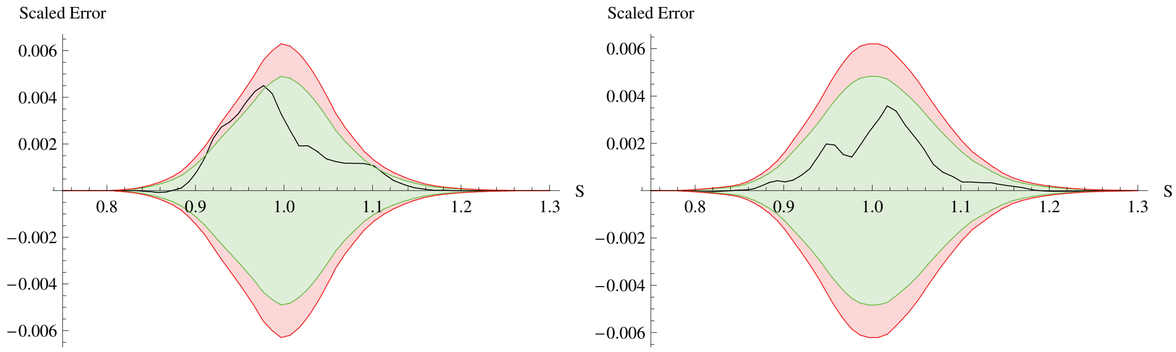

Figure 12. *The black line represents the difference, scaled to the at-the-money price, between the parametrix call price and an MC call price in the Hobson–Rogers model for* $S_0 \in \left[\frac{7}{10}, \frac{13}{10}\right]$ *and* $\psi(t) = e^t$, $\nu(x) = \sigma\sqrt{2x^2+1}$, $\sigma = 20\%$, $T = 0.25$, $r = 5\%$, *and* $K = 1$. *The* 95% *and* 99% *MC confidence regions are marked, respectively, by green and red. A* 100*-step Euler discretization of the SDEs and an MC with* 500.000 *simulations have been used. Moreover,* $A_0 = Z_0$ *(left) and* $A_0 = Z_0 - \frac{1}{5}$ *(right).*

Table 2

MC and parametrix call prices in the Hobson–Rogers model with $\psi(t) = e^t$, $\nu(x) = \sigma\sqrt{2x^2+1}$, $\sigma = 20\%$, $T = 0.25$, $r = 5\%$, *and* $K = 1$ *in the case* $A_0 = Z_0$ *(left) and* $A_0 = Z_0 - \frac{1}{5}$ *(right).*

	Call prices			
	$A_0 = Z_0$		$A_0 = Z_0 - \frac{1}{5}$	
	MC	Parametrix	MC	Parametrix
$S = 0.8$	0.000646705	0.000679473	0.000612704	0.000696547
$S = 0.9$	0.00820041	0.00819487	0.00904307	0.00926125
$S = 1$	0.0400307	0.0399139	0.0430224	0.0429977
$S = 1.1$	0.0998243	0.0998026	0.102566	0.102507
$S = 1.2$	0.168063	0.168108	0.16929	0.169278
$S = 1.3$	0.230935	0.230946	0.231299	0.231269

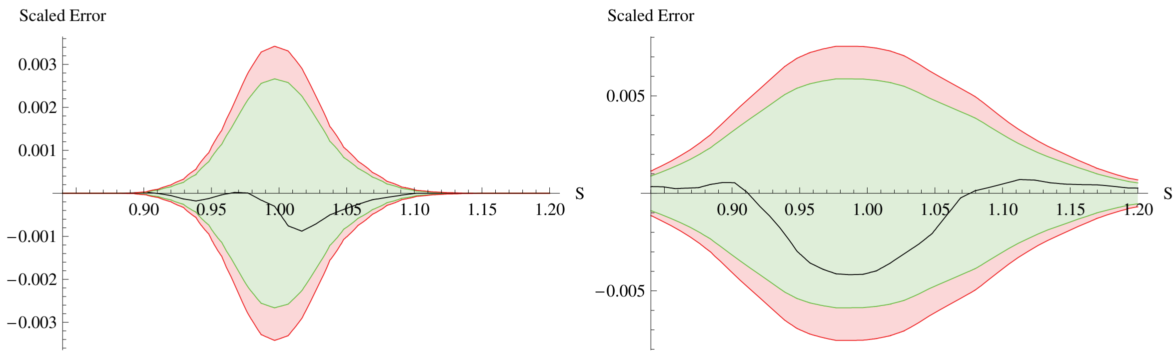

Figure 13. *The black line represents the difference, scaled to the at-the-money price, between the parametrix call price and an MC call price in the Hobson–Rogers model for* $S_0 \in \left[\frac{8}{10}, \frac{12}{10}\right]$ *and* $\psi(t) = e^t$, $\nu(x) = \sigma\sqrt{2x^2+1}$, $\sigma = 10\%$, $A_0 = Z_0$, $r = 5\%$, *and* $K = 1$. *The* 95% *and* 99% *MC confidence regions are marked, respectively, by green and red. A* 100*-step Euler discretization of the SDEs and an MC with* 500.000 *simulations have been used. Moreover,* $T = \frac{1}{12}$ *(left) and* $T = \frac{6}{12}$ *(right).*

5. Error bounds. In this section we present the parametrix expansion in its full generality and derive easily computable error estimates. We consider a parabolic differential equation in the form

$$(5.1)\qquad Lu := \sum_{i,j=1}^{N} a_{ij}(z)\partial_{x_i x_j} u + \sum_{i=1}^{N} b_i(z)\partial_{x_i} u + c(z)u - \partial_t u = 0, \qquad z = (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $A(z) = (a_{ij}(z))$ is a symmetric and positive definite matrix. Throughout this section we systematically denote by $z = (x,t)$ and $\zeta = (\xi,\tau)$ the points in $\mathbb{R}^{N+1}$. We also denote by $\lambda_1(z), \dots, \lambda_N(z)$ the eigenvalues of $A(z)$ and set[2]

$$m := \inf_{\substack{i=1,\dots,N \\ z\in\mathbb{R}^{N+1}}} \lambda_i(z), \qquad M := \sup_{\substack{i=1,\dots,N \\ z\in\mathbb{R}^{N+1}}} \lambda_i(z)\mu(z).$$

Our main hypotheses are the following.

[H1] m, M are positive and real numbers.

[H2] The coefficients of L are bounded functions and

$$(5.2)\qquad |a_{ij}(x,t) - a_{ij}(\xi,\tau)| \le \alpha\left(|x-\xi| + |t-\tau|^{\frac{1}{2}}\right), \qquad (x,t),(\xi,\tau) \in \mathbb{R}^{N+1},$$

for $i,j = 1,\dots,N$ and for some positive constant α.

As a consequence of **[H1]** we have the usual uniform parabolicity condition:

$$(5.3)\qquad m|\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}(z)\xi_i\xi_j \le M|\xi|^2, \qquad \xi \in \mathbb{R}^N,\ z \in \mathbb{R}^{N+1}.$$

5.1. Forward parametrix. In this section for brevity we consider only the classical case, corresponding to $\bar{b} = \bar{c} = 0$ in (2.4). Given $w \in \mathbb{R}^{N+1}$, we denote by $\Gamma_w(z;\zeta)$ the fundamental solution to the *frozen operator* L_w defined by

$$(5.4)\qquad L_w = \sum_{i,j=1}^{N} a_{ij}(w)\partial_{x_i x_j} - \partial_t.$$

We recall that $\Gamma_w(z;\zeta) = \Gamma_w(z-\zeta)$, where

$$(5.5)\qquad \Gamma_w(x,t) := \Gamma_w(x,t;0,0) = \frac{(4\pi t)^{-\frac{N}{2}}}{\sqrt{\det A(w)}} \exp\left(-\frac{\langle A^{-1}(w)x, x\rangle}{4t}\right), \qquad x \in \mathbb{R}^N,\ t > 0.$$

We define the *forward parametrix*

$$(5.6)\qquad Z(z;\zeta) = \Gamma_\zeta(z;\zeta).$$

[2]Equivalently we may use $m := \inf_{z\in\mathbb{R}^{N+1}} \mu(z)$ and $M := \sup_{z\in\mathbb{R}^{N+1}} \mu(z)$, where $\mu(z)$ is the Euclidean norm of $A(z)$ in $\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$ (and also equal to the Euclidean norm of $(\lambda_1(z),\dots,\lambda_N(z))$, the vector of the eigenvalues of $A(z)$). This gives less precise, but more easily computable, estimates.

We recall Notation 2.1 and remark explicitly that

$$L_{\zeta}^{(z)} Z(z;\zeta) = 0 \qquad \text{for } z \neq \zeta. \tag{5.7}$$

The following classical result (cf., for instance, Theorem 1.4 in [11]) states the existence of a fundamental solution Γ of operator L.

Theorem 5.1. *Assume hypotheses* **[H1]** *and* **[H2]**. *Then for every $\zeta \in \mathbb{R}^{N+1}$, the function defined by*

$$\Gamma(z;\zeta) = Z(z;\zeta) + \sum_{n=1}^{+\infty} \int_{\tau}^{t} \int_{\mathbb{R}^N} Z(z;w)(LZ)_n(w;\zeta)dw, \qquad t > \tau, \tag{5.8}$$

is a fundamental solution of L in (5.1). In (5.8) we have

$$(LZ)_1(w;\zeta) = L^{(w)} Z(w;\zeta), \qquad w = (y,s),$$
$$(LZ)_{n+1}(w;\zeta) = \int_{\tau}^{s} \int_{\mathbb{R}^N} L^{(w)} Z(w;z_0)(LZ)_n(z_0;\zeta)dz_0, \qquad n \geq 1,$$

and, for every $T > 0$, the series

$$\Phi(w;\zeta) := \sum_{n=1}^{+\infty} (LZ)_n(w;\zeta) \tag{5.9}$$

converges uniformly for $w \in \mathbb{R}^N \times]\tau, \tau + T[$.

Our first result is the following global estimate for the parametrix approximation truncated at the nth term.

Theorem 5.2. *Under the assumptions of Theorem 5.1, for every positive ε we have*

$$\begin{aligned} &\left| \Gamma(z;\zeta) - Z(z;\zeta) - \sum_{k=1}^{n-1} \int_{\tau}^{t} \int_{\mathbb{R}^N} Z(z;w)(LZ)_k(w;\zeta)dw \right| \\ &\qquad \leq \sqrt{\frac{2}{\pi}} \left(\frac{M+\varepsilon}{m} \right)^{\frac{N}{2}} f_n\left(\eta_{\varepsilon,T} \sqrt{2\pi(t-\tau)} \right) \Gamma^{M+\varepsilon}(z;\zeta) \end{aligned} \tag{5.10}$$

for $t \in]\tau, \tau + T[$, where $\Gamma^{M+\varepsilon}$ is the Gaussian density defined in (A.1)–(A.2), $\eta_{\varepsilon,T}$ is the constant defined in (A.7), and

$$f_n(\eta) = e^{\frac{\eta^2}{2}} (\eta + 1) \frac{\left(\frac{\eta^2}{2} \right)^{\left[\frac{n+1}{2} \right]}}{\left[\frac{n+1}{2} \right]!}, \tag{5.11}$$

with $[a]$ denoting the integer part of $a \in \mathbb{R}$.

Remark 5.3. *We remark explicitly that when $\eta = \eta_{\varepsilon,T} \sqrt{2\pi(t-\tau)} < 1$ in (5.11),*

$$f_n(\eta) \leq C \frac{\eta^n}{\left(\frac{n}{2} \right)!}, \qquad n \in \mathbb{N},$$

for some positive constant C so that the convergence of the parametrix approximation is extremely fast. This is the case, for instance, when $t-\tau \ll 1$, i.e., for short time to maturity. Also note that (5.10) *is a* global estimate *with respect to the spatial variables.*

Proof of Theorem 5.2. Theorem 5.2 is based on several results whose proofs are postponed to the appendix. We have

$$\left|\Gamma(z;\zeta) - Z(z;\zeta) - \sum_{k=1}^{n-1}\int_\tau^t \int_{\mathbb{R}^N} Z(z;w)(LZ)_k(w;\zeta)dw\right| \le \sum_{k=n}^{\infty}\int_\tau^t \int_{\mathbb{R}^N} Z(z;w)\,|(LZ)_k(w;\zeta)|\,dw$$

(by Lemma A.1, estimate (A.8), and the reproduction property)

$$\le \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}} \Gamma^{M+\varepsilon}(z;\zeta)\sum_{k=n}^{\infty}\int_\tau^t \frac{\Gamma_E\left(\frac{1}{2}\right)^k}{\Gamma_E\left(\frac{k}{2}\right)} \frac{\eta_{\varepsilon,T}^k}{(s-\tau)^{1-\frac{k}{2}}}ds$$

(using the properties of the Gamma function[3])

$$= \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}} \Gamma^{M+\varepsilon}(z;\zeta)\sqrt{\frac{2}{\pi}}\sum_{k=n}^{\infty} \frac{\left(\eta_{\varepsilon,T}\sqrt{2\pi(t-\tau)}\right)^k}{k!!}. \tag{5.12}$$

Then estimate (5.10) follows from some elementary computation. Indeed, if n is even, then $\left[\frac{n+1}{2}\right] = \frac{n}{2}$ and we have

$$\sum_{k=n}^{\infty}\frac{\eta^k}{k!!} = \sum_{k=\frac{n}{2}}^{\infty}\frac{\eta^{2k}}{(2k)!!} + \sum_{k=\frac{n}{2}+1}^{\infty}\frac{\eta^{2k-1}}{(2k-1)!!} \le \sum_{k=\frac{n}{2}}^{\infty}\frac{\eta^{2k}}{(2k)!!} + \sum_{k=\frac{n}{2}+1}^{\infty}\frac{\eta^{2k-1}}{(2k-2)!!}$$

(since $(2k)!! = 2^k k!$)

$$\le \sum_{k=\frac{n}{2}}^{\infty}\frac{1}{k!}\left(\frac{\eta^2}{2}\right)^k + \sum_{k=\frac{n}{2}}^{\infty}\frac{\eta^{2k+1}}{2^k k!} = f_n(\eta),$$

with f_n as in (5.11) and using the fact that

$$\sum_{k=n}^{\infty}\frac{\eta^k}{k!} = \frac{e^\eta \eta^n}{n!}.$$

The case of n odd can be treated analogously and is omitted. ■

From Theorem 5.2 we deduce the following forward parametrix expansion for solutions to the Cauchy problem for L.

[3]Recall that

$$\frac{\Gamma_E\left(\frac{1}{2}\right)^k}{\Gamma_E\left(\frac{k}{2}\right)} = \frac{(2\pi)^{\frac{k-1}{2}}}{(k-2)!!},$$

where $n!!$ is the double factorial defined by $n!! = 2\cdot 4\cdot 6\cdots n$ if n is even and $n!! = 1\cdot 3\cdot 5\cdots n$ if n is odd.

Theorem 5.4. *The solution to the Cauchy problem*

$$\begin{cases} Lu(x,t) = 0, & x \in \mathbb{R}^N, \ t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^N, \end{cases} \tag{5.13}$$

has an expansion of the form (2.17)–(2.18)–(2.19).

Proof. For simplicity let us consider only the one-dimensional case. By formulae (2.3) and (5.8) we have

$$u(z) = \int_{\mathbb{R}} Z(z;\xi,0)\varphi(\xi)d\xi + \sum_{n=1}^{+\infty} \int_{\mathbb{R}} \left(\int_{\tau}^{t} \int_{\mathbb{R}} Z(z;w)(LZ)_n(w;\xi,0)dw \right) \varphi(\xi)d\xi.$$

Using the expression of $(LZ)_1$ in (5.1), we have

$$u_1(z) = \int_{\mathbb{R}} \varphi(\xi) \int_0^t \int_{\mathbb{R}} Z(z;z_0)LZ(z_0;\xi,0)dz_0 d\xi = \int_0^t \int_{\mathbb{R}} Z(z;z_0)L \underbrace{\int_{\mathbb{R}} \varphi(\xi)Z(z_0;\xi,0)d\xi}_{=u_0(z_0)} dz_0.$$

Moreover,

$$u_2(z) = \int_{\mathbb{R}} \varphi(\xi) \int_0^t \int_{\mathbb{R}} Z(z;z_1) \int_0^{t_1} \int_{\mathbb{R}} LZ(z_1;z_0)LZ(z_0;\xi,0)dz_0 dz_1 d\xi$$

(changing the order of integration)

$$= \int_0^t \int_{\mathbb{R}} Z(z;z_1) \int_0^{t_1} \int_{\mathbb{R}} LZ(z_1;z_0)L \underbrace{\int_{\mathbb{R}} \varphi(\xi)Z(z_0;\xi,0)d\xi}_{=u_0(z_0)} dz_0 dz_1$$

$$= \int_0^t \int_{\mathbb{R}} Z(z;z_1) \Bigg(L \underbrace{\int_0^{t_1} \int_{\mathbb{R}} Z(z_1;z_0)Lu_0(z_0)dz_0}_{=u_1(z_1)} + Lu_0(z_1) \Bigg) dz_1,$$

and this proves (2.19) for $n = 2$. The general case can be proved by induction. ■

As a byproduct of the parametrix method, we also obtain the following upper Gaussian estimate of the fundamental solution.

Corollary 5.5. *For every $\varepsilon, T > 0$, we have*

$$\Gamma(z;\zeta) \le \left(\frac{M+\varepsilon}{m} \right)^{\frac{N}{2}} \left(1 + \eta_{\varepsilon,T} \sqrt{2\pi(t-\tau)} \right) e^{\pi(t-\tau)\eta_{\varepsilon,T}^2} \, \Gamma^{M+\varepsilon}(z;\zeta)$$

for $z, \zeta \in \mathbb{R}^{N+1}$, $t \in]\tau, \tau+T[$, where $\Gamma^{M+\varepsilon}$ is the Gaussian density defined in (A.1)–(A.2) *and $\eta_{\varepsilon,T}$ is the constant in* (A.7).

Proof. By Theorem 5.2 we have

$$\Gamma(z;\zeta) = Z(z;\zeta) + \sum_{k=1}^{\infty} \int_{\tau}^{t} \int_{\mathbb{R}^N} Z(z;w)(LZ)_k(w;\zeta)dw;$$

therefore, as in (5.12), we get

$$\Gamma(z;\zeta) \le \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}} \Gamma^{M+\varepsilon}(z;\zeta) \sum_{k=0}^{\infty} \frac{\left(\eta_{\varepsilon,T}\sqrt{2\pi(t-\tau)}\right)^k}{k!!},$$

and the thesis follows since

$$\sum_{k=0}^{\infty} \frac{\eta^k}{k!!} \le (1+\eta)e^{\frac{\eta^2}{2}}$$

for $\eta > 0$. ■

5.2. Backward parametrix. We assume the following additional hypothesis, which allows us to introduce the adjoint operator of L.

[H3] The derivatives $\partial_{x_i} a_{ij}, \partial_{x_i x_j} a_{ij}, \partial_{x_i} b_i$ are bounded functions.

We define the adjoint operator $\widetilde{L}$ of L as usual:

$$\widetilde{L}u = \sum_{i,j=1}^{N} a_{ij}\partial_{x_i x_j} u + \sum_{i=1}^{N} \widetilde{b}_i \partial_{x_i} u + \widetilde{c}u + \partial_t u, \tag{5.14}$$

where

$$\widetilde{b}_i = -b_i + 2\sum_{j=1}^{N} \partial_{x_i} a_{ij}, \qquad \widetilde{c} = c + \sum_{i,j=1}^{N} \partial_{x_i x_j} a_{ij} - \sum_{i=1}^{N} \partial_{x_i} b_i. \tag{5.15}$$

Thus we have

$$\int_{\mathbb{R}^{N+1}} \varphi L\psi = \int_{\mathbb{R}^{N+1}} \psi \widetilde{L}\varphi, \qquad \varphi, \psi \in C_0^\infty(\mathbb{R}^{N+1}),$$

and the following classical result holds (cf., for instance, [17, Chap. 1, Theor. 15]).

Theorem 5.6. *There exists a fundamental solution $\widetilde{\Gamma}$ of $\widetilde{L}$, and we have*

$$\Gamma(z;\zeta) = \widetilde{\Gamma}(\zeta;z), \qquad z,\zeta \in \mathbb{R}^{N+1}, \ z \ne \zeta. \tag{5.16}$$

For fixed $w \in \mathbb{R}^{N+1}$, we define the frozen operator

$$\widetilde{L}_w^{(\zeta)} = \sum_{i,j=1}^{N} a_{ij}(w)\partial_{\xi_i \xi_j} + \partial_\tau$$

and denote by $P(z;\zeta)$ the *backward parametrix* defined as the fundamental solution of $\widetilde{L}_w^{(\zeta)}$ with $w = z$, or, more precisely,

$$P(z;\zeta) = \widetilde{\Gamma}_z(\zeta;z) = \Gamma_z(z-\zeta) \tag{5.17}$$

for Γ_z as in (5.5). Analogously to (5.7), we have

$$\widetilde{L}_z^{(\zeta)} P(z;\zeta) = 0 \qquad \text{for } z \ne \zeta.$$

Our main result reads as follows.

Theorem 5.7. *Assume hypotheses* **[H1]**, **[H2]**, *and* **[H3]**. *Then, for every* $\zeta \in \mathbb{R}^{N+1}$, *the following expansion of the fundamental solution* Γ *holds:*

$$\Gamma(z;\zeta) = P(z;\zeta) + \int_\tau^t \int_{\mathbb{R}^N} P(z;w)\Psi(w;\zeta)dw, \qquad t > \tau, \tag{5.18}$$

where

$$\Psi(z;\zeta) = \sum_{k=1}^{+\infty} (LP)_k(z;\zeta), \tag{5.19}$$

with

$$(LP)_1(z;\zeta) = L^{(z)}P(z,\zeta),$$
$$(LP)_{k+1}(z;\zeta) = \int_\tau^t \int_{\mathbb{R}^N} L^{(z)}Z(z;w)(LP)_k(w;\zeta)dw,$$

and, for every $T > 0$, *the series in* (5.9) *converges uniformly in the strip* $\mathbb{R}^N \times]\tau, \tau + T[$. *Moreover, for every positive* ε, *we have the following estimate for the approximation truncated at the* n*th term:*

$$\begin{aligned} &\left| \Gamma(z;\zeta) - P(z;\zeta) - \sum_{k=1}^{n-1} \int_\tau^t \int_{\mathbb{R}^N} P(z;w)(LP)_k(w;\zeta)dw \right| \\ &\le \sqrt{\frac{2}{\pi}} \left(\frac{M+\varepsilon}{m} \right)^{\frac{N}{2}} f_n\left(\widetilde{\eta}_{\varepsilon,T}\sqrt{2\pi(t-\tau)} \right) \Gamma^{M+\varepsilon}(z;\zeta) \end{aligned} \tag{5.20}$$

for $t \in]\tau, \tau + T[$, *where* $\widetilde{\eta}_{\varepsilon,T}$ *is defined in* (A.10) *and* f_n *in* (5.11). *As a consequence, the solution to the Cauchy problem* (5.13) *has an expansion of the form* (2.17)–(2.20)–(2.21).

Proof. Proceeding as in the forward case, one can prove that

$$\Gamma(z;\zeta) = \widetilde{\Gamma}(\zeta;z) = \widetilde{\Gamma}_z(\zeta;z) + \int_\tau^t \int_{\mathbb{R}^N} \widetilde{\Gamma}_w(\zeta;w)\widetilde{\Phi}(w;z)dw, \qquad t > \tau, \tag{5.21}$$

where

$$\widetilde{\Phi}(\zeta;z) = \sum_{k=1}^{+\infty} I_k(\zeta;z), \tag{5.22}$$

with

$$I_1(\zeta;z) = \widetilde{L}^{(\zeta)}\widetilde{\Gamma}_z(\zeta;z),$$
$$I_{k+1}(\zeta;z) = \int_\tau^t \int_{\mathbb{R}^N} \widetilde{L}^{(\zeta)}\widetilde{\Gamma}_w(\zeta;w)I_k(w;z)dw,$$

and the series converges uniformly on the strips. Moreover, error estimate (5.20) holds true. In order to conclude the proof, it suffices to invoke Theorem 5.6 to prove that the terms of the expansions (5.18)–(5.19) and (5.21)–(5.22) coincide, that is,

$$\int_\tau^t \int_{\mathbb{R}^N} P(z;w)(LP)_k(w;\zeta)dw = \int_\tau^t \int_{\mathbb{R}^N} \widetilde{\Gamma}_w(\zeta;w)I_k(w;z)dw \tag{5.23}$$

for every $k \in \mathbb{N}$.

For $k = 1$, recalling (5.17), we have

$$\int_\tau^t \int_{\mathbb{R}^N} \widetilde{\Gamma}_w(\zeta;w)I_1(w;z)dw = \int_\tau^t \int_{\mathbb{R}^N} P(w;\zeta)\widetilde{L}^{(w)}P(z;w)dw,$$

so that the thesis follows immediately by integrating by parts since we have no contribution at borders. Indeed, denoting $w = (y,s)$, *formally* we have

$$\int_\tau^t \int_{\mathbb{R}^N} \Gamma_w(w;\zeta)\partial_s\Gamma_z(z;w)dw = \bar{I} - \int_\tau^t \int_{\mathbb{R}^N} \partial_s\Gamma_w(w;\zeta)\Gamma_z(z;w)dw,$$

where

$$\bar{I} = \int_{\mathbb{R}^N} \Gamma_{(y,t)}(y,t;\xi,\tau)\Gamma_{(x,t)}(x,t;y,t)dy - \int_{\mathbb{R}^N} \Gamma_{(y,\tau)}(y,\tau;\xi,\tau)\Gamma_{(x,t)}(x,t;y,\tau)dy = 0$$

since $\Gamma_{(x,t)}(x,t;y,t) = \delta_x(y)$ and $\Gamma_{(y,\tau)}(y,\tau;\xi,\tau) = \delta_\xi(y)$. On the other hand, the above argument can be made rigorous by performing the integration by parts on a thinner strip $S_{\tau+\delta,t-\delta}$ and then applying the dominated convergence theorem as $\delta \to 0^+$ combined with the summability estimate (A.9).

For $k = 2$, we have

$$\begin{aligned}
&\int_\tau^t \int_{\mathbb{R}^N} \Gamma_{z_0}(z_0;\zeta) \int_{t_0}^t \int_{\mathbb{R}^N} \widetilde{L}^{(z_0)}\Gamma_{z_1}(z_1;z_0)\widetilde{L}^{(z_1)}\Gamma_z(z;z_1)dz_1dz_0 \\
&= \int_\tau^t \int_{\mathbb{R}^N} \Gamma_{z_0}(z_0;\zeta) \left(\widetilde{L}^{(z_0)} \int_{t_0}^t \int_{\mathbb{R}^N} \Gamma_{z_1}(z_1;z_0)\widetilde{L}^{(z_1)}\Gamma_z(z;z_1)dz_1 \right. \\
&\left. + \int_{\mathbb{R}^N} \Gamma_{(y,t_0)}(y,t_0;z_0)\widetilde{L}^{(y,t_0)}\Gamma_z(z;y,t_0)dy \right) dz_0 \equiv J_1 + J_2,
\end{aligned}$$

where, using again that $\Gamma_{(y,t_0)}(y,t_0;z_0) = \delta_{x_0}(y)$, we get

$$J_2 = \int_\tau^t \int_{\mathbb{R}^N} \Gamma_{z_0}(z_0;\zeta)\widetilde{L}^{(z_0)}\Gamma_z(z;z_0)dz_0$$

(proceeding as in the case $k = 1$)

$$= \int_\tau^t \int_{\mathbb{R}^N} L^{(z_0)}\Gamma_{z_0}(z_0;\zeta)\Gamma_z(z;z_0)dz_0;$$

on the other hand,

$$J_1 = \int_\tau^t \int_{\mathbb{R}^N} \Gamma_{z_0}(z_0;\zeta)\widetilde{L}^{(z_0)} \int_{t_0}^t \int_{\mathbb{R}^N} \Gamma_{z_1}(z_1;z_0)\widetilde{L}^{(z_1)}\Gamma_z(z;z_1)dz_1dz_0$$

(by parts as before)

$$= \int_\tau^t \int_{\mathbb{R}^N} L^{(z_0)}\Gamma_{z_0}(z_0;\zeta) \int_{t_0}^t \int_{\mathbb{R}^N} L^{(z_1)}\Gamma_{z_1}(z_1;z_0)\Gamma_z(z;z_1)dz_1dz_0$$
$$- \int_{\mathbb{R}^N} \Gamma_{(y,\tau)}(y,\tau;\xi,\tau) \int_\tau^t \int_{\mathbb{R}^N} L^{(z_1)}\Gamma_{z_1}(z_1;y,\tau)\Gamma_z(z;z_1)dz_1dy$$

(since $\Gamma_{(y,\tau)}(y,\tau;\xi,\tau) = \delta_\xi(y)$)

$$= \int_\tau^t \int_{\mathbb{R}^N} L^{(z_0)}\Gamma_{z_0}(z_0;\zeta) \int_{t_0}^t \int_{\mathbb{R}^N} L^{(z_1)}\Gamma_{z_1}(z_1;z_0)\Gamma_z(z;z_1)dz_1dz_0$$
$$- \int_\tau^t \int_{\mathbb{R}^N} L^{(z_1)}\Gamma_{z_1}(z_1;\zeta)\Gamma_z(z;z_1)dz_1.$$

Combining the expressions of J_1 and J_2, eventually we obtain

$$\int_\tau^t \int_{\mathbb{R}^N} \Gamma_{z_0}(z_0;\zeta) \int_{t_0}^t \int_{\mathbb{R}^N} \widetilde{L}^{(z_0)}\Gamma_{z_1}(z_1;z_0)\widetilde{L}^{(z_1)}\Gamma_z(z;z_1)dz_1dz_0$$
$$= \int_\tau^t \int_{\mathbb{R}^N} L^{(z_0)}P(z_0;\zeta) \int_{t_0}^t \int_{\mathbb{R}^N} L^{(z_1)}P(z_1;z_0)P(z;z_1)dz_1dz_0,$$

which concludes the proof. As before, the previous argument should be made rigorous by some approximating procedure. The general case can be achieved by induction. ∎

Appendix. We collect several lemmas that are preliminary to the proofs of Theorems 5.2 and 5.7. These lemmas are essentially estimates of Γ_w in (5.5) and its derivatives in terms of the fundamental solution of the heat equation.

Given a constant $\mu > 0$, we denote by Γ^μ the fundamental solution to the heat operator

$$\mu \sum_{i=1}^N \partial_{x_i x_i} - \partial_t. \tag{A.1}$$

Lemma A.1. *For every $z, \zeta, w \in \mathbb{R}^{N+1}$ with $z \neq \zeta$, we have*

$$\left(\frac{m}{M}\right)^{\frac{N}{2}} \Gamma^m(z;\zeta) \le \Gamma_w(z;\zeta) \le \left(\frac{M}{m}\right)^{\frac{N}{2}} \Gamma^M(z;\zeta).$$

Proof. We prove only the second inequality in the case $\zeta = 0$. Keeping in mind formula (5.5), we see that the thesis follows directly from condition (5.3). Indeed, we have

$$\Gamma_w(z) \le \frac{1}{(4\pi t m)^{\frac{N}{2}}} \exp\left(-\frac{|x|^2}{4tM}\right) = \left(\frac{M}{m}\right)^{\frac{N}{2}} \Gamma^M(z). \tag{A.2}$$ ∎

Lemma A.2. *For every $\varepsilon, \mu > 0$ and $n \in \mathbb{N} \cup \{0\}$ we have*

$$\left(\frac{|x|}{\sqrt{t}}\right)^n \Gamma^{\mu}(x,t) \le \left(\frac{n}{\varepsilon}\right)^{\frac{n}{2}} (\mu+\varepsilon)^n \left(\frac{\mu+\varepsilon}{\mu}\right)^{\frac{N}{2}} \Gamma^{\mu+\varepsilon}(x,t)$$

for any $x \in \mathbb{R}^N$ and $t > 0$.

Proof. Setting $a = \frac{|x|}{\sqrt{t}}$, we have

$$\left(\frac{|x|}{\sqrt{t}}\right)^n \Gamma^{\mu}(z,0) = a^n (4\pi\mu t)^{-\frac{N}{2}} \exp\left(-\frac{a^2}{4\mu}\right) \le (4\pi\mu t)^{-\frac{N}{2}} \exp\left(-\frac{a^2}{4(\mu+\varepsilon)}\right) \sup_{\mathbb{R}_+} G,$$

where

$$G(a) = a^n \exp\left(-\left(\frac{1}{4\mu} - \frac{1}{4(\mu+\varepsilon)}\right)a^2\right). \tag{A.3}$$

The thesis follows by a straightforward computation, since G attains a global maximum at $\bar{a} = \sqrt{\frac{2n\mu(\mu+\varepsilon)}{\varepsilon}}$ and

$$G(\bar{a}) = \left(\frac{2n\mu(\mu+\varepsilon)}{e\varepsilon}\right)^{\frac{n}{2}} \le \left(\frac{n}{\varepsilon}\right)^{\frac{n}{2}} (\mu+\varepsilon)^n. \qquad \blacksquare$$

Lemma A.3. *For every $\varepsilon > 0$ and $i, j = 1, \dots, N$ we have*

$$|\partial_{x_i} \Gamma_w(z;\zeta)| \le \frac{1}{2\sqrt{\varepsilon(t-\tau)}} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+1} \Gamma^{M+\varepsilon}(z;\zeta), \tag{A.4}$$

$$\left|\partial_{x_i x_j} \Gamma_w(z;\zeta)\right| \le \frac{1}{\varepsilon(t-\tau)} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+2} \Gamma^{M+\varepsilon}(z;\zeta) \tag{A.5}$$

for any $z, \zeta, w \in \mathbb{R}^{N+1}$ with $t > \tau$.

Proof. For the sake of simplicity, we prove the above estimates in the case $\zeta = 0$. We have

$$|\partial_{x_i} \Gamma_w(z)| = \frac{1}{2} \frac{\left|\left(A^{-1}(w)x\right)_i\right|}{t} \Gamma_w(z)$$

(by Lemma A.1)

$$\le \frac{1}{2m\sqrt{t}} \left(\frac{M}{m}\right)^{\frac{N}{2}} \frac{|x|}{\sqrt{t}} \Gamma^M(z),$$

and (A.5) follows by applying Lemma A.2 with $\mu = M$ and $n = 1$.

Moreover,

$$\left|\partial_{x_i x_j} \Gamma_w(z)\right| = \frac{1}{2t} \left|A^{-1}(w)_{ij} + \frac{1}{2t}\left(A^{-1}(w)x\right)_i \left(A^{-1}(w)x\right)_j\right| \Gamma_w(z) \le \frac{1}{2t}\left(\frac{1}{m} + \frac{|x|^2}{2m^2 t}\right)\Gamma_w(z),$$

and (A.4) easily follows by Lemmas A.1 and A.2 with $\mu = M$. $\blacksquare$

Lemma A.4. *For every positive ε and T, we have*

$$\left|L^{(z)}Z(z;\zeta)\right| \le \frac{\eta_{\varepsilon,T}}{\sqrt{t-\tau}}\Gamma^{M+\varepsilon}(z;\zeta), \qquad z,\zeta\in\mathbb{R}^{N+1},\ t\in\,]\tau,\tau+T[, \tag{A.6}$$

where

$$\begin{aligned}\eta_{\varepsilon,T} := \alpha N^2\left(\frac{2}{\varepsilon}\right)^{\frac{3}{2}}\left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+2}\left(M+\varepsilon+\sqrt{\frac{\varepsilon}{2}}\right)+\beta\frac{N}{2\sqrt{\varepsilon}}\left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+1}\\ +\gamma\left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}}\sqrt{T}\end{aligned} \tag{A.7}$$

and

$$\beta := \sup_{\substack{i=1,\dots,N\\ z\in\mathbb{R}^{N+1}}}|b_i(z)|, \qquad \gamma := \sup_{z\in\mathbb{R}^{N+1}}|c(z)|,$$

and α is the constant in (5.2).

Proof. For $t>\tau$, we have

$$|LZ(z;\zeta)| = |(L-L_\zeta)Z(z;\zeta)| \le I_1+I_2+I_3,$$

where

$$I_1 = \sum_{i,j=1}^{N}|a_{ij}(z)-a_{ij}(\zeta)|\left|\partial_{x_ix_j}Z(z;\zeta)\right|$$

(by (5.2))

$$\le \alpha N^2\left(|x-\xi|+\sqrt{t-\tau}\right)\max_{i,j}\left|\partial_{x_ix_j}Z(z;\zeta)\right|$$

(by Lemma A.3)

$$\le \frac{\alpha N^2}{\varepsilon}\left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+2}\left(1+\frac{|x-\xi|}{\sqrt{t-\tau}}\right)\Gamma^{M+\varepsilon}(z;\zeta)$$

(by Lemma A.2)

$$\begin{aligned}&\le \frac{\alpha N^2}{\varepsilon}\left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+2}\left(\frac{M+2\varepsilon}{M+\varepsilon}\right)^{\frac{N}{2}}\left(1+\frac{M+2\varepsilon}{\sqrt{\varepsilon}}\right)\Gamma^{M+2\varepsilon}(z;\zeta)\\ &\le \frac{\alpha N^2}{\varepsilon^{\frac{3}{2}}}\left(\frac{M+2\varepsilon}{m}\right)^{\frac{N}{2}+2}\left(M+2\varepsilon+\sqrt{\varepsilon}\right)\Gamma^{M+2\varepsilon}(z;\zeta).\end{aligned}$$

Moreover, by Lemma A.3, we have

$$I_2 = \sum_{i=1}^{N}|b_i(z)|\,|\partial_{x_i}Z(z;\zeta)| \le \beta\frac{N}{2\sqrt{\varepsilon(t-\tau)}}\left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+1}\Gamma^{M+\varepsilon}(z;\zeta);$$

finally, by Lemma A.1, we have

$$I_3 = |c(z)|Z(z;\zeta) \le \gamma\left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}}\Gamma^{M+\varepsilon}(z;\zeta). \qquad \blacksquare$$

Lemma A.5. *For every $\varepsilon > 0$ and $k \geq 1$ the following estimate for the term $(LZ)_k$ in (5.9) holds:*

$$|(LZ)_k(z;\zeta)| \leq \frac{\Gamma_E\left(\frac{1}{2}\right)^k}{\Gamma_E\left(\frac{k}{2}\right)} \frac{\eta^k_{\varepsilon,T}}{(t-\tau)^{1-\frac{k}{2}}} \Gamma^{M+\varepsilon}(z;\zeta), \qquad z,\zeta \in \mathbb{R}^{N+1},\ t > \tau, \tag{A.8}$$

where $\eta_{\varepsilon,T}$ is defined in (A.7) and Γ_E denotes Euler's Gamma function.

Proof. We prove (A.8) by induction on k. The case $k = 1$ was proved in Lemma A.4. Let us now assume that (A.8) holds for k and prove it for $k+1$. We have

$$|(LZ)_{k+1}(z;\zeta)| = \left| \int_\tau^t \int_{\mathbb{R}^N} L^{(z)} Z(z;w)(LZ)_k(w;\zeta) dw \right|$$

(by Lemma A.4, the inductive hypothesis, and denoting $(y,s) = w$)

$$\leq \eta^{k+1}_{\varepsilon,T} \frac{\Gamma_E\left(\frac{1}{2}\right)^k}{\Gamma_E\left(\frac{k}{2}\right)} \int_\tau^t \frac{1}{\sqrt{t-s}(s-\tau)^{1-\frac{k}{2}}} \int_{\mathbb{R}^N} \Gamma^{M+\varepsilon}(x,t;y,s)\Gamma^{M+\varepsilon}(y,s;\xi,\tau) dy ds$$

(by the reproduction property[4] for $\Gamma^{M+\varepsilon}$ and by the change of variable $s = (1-r)\tau + rt$)

$$= \frac{\eta^{k+1}_{\varepsilon,T}}{(t-\tau)^{1-\frac{k+1}{2}}} \frac{\Gamma_E\left(\frac{1}{2}\right)^k}{\Gamma_E\left(\frac{k}{2}\right)} \int_0^1 \frac{1}{r^{1-\frac{k}{2}}\sqrt{1-r}} dr\, \Gamma^{M+\varepsilon}(z;\zeta),$$

and the thesis follows by the known properties[5] of Euler's Gamma function. ∎

Finally, we recall Notation 2.1 and state the dual version of Lemmas A.3 and A.4. Proofs are omitted since they are analogous.

Lemma A.6. *For every $\varepsilon > 0$ and $i,j = 1,\dots,N$ we have*

$$|\partial_{\xi_i} \Gamma_w(z;\zeta)| \leq \frac{1}{2\sqrt{\varepsilon(t-\tau)}} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+1} \Gamma^{M+\varepsilon}(z;\zeta),$$

$$\left|\partial_{\xi_i\xi_j} \Gamma_w(z;\zeta)\right| \leq \frac{1}{\varepsilon(t-\tau)} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+2} \Gamma^{M+\varepsilon}(z;\zeta)$$

for any $z,\zeta,w \in \mathbb{R}^{N+1}$ with $t > \tau$.

Lemma A.7. *Under hypotheses* **[H1]**–**[H3]**, *for every positive ε, we have*

$$\left|\widetilde{L}^{(\zeta)} P(z;\zeta)\right| \leq \frac{\widetilde{\eta}_{\varepsilon,T}}{\sqrt{t-\tau}} \Gamma^{M+\varepsilon}(z;\zeta), \qquad z,\zeta \in \mathbb{R}^{N+1},\ t > \tau, \tag{A.9}$$

[4]For every $x,\xi \in \mathbb{R}^N$ and $\tau < s < t$, we have

$$\int_{\mathbb{R}^N} \Gamma^{M+\varepsilon}(z;y,s)\Gamma^{M+\varepsilon}(y,s;\zeta) dy = \Gamma^{M+\varepsilon}(z;\zeta).$$

[5]It holds that

$$\int_0^1 \frac{1}{r^{1-\frac{k}{2}}\sqrt{1-r}} dr = \frac{\Gamma_E\left(\frac{1}{2}\right)\Gamma_E\left(\frac{k}{2}\right)}{\Gamma_E\left(\frac{k+1}{2}\right)}.$$

where

$$\widetilde{\eta}_{\varepsilon,T} := \alpha N^2 \left(\frac{2}{\varepsilon}\right)^{\frac{3}{2}} \left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+2} \left(M+\varepsilon+\sqrt{\frac{\varepsilon}{2}}\right) + \widetilde{\beta}\frac{N}{2\sqrt{\varepsilon}}\left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}+1} + \widetilde{\gamma}\left(\frac{M+\varepsilon}{m}\right)^{\frac{N}{2}}\sqrt{T}, \tag{A.10}$$

where

$$\widetilde{\beta} := \sup_{\substack{i=1,\dots,N\\ z\in\mathbb{R}^{N+1}}} |\widetilde{b}_i(z)|, \qquad \widetilde{\gamma} := \sup_{z\in\mathbb{R}^{N+1}} |\widetilde{c}(z)|.$$

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