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Free boundary and optimal stopping problems for American Asian options

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Abstract We give a complete and self-contained proof of the existence of a strong solution to the free boundary and optimal stopping problems for pricing American path-dependent options. The framework is sufficiently general to include geometric Asian options with nonconstant volatility and recent path-dependent volatility models.

Keywords American option · Asian option · Free boundary problem · Optimal stopping

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JEL Classification C02

1 Introduction

In modern finance theory, the valuation of options with early exercise leads to optimal stopping problems which are equivalent to parabolic free boundary problems. Precisely, the price of an American option, with expiry date T and payoff function φ, is the solution to the optimal stopping problem

$$u(t,x)=\sup_{\tau\in\mathcal{T}_{t,T}} E\big[\varphi\big(X_\tau^{t,x}\big)\big], \tag{1.1}$$

where X is the diffusion starting from (t,x) that describes the dynamics of the underlying assets, and $\mathcal{T}_{t,T}$ denotes the set of all stopping times with values in $[t,T]$.

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Equivalently, the price is determined by the solution to the free boundary problem

$$\begin{cases} \max\{Lu, \varphi - u\} = 0 & \text{in }]0, T[\times \mathbb{R}^N, \\ u(T, \cdot) = \varphi(T, \cdot) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

where L is the Kolmogorov operator of X.

A rigorous theory of American options was first developed by Bensoussan [5], Karatzas [22], and Jaillet et al. [20] by using the classical results by van Moerbeke [32], Bensoussan and Lions [6], Kinderlehrer and Stampacchia [24], and Friedman [13] in the framework of parabolic PDEs. However, there are relevant kinds of American options, commonly traded in financial markets, whose modeling involves equations that are not uniformly parabolic and to which the classical theory does not apply. Two remarkable examples are given by average-rate (more commonly called Asian) options and recent path-dependent volatility models such as the stochastic volatility model by Hobson and Rogers [16].

While there are several papers on the valuation of Asian options with early exercise (for instance, Barraquand and Pudet [2], Barles [1], Hansen and Jørgensen [15], Meyer [28], Wu et al. [33], Fu and Wu [14], Jiang and Dai [21], Ben-Ameuret et al. [4], Marcozzi [27], Dai and Kwok [8], and Huang and Thulasiram [18]), most of these are devoted to numerical issues (i.e., the development of numerical techniques for pricing and determining the exercise boundary) by some means, assuming as established the existence and regularity of the solution to the free boundary or optimal stopping problem. To a certain extent, using the weak notion of viscosity solution, it is possible to obtain general existence results. Using the same techniques, it is also possible to prove the uniform convergence of numerical schemes (cf. Barles [1] and Jiang and Dai [21]), though without having adequate control over the errors and the rate of convergence. As a matter of fact, if L is a uniformly parabolic operator, problem (1.2) is classically solved within the natural framework of the theory of Sobolev spaces and admits a *strong solution*. Indeed, it is well known that even in the Black–Scholes setting, a free boundary problem generally does not have a classical smooth solution and the regularity in some suitable Sobolev space is optimal.

In this paper, we consider a quite general financial model, possibly corresponding to a degenerate PDE, that includes Asian options and path-dependent volatility models as particular cases. We introduce a suitable functional setting and in this framework prove the existence and uniqueness of a strong solution u to the free boundary and optimal stopping problems. The regularity properties of u are precisely stated in Sect. 4: roughly speaking, u has weak second-order derivatives in L^p_{loc} for any $p \geq 1$ and locally Hölder-continuous first-order derivatives.

The outline for this paper is as follows. In Sect. 2, we briefly recall some known results for American Asian options in the Black–Scholes setting. In Sect. 3, we state the assumptions and examine some examples. Section 4 contains our main results regarding the existence of a strong solution to problem (1.2) and a Feynman–Kač-type theorem connecting the free boundary and optimal stopping problems. In the Appendix, we review some basic facts about Kolmogorov PDEs associated with linear SDEs and describe a functional setting suitable for our study.

2 American Asian options in the Black–Scholes model

Asian options are averaging options whose terminal payoff depends on some form of averaging prices of the underlying asset over a part or the whole life of the option. Let r denote the constant interest rate and assume that the price of the underlying asset is modeled by a geometric Brownian motion

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t.$$

We denote by M the path-dependent variable: For an *arithmetic Asian option,* we set

$$M_t = \frac{A_t}{t}, \qquad A_t = \int_0^t S_s\,ds, \quad \text{with } M_0 = S_0, \tag{2.1}$$

and for a *geometric Asian option,*

$$M_t = \exp\left(\frac{G_t}{t}\right), \qquad G_t = \int_0^t \log S_s\,ds, \quad \text{with } M_0 = S_0. \tag{2.2}$$

Then the payoff function of a *fixed-strike* call Asian option is given by

$$\varphi(t, S_t, M_t) = (M_t - K)^+, \tag{2.3}$$

where K is the strike price; for a *floating-strike* call Asian option, the payoff is

$$\varphi(t, S_t, M_t) = (S_t - M_t)^+. \tag{2.4}$$

Arithmetic and geometric Asian options are commonly traded in specific markets (for instance, currency and commodity markets, cf. [15]) and were introduced to avoid the well-known problems of European options that can be subject to price manipulations of the underlying asset near the maturity.

For the arithmetic average (2.1), by the usual no-arbitrage arguments we obtain the pricing PDE

$$\frac{\sigma^2 S^2}{2}\partial_{SS}u + rS\partial_S u + S\partial_A u + \partial_t u - ru = 0 \tag{2.5}$$

for the option price process $u(t, S_t, A_t)$. As usual, state augmentation converts the path-dependent problem for the Asian option into an equivalent path-independent and Markovian problem. However, increasing the dimension generally leads to degenerate PDEs which are not uniformly parabolic. This is the case for (2.5) which only contains the second-order partial derivatives with respect to one of the two “spatial variables.”

On the other hand, it is known that, in the particular Black–Scholes setting and for specific payoff functions, it is possible to reduce the study of an Asian option to a PDE with only one state variable. More precisely, for the floating-strike Asian option, Ingersoll [19] proposes the change of variable $x = \frac{A}{S}$. It is straightforward to show that $u = u(t, S, A)$ solves the Cauchy problem for (2.5) with final condition

$$u(T, S_T, A_T) = \left(S_T - \frac{A_T}{T}\right)^+$$

if and only if the function $U = U(t, x)$ defined by $u(t, S, A) = SU(t, \frac{A}{S})$ solves the bidimensional parabolic Cauchy problem

$$\begin{cases} \frac{\sigma^2 x^2}{2} \partial_{xx} U + (1 - rx)\partial_x U + \partial_t U = 0, & t \in]0, T[,\ x > 0, \\ U(T, x) = (1 - \frac{x}{T})^+, & x > 0. \end{cases}$$

A similar result holds for the corresponding free boundary problem in the early exercise case.

Analogously, for the fixed-strike Asian option, Rogers and Shi [30] implicitly propose the change of variable $x = \frac{A/T - K}{S}$. In this case, u solves the Cauchy problem for (2.5) with final condition

$$u(T, S_T, A_T) = \left(\frac{A_T}{T} - K \right)^+$$

if and only if the function $U = U(t, x)$ defined by

$$u(t, S, A) = SU\left(t, \frac{\frac{A}{T} - K}{S} \right)$$

solves the degenerate Cauchy problem in $\mathbb{R}^2$

$$\begin{cases} \frac{\sigma^2 x^2}{2} \partial_{xx} u + (\frac{1}{T} - rx)\partial_x u + \partial_t u = 0, & t \in]0, T[,\ x \in \mathbb{R}, \\ u(T, x) = x^+, & x \in \mathbb{R}. \end{cases} \tag{2.6}$$

Note that the PDE in (2.6) is not parabolic and degenerates at $x = 0$.

We emphasize that reduction to a lower-dimensional PDE is possible only under rather restrictive hypotheses; namely, assuming that S is a geometric Brownian motion and for the specific payoff functions in (2.3) and (2.4). More generally, reduction is possible if the payoff function has suitably homogeneity properties, for instance, $\varphi(t, S, M) = S\varphi(t, 1, M/S)$. The idea that degenerate diffusions can be reduced to lower-dimensional nondegenerate diffusions on a sub-manifold of the underlying asset space was carried on by Barraquand and Pudet [2].

For geometric Asian options, the pricing PDE for the value function $u = u(t, S, G)$ reads

$$\frac{\sigma^2 S^2}{2} \partial_{SS} u + rS\partial_S u + \log(S)\partial_G u + \partial_t u - ru = 0, \quad t \in]0, T[,\ S, G > 0. \tag{2.7}$$

By the change of variables (cf. [3])

$$f(t, x, y) = e^{x \frac{2r - \sigma^2}{2\sqrt{2}\sigma} + t (\frac{2r + \sigma^2}{2\sqrt{2}\sigma})^2} u\left(t, e^{\frac{\sigma x}{\sqrt{2}}}, \frac{\sigma y}{\sqrt{2}} \right),$$

u solves (2.7) if and only if f is a solution to

$$\partial_{xx} f + x\partial_y f + \partial_t f = 0, \quad t \in]0, T[,\ (x, y) \in \mathbb{R}^2. \tag{2.8}$$

Even in this case, it seems that a reduction to a one-dimensional problem is not generally possible. On the other hand, in the next section, we show that the process

(S_t, G_t) is nondegenerate and has an explicit strictly positive transition density that is the fundamental solution of (2.7).

3 Assumptions, preliminaries, and examples

We consider a quite general Markov-type financial model where the dynamics of the N state variables is driven by the stochastic differential equation

$$dX_t = \big(BX_t + b(t, X_t)\big)\,dt + \sigma(t, X_t)\,dW_t. \tag{3.1}$$

In (3.1), W denotes a d-dimensional standard Wiener process with $d \le N$. To fix ideas, for an Asian option, we have $N = 2$ and X is the two-dimensional process whose components are the underlying price (in logarithmical scale) and related average. We refer to Subsect. 3.1 for further examples.

We assume the following structural condition:

Assumption 3.1 $\sigma = \sigma(t, x)$ is an $N \times d$ matrix whose entries are bounded Hölder-continuous functions. Moreover, $B = (b_{ij})$ is an $N \times N$ constant matrix, and $b = (b_1, \dots, b_N)$ is a bounded Hölder-continuous vector-valued function such that

$$0 = b_{d+1} = \cdots = b_N. \tag{3.2}$$

By Remark 5.2 below, the standard Hölder-continuity hypothesis is equivalent to the more natural assumption that $a_{ij}, b_i \in C^\alpha_B$ for some $\alpha \in]0, 1[$, where the Hölder space C^α_B is defined in Subsect. 5.2 below. We also remark that our results straightforwardly generalize to the case $B = B(t)$ and $b_{d+1}(t), \dots, b_N(t)$ measurable and bounded functions of time. Clearly, for $d = N$, condition (3.2) can be neglected.

Before stating the other main hypotheses, we recall some well-known facts about linear SDEs; for more details, we refer, for instance, to [23], Chap. 5.6. Let I_d denote the identity matrix in $\mathbb{R}^d$ and consider the $N \times d$ constant matrix defined, in block form, by

$$\sigma_0 = \begin{pmatrix} I_d \\ 0 \end{pmatrix}. \tag{3.3}$$

Then, for fixed $(y, s) \in \mathbb{R}^{N+1}$, the solution of the *linear* SDE

$$dX^{s,y}_t = BX^{s,y}_t\,dt + \sigma_0\,dW_t, \qquad X^{s,y}_s = y, \tag{3.4}$$

is a Gaussian process with mean vector

$$E\big[X^{s,y}_t\big] = e^{(t-s)B}y$$

and covariance matrix $\mathcal{C}_0(t-s)$, where

$$\mathcal{C}_0(t) = \int_0^t e^{(t-\rho)B}\sigma_0\sigma_0^* e^{(t-\rho)B^*}\,d\rho, \qquad t \ge 0.$$

Since σ_0 has dimension $N \times d$, the matrix $\mathcal{C}_0(t)$ is generally only positive *semi-definite* in $\mathbb{R}^N$, that is, $X_t^{s,y}$ has a possibly *degenerate* multivariate normal distribution. The well-known Kalman condition from control theory (see [23], Proposition 5.6.5) provides an operative criterion for the positivity of $\mathcal{C}_0(t)$.

Theorem 3.2 (Kalman rank condition) *The matrix $\mathcal{C}_0(t)$ is positive definite for $t > 0$ if and only if*

$$\operatorname{rank}\big[\sigma_0,\ B\sigma_0,\ B^2\sigma_0,\ \dots,\ B^{N-1}\sigma_0\big] = N. \tag{3.5}$$

Incidentally, the previous result shows that $\mathcal{C}_0(t) > 0$ for some $t > 0$ if and only if $\mathcal{C}_0(t) > 0$ for every $t > 0$. Then (3.5) ensures that $X^{s,y}$ has the Gaussian transition density

$$\begin{aligned} G(s, y; t, x) = {} & \frac{1}{\sqrt{(2\pi)^N \det \mathcal{C}_0(t-s)}} \\ & \times \exp\left(-\frac{1}{2}\big\langle \mathcal{C}_0^{-1}(t-s)\big(x - e^{(t-s)B}y\big), x - e^{(t-s)B}y\big\rangle\right). \end{aligned} \tag{3.6}$$

Furthermore, G is the fundamental solution of the Kolmogorov PDE associated to (3.4), i.e.,

$$Ku(t,x) := \frac{1}{2}\sum_{i,j=1}^{d} \partial_{x_i x_i} u(t,x) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(t,x) + \partial_t u(t,x) = 0, \tag{3.7}$$

which, in compact form, reads

$$Ku = \frac{1}{2}\Delta_{\mathbb{R}^d} u + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u + \partial_t u = 0.$$

We emphasize that generally (3.7) is not a uniformly parabolic PDE since $d \le N$. In the Appendix, we briefly review some basic results about Kolmogorov equations related to linear SDEs and verify that (3.5) is equivalent to the Hörmander condition [17], which is a nondegeneracy criterion well-known in PDE theory.

Now we consider the general SDE (3.1) and state the second hypothesis:

Assumption 3.3 The matrix $\sigma\sigma^*$ takes the block form

$$\sigma\sigma^* = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.8}$$

where $A = (a_{ij})_{i,j=1,\dots,d}$ is uniformly positive definite, i.e., there exists a positive constant Λ such that

$$\Lambda^{-1}|\eta|^2 < \big\langle A(t,x)\eta, \eta\big\rangle < \Lambda|\eta|^2 \tag{3.9}$$

for any $\eta \in \mathbb{R}^d$ and $(t,x) \in \mathbb{R}^{N+1}$. Moreover, the matrix B satisfies the Kalman condition (3.5) with σ_0 as in (3.3).

Combining the results by Stroock and Varadhan [31] and Di Francesco and Pascucci [9], Assumptions 3.1 and 3.3 ensure the existence and uniqueness of a weak solution $(\Omega, \mathcal{F}, P, (\mathcal{F}_s), W, X)$ to the SDE (3.1). Specifically, it is proved in [9], Theorem 1.4, that the Kolmogorov operator of (3.1), that is

$$\begin{aligned} Lu(t,x) = {} & \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\partial_{x_i x_j} u(t,x) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(t,x) \\ & + \sum_{i=1}^{d} b_i(t,x)\partial_{x_i} u(t,x) + \partial_t u(t,x), \end{aligned}$$

has a fundamental solution $\Gamma = \Gamma(s,y;t,x)$ which is the transition density of the weak solution of (3.1). Moreover, the following Gaussian upper bound holds:

$$\Gamma(s,y;t,x) \le C G_0(s,y;t,x), \quad s<t,\ x,y\in\mathbb{R}^N, \tag{3.10}$$

where G_0 denotes a density of the form (3.6), and C is a positive constant only depending on L and $t-s$. Precisely, G_0 is the transition density related to the linear SDE (3.4) with

$$\sigma_0 = \begin{pmatrix} \Lambda I_d \\ 0 \end{pmatrix}$$

and Λ as in (3.9). For convenience, we rewrite the operator L in compact form as

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\partial_{x_i x_j} + \langle Bx + b, \nabla\rangle + \partial_t. \tag{3.11}$$

We consider the free boundary problem

$$\begin{cases} \mathcal{L}u := \max\{Lu - ru - f, \varphi - u\} = 0 & \text{in } \mathcal{S}_T :=]0,T[\times\mathbb{R}^N, \\ u(T,\cdot) = \varphi(T,\cdot) & \text{in } \mathbb{R}^N. \end{cases} \tag{3.12}$$

In (3.12), r and f are nonconstant coefficients typically representing the locally risk-free interest rate and some transaction costs, respectively.

We assume the following conditions on the coefficients.

Assumption 3.4 The coefficients r, f are bounded and Hölder-continuous. The payoff function φ is locally Lipschitz-continuous on $\overline{\mathcal{S}_T}$, and for any compact subset H of $\mathcal{S}_T$, there exists a constant $c \in \mathbb{R}$ such that

$$\sum_{i,j=1}^{d} \eta_i \eta_j \partial_{x_i x_j}\varphi \ge c|\eta|^2 \quad \text{in } H \text{ for } \eta \in \mathbb{R}^d \tag{3.13}$$

in the distributional sense, that is,

$$\sum_{i,j=1}^{d} \eta_i \eta_j \int_H \partial_{x_i x_j}\psi(z)\varphi(z)\,dz \ge c|\eta|^2 \int_H \psi(z)\,dz$$

for any $\eta \in \mathbb{R}^d$ and $\psi \in C_0^\infty(H)$, $\psi \ge 0$.

The financial intuition underlying condition (3.13) is that the payoff φ has to be a convex function (in a very weak sense) with respect to the first d state variables corresponding, as we shall see in the examples, to the prices of the underlying assets for the option.

We explicitly remark that *any* C^2 *function* satisfies condition (3.13) as well as any Lipschitz-continuous function that is *convex with respect to the first d variables.* In particular, the payoff functions of standard call and put Asian options are included. Note that $x \mapsto (x-K)^+$ satisfies (3.13), since its second-order distributional derivative in K is a Dirac delta which is "nonnegative"; on the contrary, the function $x \mapsto -(x-K)^+$ does not satisfy (3.13), since its second-order distributional derivative in K is a minus Dirac delta which is not "bounded from below."

3.1 Some examples

Example 3.5 (Geometric Asian option with local volatility) In a local volatility model, we assume that the logarithm Z of the underlying asset has the dynamics

$$dZ_t = \mu(t, Z_t)\,dt + \sigma(t, Z_t)\,dW_t,$$

where W is a standard one-dimensional Brownian motion. Then the average process G in (2.2) for a geometric Asian option satisfies

$$dG_t = Z_t\,dt.$$

Adopting the notation (3.1–3.3), we have $1 = d < N = 2$ and

$$X_t = \begin{pmatrix} Z_t \\ G_t \end{pmatrix}, \qquad b = \begin{pmatrix} \mu \\ 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Moreover, condition (3.9) is clearly satisfied with $A = \sigma^2$ whenever σ is a (uniformly strictly) positive and bounded function. Finally, we have

$$[\sigma_0,\ B\sigma_0] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so that the Kalman rank condition and Assumption 3.3 are verified. We remark that our results can be generalized to accomodate the arithmetic average case as well; however, this requires some nontrivial (yet merely technical) question to be addressed. For this reason, we treat that topic separately and plan to complete the study in a forthcoming paper.

Example 3.6 (Geometric Asian option in the Heston stochastic volatility model) In the Heston stochastic volatility model, we have $2 = d < N = 3$ and

$$\begin{aligned} dZ_t &= \left(\mu - \frac{\nu_t}{2}\right) dt + \sigma\sqrt{\nu_t}\,dW_t^1, \\ d\nu_t &= (a - \nu_t)\,dt + \eta\sqrt{\nu_t}\,dW_t^2, \\ dG_t &= Z_t\,dt, \end{aligned}$$

where (W^1, W^2) is a two-dimensional Brownian motion, and μ, σ, a, η are positive parameters. In this case,

$$X_t = \begin{pmatrix} Z_t \\ \nu_t \\ G_t \end{pmatrix}, \qquad b(t, z, \nu) = \begin{pmatrix} \mu \\ a \\ 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and again Assumption 3.3 is easily verified by the rank condition. We refer to the paper by Parrott and Clarke [29] for a numerical study of American Asian options under stochastic volatility.

Example 3.7 (Path-dependent volatility) We consider an extension of the local volatility model in which the volatility is defined as a function of the whole trajectory of the underlying asset, not only in terms of the spot price. Path-dependent volatility was first introduced by Hobson and Rogers [16] and then generalized by Foschi and Pascucci [11]; the main feature is that this generally leads to a complete market model. We refer to [11] for an empirical analysis which shows the effectiveness of the model and compares the hedging performance with respect to standard stochastic volatility models.

Let ψ be an average weight, that is, a nonnegative, piecewise-continuous, and integrable function on $]-\infty, T]$. We assume that ψ is strictly positive in $[0, T]$ and set

$$\Psi(t) = \int_{-\infty}^{t} \psi(s)\, ds.$$

Then we define the average process as

$$M_t = \frac{1}{\Psi(t)} \int_{-\infty}^{t} \psi(s) Z_s\, ds, \qquad t \in]0, T],$$

where $Z_t = \log(e^{-rt} S_t)$ denotes the logarithmic discounted price process. The Hobson and Rogers model corresponds to the specification $\psi(t) = e^{\lambda t}$ for some positive parameter λ. Then by Itô's formula we have

$$dM_t = \frac{\varphi(t)}{\Phi(t)} (Z_t - M_t)\, dt.$$

Assuming for the log-price the dynamics

$$dZ_t = \mu(Z_t - M_t)\, dt + \sigma(Z_t - M_t)\, dW_t,$$

we obtain the pricing PDE

$$\frac{\sigma^2(z-m)}{2} (\partial_{zz} f - \partial_z f) + \frac{\varphi(t)}{\Phi(t)} (z - m) \partial_m f + \partial_t f = 0, \qquad (t, z, m) \in]0, T[\times \mathbb{R}^2.$$

In this case, $1 = d < N = 2$ and

$$X_t = \begin{pmatrix} Z_t \\ M_t \end{pmatrix}, \qquad b = \begin{pmatrix} \mu \\ 0 \end{pmatrix}, \qquad B = \frac{\varphi(t)}{\Phi(t)} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \qquad \sigma_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Assumptions 3.1 and 3.3 are readily verified.

4 Free boundary and optimal stopping problems

In this section, we prove our main results regarding the free boundary problem (3.12) and related optimal stopping problem for SDE (3.1). Throughout this section, Assumptions 3.1, 3.3, and 3.4 are supposed to hold.

In order to introduce the notion of *strong (super-)solution*, we recall the definition of the Sobolev space $\mathcal{S}^p$ given in Subsect. 5.2 below. For $p \geq 1$, $\mathcal{S}^p$ denotes the space of functions $u \in L^p$ such that $\partial_{x_i} u$, $\partial_{x_i x_j} u$ for $i = 1, \dots, d$ and

$$Yu := \langle Bx, \nabla u \rangle + \partial_t u = \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u + \partial_t u \tag{4.1}$$

belong to L^p. As usual, given a domain D in $\mathbb{R}^{N+1}$, $\mathcal{S}^p_{\mathrm{loc}}(D)$ denotes the space of functions $u \in \mathcal{S}^p(D_0)$ for any compact subset D_0 of D. Let us also recall the notation

$$\mathcal{S}_T =]0, T[\times \mathbb{R}^N$$

for a strip in $\mathbb{R}^{N+1}$.

Definition 4.1 A function $u \in \mathcal{S}^1_{\mathrm{loc}}(\mathcal{S}_T) \cap C(\mathbb{R}^N \times]0, T])$ is a strong solution of problem (3.12) if $\mathcal{L}u = 0$ almost everywhere in $\mathcal{S}_T$ and u attains the final datum pointwise. A function $u \in \mathcal{S}^1_{\mathrm{loc}}(\mathcal{S}_T) \cap C(\mathbb{R}^N \times]0, T])$ is a strong supersolution of problem (3.12) if $\mathcal{L}u \leq 0$.

In Subsect. 4.2, we prove the following existence result.

Theorem 4.2 *If there exists a strong supersolution $\bar{u}$ of problem* (3.12), *then there also exists a strong solution u of* (3.12) *such that $u \leq \bar{u}$ in $\mathcal{S}_T$. Moreover, $u \in \mathcal{S}^p_{\mathrm{loc}}(\mathcal{S}_T)$ for any $p \geq 1$ and, consequently, by the embedding theorem in Subsect.* 5.2, *$u \in C^{1,\alpha}_{B,\mathrm{loc}}(\mathcal{S}_T)$ for any $\alpha \in]0, 1[$.*

We remark that a supersolution to problem (3.12) exists in the case of put options and, more generally, whenever φ is a bounded function and $f \geq 0$. Indeed, in this case, we can simply set $\bar{u}(x, t) = e^{-t\|r\|_\infty} \|\varphi\|_\infty$.

For unbounded payoffs, one can usually rely upon some financial consideration based on no-arbitrage arguments. For instance, after the usual change of variable for the asset price $S = e^x$, a supersolution for an American call option with payoff $(e^x - K)^+$ is simply given by $\bar{u}(t, x) = e^x$.

Regarding the regularity of the solution, recalling the definition of $C_B^{1,\alpha}$ in Subsect. 5.2 below, Theorem 4.2 shows that u and its first derivatives $\partial_{x_1}u, \dots, \partial_{x_d}u$ are Hölder-continuous of exponent α for any $\alpha \in]0, 1[$. Since in [10] it is proved that strong solutions are also solutions in the viscosity sense, then Theorem 4.2 improves the known regularity results.

Now by $X^{t,x}$ we denote the solution to SDE (3.1) starting at time t from $x \in \mathbb{R}^N$ and defined on the Wiener space $(\Omega, \mathcal{F}, P, (\mathcal{F}_s), W)$. We recall the following standard maximal estimate (cf., for instance, Chap. 5 in [12]):

$$E\Big[\exp\Big(\lambda \sup_{t\le s\le T}\big|X_s^{t,x}\big|^2\Big)\Big] < \infty \tag{4.2}$$

for suitably small positive constant $\lambda = \lambda(T, B, b, \Lambda, N)$; more explicitly, it suffices to take

$$\lambda \le \frac{e^{-2T(\|B\|+\|b\|_\infty)}}{2TN\Lambda}.$$

The following representation theorem for strong solutions holds.

Theorem 4.3 *Let u be a strong solution to the free boundary problem (3.12) such that*

$$\big|u(t,x)\big| \le Ce^{\lambda|x|^2}, \quad (t,x) \in \mathcal{S}_T, \tag{4.3}$$

for some constants C, λ with λ sufficiently small so that (4.2) holds. Then we have

$$u(t,x) = \sup_{\tau\in\mathcal{T}_{t,T}} E\Big[\varphi\big(\tau, X_\tau^{t,x}\big)e^{-\int_t^\tau r(s,X_s^{t,x})\,ds} - \int_t^\tau f\big(s, X_s^{t,x}\big)e^{-\int_t^s r(\rho,X_\rho^{t,x})d\rho}\,ds\Big], \tag{4.4}$$

where

$$\mathcal{T}_{t,T} = \big\{\tau \in \mathcal{T} \mid \tau \in [t,T]\ a.s.\big\},$$

and $\mathcal{T}$ is the set of all stopping times with respect to the filtration $(\mathcal{F}_s)$. In particular, such a solution is unique.

In the next subsections, we prove Theorem 4.3 and present a detailed outline of the proof of Theorem 4.2. For a complete study of the related obstacle problem, we refer to the recent paper [10].

4.1 Proof of Theorem 4.3

For simplicity, we only consider the case $r = f = 0$. As in the classical case, the proof is based on Itô's formula. However, we remark that a strong solution u need not have the required regularity in order to apply the Itô formula directly. Then in order to exploit the weak interior regularity properties of u, we employ a truncation and regularization technique.

We set $B_R = \{x \in \mathbb{R}^N \mid |x| < R\}$, $R > 0$ and, for $x \in B_R$, denote by τ_R the first exit time of $X^{t,x}$ from B_R. It is well known that, since σ is not totally degenerate, $E[\tau_R]$ is finite.

As a first step, we prove the following result. For every $(t,x)\in\,]0,T[\times B_R$ and $\tau\in\mathcal{T}$ such that $t\le\tau\le\tau_R$ a.s., we have

$$u(t,x)=E\left[u\big(\tau,X_\tau^{t,x}\big)-\int_t^\tau Lu\big(s,X_s^{t,x}\big)\,ds\right]. \tag{4.5}$$

Indeed, for fixed ε positive and suitably small, we consider a function $u^{\varepsilon,R}$ on $\mathbb{R}^{N+1}$ with compact support such that $u^{\varepsilon,R}=u$ on $]t,T-\varepsilon[\times B_R$. Moreover, we denote by $(u^{\varepsilon,R,n})_{n\in\mathbb{N}}$ the regularizing sequence obtained by convolution of $u^{\varepsilon,R}$ with the usual mollifiers. Then, for any $p\ge1$, $u^{\varepsilon,R,n}\in\mathcal{S}^p(\mathbb{R}^{N+1})$ and

$$\lim_{n\to\infty}\big\|Lu^{\varepsilon,R,n}-Lu^{\varepsilon,R}\big\|_{L^p(]t,T-\varepsilon[\times B_R)}=0. \tag{4.6}$$

By the standard Itô formula applied to the smooth function $u^{\varepsilon,R,n}$ we have

$$\begin{aligned}u^{\varepsilon,R,n}\big(\tau,X_\tau^{t,x}\big)&=u^{\varepsilon,R,n}(t,x)+\int_t^\tau Lu^{\varepsilon,R,n}\big(s,X_s^{t,x}\big)\,ds\\&\quad+\int_t^\tau\nabla u^{\varepsilon,R,n}\big(s,X_s^{t,x}\big)\sigma\big(s,X_s^{t,x}\big)\,dW_s\end{aligned} \tag{4.7}$$

for $\tau\in\mathcal{T}$ such that $t\le\tau\le\tau_R\wedge(T-\varepsilon)$ a.s. Since $(\nabla u^{\varepsilon,R,n})\sigma$ is a bounded function on $]t,T-\varepsilon[\times B_R$, we have

$$E\left[\int_t^\tau\nabla u^{\varepsilon,R,n}\big(s,X_s^{t,x}\big)\sigma\big(s,X_s^{t,x}\big)\,dW_s\right]=0.$$

Moreover, we have

$$\lim_{n\to\infty}u^{\varepsilon,R,n}(t,x)=u^{\varepsilon,R}(t,x)$$

and, by dominated convergence,

$$\lim_{n\to\infty}E\big[u^{\varepsilon,R,n}\big(\tau,X_\tau^{t,x}\big)\big]=E\big[u^{\varepsilon,R}\big(\tau,X_\tau^{t,x}\big)\big].$$

Next, we prove the convergence of the ds-integral in (4.7). First we remark that by the Gaussian estimate (3.10) the transition density of $X^{t,x}$ satisfies

$$\Gamma(t,x;\cdot,\cdot)\in L^{\bar q}\big(]t,T[\times B_R\big) \tag{4.8}$$

for some $\bar q>1$. We show (4.8) at the end of this subsection and, taking it for granted, first conclude the proof of the theorem. We have, using $\tau\le\tau_R$ and Hölder's inequality and denoting by $\bar p$ the conjugate exponent of $\bar q$ in (4.8),

$$\begin{aligned}&\left|E\left[\int_t^\tau Lu^{\varepsilon,R,n}\big(s,X_s^{t,x}\big)\,ds\right]-E\left[\int_t^\tau Lu^{\varepsilon,R}\big(s,X_s^{t,x}\big)\,ds\right]\right|\\&\quad\le E\left[\int_t^\tau\big|Lu^{\varepsilon,R,n}\big(s,X_s^{t,x}\big)-Lu^{\varepsilon,R}\big(s,X_s^{t,x}\big)\big|\,ds\right]\end{aligned}$$

$$
\begin{aligned}
&\le E\left[\int_t^{T-\varepsilon} \left|Lu^{\varepsilon,R,n}\left(s, X_s^{t,x}\right) - Lu^{\varepsilon,R}\left(s, X_s^{t,x}\right)\right| \mathbb{1}_{\{|X_s^{t,x}| \le R\}} \, ds\right] \\
&= \int_t^{T-\varepsilon} \int_{B_R} \left|Lu^{\varepsilon,R,n}(s,y) - Lu^{\varepsilon,R}(s,y)\right| \Gamma(t,x;s,y) \, dy \, ds \\
&\le \left\| Lu^{\varepsilon,R,n} - Lu^{\varepsilon,R} \right\|_{L^{\bar{p}}(]t,T-\varepsilon[\times B_R)} \left\| \Gamma(t,x;\cdot,\cdot) \right\|_{L^{\bar{q}}(]t,T-\varepsilon[\times B_R)}
\end{aligned}
$$

and, by (4.6) and (4.8), we obtain

$$
\lim_{n\to\infty} E\left[\int_t^{\tau} Lu^{\varepsilon,R,n}\left(s, X_s^{t,x}\right) ds\right] = E\left[\int_t^{\tau} Lu^{\varepsilon,R}\left(s, X_s^{t,x}\right) ds\right].
$$

This concludes the proof of (4.5), since $u^{\varepsilon,R} = u$ on $]t, T-\varepsilon[\times B_R$ and $\varepsilon > 0$ is arbitrary.

Next, since $Lu \le 0$ a.e. and the law of $X^{t,x}$ is absolutely continuous with respect to the Lebesgue measure, we have

$$
E\left[\int_t^{\tau} Lu\left(s, X_s^{t,x}\right) ds\right] \le 0
$$

for any $\tau \in \mathcal{T}_{t,T}$ and from (4.5) we infer

$$
u(t,x) \ge E\left[u\left(\tau \wedge \tau_R, X_{\tau\wedge\tau_R}^{t,x}\right)\right] \tag{4.9}
$$

for any $\tau \in \mathcal{T}_{t,T}$. Next, we pass to the limit as $R \to +\infty$; we have

$$
\lim_{R\to+\infty} \tau \wedge \tau_R = \tau
$$

pointwise and, by the growth assumption (4.3),

$$
\left|u\left(\tau \wedge \tau_R, X_{\tau\wedge\tau_R}^{t,x}\right)\right| \le C \exp\left(\lambda \sup_{t\le s\le T} \left|X_s^{t,x}\right|^2\right).
$$

By (4.2) the right-hand side of the previous estimate is integrable; thus, Lebesgue's theorem applies, and from (4.9) we deduce that, as $R \to +\infty$,

$$
u(t,x) \ge E\left[u\left(\tau, X_\tau^{t,x}\right)\right] \ge E\left[\varphi\left(\tau, X_\tau^{t,x}\right)\right].
$$

This shows that

$$
u(t,x) \ge \sup_{\tau\in\mathcal{T}_{t,T}} E\left[\varphi\left(\tau, X_\tau^{t,x}\right)\right].
$$

We conclude the proof by putting

$$
\tau_0 = \inf\left\{s \in [t,T] \mid u\left(s, X_s^{t,x}\right) = \varphi\left(s, X_s^{t,x}\right)\right\}.
$$

Since $Lu = 0$ a.e., where $u > \varphi$, we have that

$$
E\left[\int_t^{\tau_0\wedge\tau_R} Lu\left(s, X_s^{t,x}\right) ds\right] = 0 \tag{4.10}
$$

and from (4.5) we infer

$$u(t,x)=E\big[u\big(\tau_0\wedge\tau_R, X^{t,x}_{\tau_0\wedge\tau_R}\big)\big].$$

Repeating the previous argument to pass to the limit in R, we obtain

$$u(t,x)=E\big[u\big(\tau_0, X^{t,x}_{\tau_0}\big)\big]=E\big[\varphi\big(\tau_0, X^{t,x}_{\tau_0}\big)\big].$$

In order to conclude the proof of Theorem 4.3, it remains to show (4.8). By estimate (3.10), it suffices to prove that $G(t,x;\cdot,\cdot)$ in (3.6) belongs to $L^q(]t,T[\times\mathbb{R}^N)$ for $q<1+\frac{2}{Q}$, where Q is the homogeneous dimension of $\mathbb{R}^N$ defined below in (5.7). We have, for a suitable constant c,

$$\begin{aligned}
&\int_t^T\int_{\mathbb{R}^N} G(t,x;s,y)^q\,dy\,ds\\
&\quad=\int_t^T\int_{\mathbb{R}^N}\frac{c}{(\det\mathcal{C}_0(s-t))^{\frac{q}{2}}}\\
&\qquad\times\exp\Big(-\frac{q}{2}\big\langle\mathcal{C}_0^{-1}(s-t)\big(y-e^{(s-t)B}x\big), y-e^{(s-t)B}x\big\rangle\Big)\,dy\,ds\\
&\quad=\int_t^T\frac{c}{(\det\mathcal{C}_0(s-t))^{\frac{q-1}{2}}}\,ds\int_{\mathbb{R}^N}e^{-\frac{|\eta|^2}{2}}\,d\eta
\end{aligned}$$

by the change of variables $\eta=\mathcal{C}_0^{-\frac{1}{2}}(s-t)(y-e^{(s-t)B}x)$. Then the statement of the theorem follows from the fact (see, for instance, Sect. 2 in [9]) that

$$\det\mathcal{C}_0(s-t)=O\big((s-t)^Q\big)\qquad\text{as } s\to t.$$

4.2 Free boundary problem

A solution of problem (3.12) can be obtained as the limit of strong solutions to a sequence of free boundary problems on bounded cylinders of the form $]0,T[\times H_n$, where (H_n) is an increasing covering of $\mathbb{R}^N$. The proof of this simple and quite general fact can be found, for instance, in [10], Theorem 4.1. Thus, we only examine here the free boundary problem on a bounded cylinder. Precisely, we prove the existence of a strong solution to the problem

$$\begin{cases}\max\{Lu-ru-f,\varphi-u\}=0 & \text{in } H(T):=]0,T[\times H,\\ u|_{\partial_P H(T)}=\varphi,\end{cases}\tag{4.11}$$

where H is a bounded domain of $\mathbb{R}^N$, and

$$\partial_P H(T):=\partial H(T)\setminus(\{T\}\times H)$$

is the parabolic boundary of $H(T)$. Hereafter we assume that $H(T)$ is a regular domain in the sense that the standard initial-boundary problem for L on $H(T)$ has a solution. A well-known sufficient condition for this is the existence of a so-called barrier function at any point of the parabolic boundary. We recall that a barrier w at $\zeta \in \partial_P H(T)$ is a smooth function defined on $V \cap \overline{H(T)}$, where V is a suitable neighborhood of ζ such that $Lw \leq -1$ in $V \cap H(T)$, $w > 0$ in $V \cap \overline{H(T)} \setminus \{\zeta\}$, and $w(\zeta) = 0$.

Theorem 4.4 *Problem* (4.11) *has a strong solution* $u \in \mathcal{S}^1_{\mathrm{loc}}(H(T)) \cap C(\overline{H(T)})$. *Moreover,* $u \in \mathcal{S}^p_{\mathrm{loc}}(H(T))$ *for any* $p > 1$ *and, by the embedding theorem in Subsect.* 5.2, $u \in C^{1,\alpha}_{B,\mathrm{loc}}(H(T))$ *for any* $\alpha \in]0, 1[$.

Proof The proof is based on a penalization technique. We consider a family $(\beta_\varepsilon)_{\varepsilon \in]0,1[}$ of smooth functions such that, for any ε, the function β_ε is increasing and bounded on $\mathbb{R}$ and has bounded first-order derivative. Moreover, $\beta_\varepsilon(0) = 0$,

$$\beta_\varepsilon(s) \leq \varepsilon, \quad s > 0, \quad \text{and} \quad \lim_{\varepsilon \to 0} \beta_\varepsilon(s) = -\infty, \quad s < 0.$$

We also denote by L^δ, $\delta > 0$, the operator obtained by regularizing the coefficients of L. Besides φ^δ, r^δ, and f^δ respectively denote the regularizations of φ, r, and f.

By a general result for quasilinear equations (see, for instance, Theorem 3.2 in [10]), there exists a classical solution $u_{\varepsilon,\delta} \in C^{2,\alpha}_B(H(T)) \cap C(\overline{H(T)})$, $\alpha \in]0, 1[$, to the penalized and regularized problem

$$\begin{cases} L^\delta u - r^\delta u = f^\delta + \beta_\varepsilon(u - \varphi^\delta) & \text{in } H(T), \\ u|_{\partial_P H(T)} = \varphi^\delta. \end{cases}$$

The crucial step consists in proving the uniform boundedness of the penalization term, i.e.,

$$\left|\beta_\varepsilon\big(u_{\varepsilon,\delta} - \varphi^\delta\big)\right| \leq c \quad \text{in } H(T) \tag{4.12}$$

with c independent of ε and δ.

Since by construction $\beta_\varepsilon \leq \varepsilon$, it suffices to prove the lower bound in (4.12). By continuity, $\beta_\varepsilon(u_{\varepsilon,\delta} - \varphi^\delta)$ has a minimum ζ in $\overline{H(T)}$, and we may suppose that

$$\beta_\varepsilon\big(u_{\varepsilon,\delta}(\zeta) - \varphi^\delta(\zeta)\big) \leq 0,$$

since otherwise there is nothing to prove. Now, if $\zeta \in \partial_P H(T)$, then

$$\beta_\varepsilon\big(u_{\varepsilon,\delta}(\zeta) - \varphi^\delta(\zeta)\big) = \beta_\varepsilon(0) = 0.$$

On the other hand, if $\zeta \in H(T)$, then we recall that β_ε is increasing; consequently $u_{\varepsilon,\delta} - \varphi^\delta$ also has a (negative) minimum in ζ. Thus, since it is not restrictive to assume that $r \geq 0$, we have

$$L^\delta u_{\varepsilon,\delta}(\zeta) - L^\delta \varphi^\delta(\zeta) \geq 0 \geq r(\zeta)\big(u_{\varepsilon,\delta}(\zeta) - \varphi^\delta(\zeta)\big). \tag{4.13}$$

Now by the weak convexity condition (3.13) on φ in Assumption 3.4 we have that $L^\delta \varphi^\delta(\zeta)$ is bounded uniformly in δ. Therefore, by (4.13), we deduce

$$\begin{aligned}&\beta_\varepsilon\big(u_{\varepsilon,\delta}(\zeta)-\varphi^\delta(\zeta)\big)\\&\quad=L^\delta u_{\varepsilon,\delta}(\zeta)-r^\delta(\zeta)u_{\varepsilon,\delta}(\zeta)-f^\delta(\zeta)\geq L^\delta\varphi^\delta(\zeta)-r^\delta(\zeta)\varphi^\delta(\zeta)-f^\delta(\zeta)\geq c,\end{aligned}$$

where c is a constant independent of ε, δ, and this proves (4.12).

Next, we use the $\mathcal{S}^p$ interior estimate (5.8) below combined with (4.12) to infer that, for every $D \Subset H(T)$ and all $p \geq 1$, the norm $\|u_{\varepsilon,\delta}\|_{\mathcal{S}^p(D)}$ is bounded uniformly in ε and δ. It follows that $(u_{\varepsilon,\delta})$ converges, as $\varepsilon, \delta \to 0$, weakly in $\mathcal{S}^p$ (and in $C_B^{1,\alpha}$ by (5.9)) on compact subsets of $H(T)$ to a function u. Moreover,

$$\limsup_{\varepsilon,\delta\to 0}\beta_\varepsilon\big(u_{\varepsilon,\delta}-\varphi^\delta\big)\leq 0,$$

so that $Lu - ru \leq f$ a.e. in $H(T)$. On the other hand, $Lu - ru = f$ a.e. on the set $\{u > \varphi\}$. Finally, it is straightforward to verify that u is in $C(\overline{H(T)})$ and assumes the initial-boundary conditions, by using standard arguments based on the maximum principle and barrier functions. □

5 Appendix

We review some basic facts about the Kolmogorov operator in (3.7) associated with the linear SDE (3.4). With compact notation, the operator takes the form

$$K=\frac{1}{2}\Delta_{\mathbb{R}^d}+Y, \tag{5.1}$$

where $1 \leq d \leq N$, and Y is as in (4.1). In the sequel, we assume that the Kalman rank condition (3.5) holds. We also systematically write $z = (t, x)$ and $\zeta = (s, y)$ to denote points in $\mathbb{R}^{N+1}$.

The main purpose of this section is to describe the non-Euclidean group and metric structures induced by K on $\mathbb{R}^{N+1}$, which provide the natural framework for the study of the regularity properties of the operator. This structure was first studied by Lanconelli and Polidoro [25]. Secondly, we define some nonstandard Sobolev and Hölder spaces adapted to this non-Euclidean setting and state the basic a-priori estimates used in the study of the free boundary problem. In Subsect. 5.3, we give an insight into the degenerate parabolic structure of K by showing that the Kalman condition (3.5) is equivalent to the Hörmander condition [17]. Hereafter we refer to the simplest nontrivial example of Kolmogorov operator, i.e.,

$$\partial_{x_1x_1}+x_1\partial_{x_2}+\partial_t,\qquad (t,x_1,x_2)\in\mathbb{R}^3, \tag{5.2}$$

as the prototype for the general class. The operator in (5.2) is of particular interest, since it arises in the valuation of geometric Asian options (cf. (2.8)).

5.1 Group and metric structure

We first remark that K in (5.1) is invariant with respect to the law

$$\zeta\circ z:=\big(t+s,x+e^{tB}y\big),\qquad z=(t,x),\ \zeta=(s,y)\in\mathbb{R}^{N+1}. \tag{5.3}$$

More precisely, defining the left translation operator ℓ by

$$\ell_\zeta u(z) = u(\zeta \circ z),$$

it is easily verified that

$$K(\ell_\zeta u) = \ell_\zeta (Ku)$$

for any $\zeta \in \mathbb{R}^{N+1}$. Correspondingly, the fundamental solution of K, whose explicit expression in given in (3.6), has the invariance property

$$G(\zeta; z) = G\big(0; \zeta^{-1} \circ z\big),$$

where $\zeta^{-1} = (-s, -e^{-sB} y)$ is the inverse of $\zeta = (s, y)$ in the law "$\circ$." It is clear that "$\circ$" reduces to the standard sum in $\mathbb{R}^N$ when K is the heat operator and $B = 0$. On the other hand, for the operator in (5.2), using the fact that

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is a nilpotent matrix, we simply have

$$(s, y_1, y_2) \circ (t, x_1, x_2) = (t + s, x_1 + y_1, x_2 + y_2 + t y_1).$$

Next we introduce a "parabolic" norm in $\mathbb{R}^{N+1}$. Using the notation of Sect. 3, for $k = 0, \dots, N$, we denote by V_k the vector space spanned by the columns of the matrix

$$\big[\sigma_0,\ B\sigma_0,\ B^2\sigma_0,\ \dots\ ,\ B^k\sigma_0\big]$$

and, for $k = 1, \dots, N$, we define the subspace W_k of $\mathbb{R}^N$ by

$$V_k = V_{k-1} \oplus W_k.$$

By the Kalman condition, there exists $m \le N$ such that $V_m = \mathbb{R}^N$; therefore, $\mathbb{R}^N$ has an obvious direct sum decomposition, and, for $x \in \mathbb{R}^N$, we have

$$x = x^{(0)} + x^{(1)} + \dots + x^{(m)},$$

where $x^{(0)} \in V_0$ and $x^{(k)} \in W_k$ for $k = 1, \dots, m$ are uniquely determined.

Definition 5.1 The B-norm of $(t, x) \in \mathbb{R}^{N+1}$ is defined as

$$\big\|(t, x)\big\|_B = |t|^{\frac{1}{2}} + \sum_{k=0}^{m} |x^{(k)}|^{\frac{1}{2k+1}}. \tag{5.4}$$

For example, if K is the heat operator, then (5.4) defines the usual parabolic norm

$$\big\|(t, x)\big\|_B = |t|^{\frac{1}{2}} + |x|. \tag{5.5}$$

This definition is in agreement with the practical rule for the heat equation that “two x-derivatives are equivalent to one t-derivative.” Note also that the heat equation is homogeneous of degree two with respect to the dilations in $\mathbb{R}^{N+1}$ defined as

$$\delta_\lambda(t,x) = \left(\lambda^2 t, \lambda x\right), \quad \lambda \in \mathbb{R},$$

and the norm in (5.5) is δ_λ-homogeneous of degree one.

Analogously, for the operator (5.2), we have

$$\left\| (t, x_1, x_2) \right\|_B = |t|^{\frac{1}{2}} + |x_1| + |x_2|^{\frac{1}{3}}, \tag{5.6}$$

so that, in this case, the practical rule reads “∂_t and ∂_{x_2} respectively correspond to second- and third-order derivatives.” Moreover, the operator (5.2) is homogeneous of degree two with respect to the dilations in $\mathbb{R}^3$ defined as

$$\delta_\lambda(t, x_1, x_2) = \left(\lambda^2 t, \lambda x_1, \lambda^3 x_2\right), \quad \lambda \in \mathbb{R},$$

and (5.6) defines a δ_λ-homogeneous norm.

In general, the natural number

$$Q = \dim(V_0) + \sum_{k=1}^{m} (2k+1) \dim(W_k) \tag{5.7}$$

is usually called the *homogeneous dimension* of $\mathbb{R}^N$ induced by K. Clearly, $Q = N$ when K is a parabolic operator, while $N = 2$ and $Q = 4$ for the operator in (5.2).

5.2 Sobolev and Hölder spaces

We introduce some functional spaces modeled on the group and metric structure previously defined. Given a bounded domain D in $\mathbb{R}^{N+1}$ and $p \geq 1$, we define the Sobolev space

$$\mathcal{S}^p(D) = \left\{ u \in L^p(D) : \partial_{x_i} u, \partial_{x_i x_j} u, Yu \in L^p(D),\ i, j = 1, \ldots, d \right\}$$

equipped with the norm

$$\|u\|_{\mathcal{S}^p(D)} = \|u\|_{L^p(D)} + \sum_{i=1}^{d} \|\partial_{x_i} u\|_{L^p(D)} + \sum_{i,j=1}^{d} \|\partial_{x_i x_j} u\|_{L^p(D)} + \|Yu\|_{L^p(D)}.$$

Moreover, for $\alpha \in]0,1[$, we denote respectively by $C^\alpha_B(D)$, $C^{1,\alpha}_B(D)$, and $C^{2,\alpha}_B(D)$ the spaces of B-Hölder-continuous functions defined by the norms

$$\|u\|_{C^\alpha_B(D)} = \sup_D |u| + \sup_{\substack{z,\zeta,\in D \\ z \neq \zeta}} \frac{|u(z) - u(\zeta)|}{\|\zeta^{-1} \circ z\|^\alpha_B},$$

$$\|u\|_{C^{1,\alpha}_B(D)} = \|u\|_{C^\alpha_B(D)} + \sum_{i=1}^{d} \|\partial_{x_i} u\|_{C^\alpha_B(D)},$$

$$\|u\|_{C_B^{2,\alpha}(D)} = \|u\|_{C_B^{1,\alpha}(D)} + \sum_{i,j=1}^{d} \|\partial_{x_i x_j} u\|_{C_B^{\alpha}(D)} + \|Yu\|_{C_B^{\alpha}(D)}.$$

Remark 5.2 Since locally we have

$$\frac{1}{c}\,|z-\zeta| \le \left\|\zeta^{-1}\circ z\right\| \le c\,|z-\zeta|^{\frac{1}{2m+1}},$$

the following inclusion relations among spaces of Hölder-continuous functions (in the usual sense) hold:

$$C^{\alpha}(D) \subseteq C_B^{\alpha}(D) \subseteq C^{\frac{\alpha}{2m+1}}(D).$$

Several classical results from functional analysis have been extended to this non-Euclidean setting in [7, 10, 26]. Here we state some fundamental embedding and a priori estimates for the variable-coefficient operator L in (3.11). These results provide basic tools for the study of the free boundary problem in Sect. 4. In the following inequalities, Q denotes the homogeneous dimension in (5.7), D_0 is a domain contained, with its closure, in D, and c is a constant only depending on L, D, D_0, and p.

- *Interior Sobolev estimates*:

$$\|u\|_{\mathcal{S}^p(D_0)} \le c\big(\|u\|_{L^p(D)} + \|Lu\|_{L^p(D)}\big). \tag{5.8}$$

- *Embedding theorem*:

$$\|u\|_{C_B^{1,\alpha}(D_0)} \le c\|u\|_{\mathcal{S}^p(D)}, \quad \alpha = 1 - \frac{Q+2}{p}, \; p > Q+2. \tag{5.9}$$

5.3 Kalman and Hörmander conditions

We show that the Kalman condition (3.5) is equivalent to the Hörmander condition which is a well-known nondegeneracy criterion in PDE theory. We first introduce some terminology. We identify any first-order differential operator Z in $\mathbb{R}^N$ of the form

$$Zf(x) = \sum_{k=1}^{N} \alpha_k(x)\partial_{x_k} f(x),$$

with the vector field $(\alpha_1, \dots, \alpha_N)$ of its coefficients. The commutator of Z with

$$U = \sum_{k=1}^{N} \beta_k \partial_{x_k}$$

is defined as

$$[Z, U] := ZU - UZ = \sum_{k=1}^{N} (Z\beta_k - U\alpha_k)\partial_{x_k}.$$

Hörmander's theorem is a very general result which, in the particular case of the operator (5.1), states that K has a smooth fundamental solution if and only if the vector space spanned by the differential operators (vectors fields)

$$\partial_{x_1}, \dots, \partial_{x_d} \quad \text{and} \quad \widetilde{Y} := \langle Bx, \nabla \rangle,$$

together with their commutators of any order, at any point x, is equal to $\mathbb{R}^N$. This is the so-called Hörmander condition.

For example, for (5.2) we simply have $\widetilde{Y} = x_1 \partial_{x_2}$. Then

$$\partial_{x_1} \sim (1, 0) \quad \text{and} \quad [\partial_{x_1}, \widetilde{Y}] = \partial_{x_2} \sim (0, 1)$$

span $\mathbb{R}^2$.

The equivalence of the Kalman and Hörmander conditions is readily verified once we note that:

(i) For $i = 1, \dots, d$, $[\partial_{x_i}, \widetilde{Y}] = \sum_{k=1}^N b_{ki} \partial_{x_k}$ is the ith column of the matrix B; moreover, $[[\partial_{x_i}, \widetilde{Y}], \widetilde{Y}]$ is the ith column of the matrix B^2, and an analogous representation of the higher-order commutators is valid;
(ii) For $k = 1, \dots, N$, $B^k \sigma_0$ appearing in (3.5) is the $N \times d$ matrix whose columns are the first d columns of B^k.

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