Analytical approximations for financial derivatives

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Abstract

We propose a new approach to the analytical approximation of transition densities typically arising in finance. This allows to obtain an expansion of the price of financial derivatives using as starting point the classical Black&Scholes formula. Explicit error estimates for the expansion truncated at any order are available. A numerical test is presented and possible applications to Monte Carlo methods are discussed.

Résumé

Approximations analytiques pour l'évaluation de produits financiers dérivés. Nous proposons ici une nouvelle approche à l'approximation analytique de certaines densités de transitions qui se présentent typiquement en finance mathématique. Ceci nous permet d'obtenir une expansion des prix des instruments financiers dérivés en partant de la formule classique de Black&Scholes. Il est possible d'obtenir des estimations explicites de l'erreur pour des expansions tronquées à tout ordre. Nous présentons aussi un test numérique et discutons des applications possibles aux méthodes Monte Carlo.

1. Introduction

Since the introduction and widespread use of diffusion-based models, several issues in the mathematical modeling in finance have led to solving partial differential equation (PDE) problems. Apart from a small number of classical models which are commonly considered poor from the statistical point of view, usual PDE problems in finance do not admit closed form solution. This urged the research of sophisticated numerical techniques aiming at determining approximate solutions to specific financial problems. In this Note, we propose a new approach to the analytical approximation of fundamental solutions (or transition

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densities) typically arising in finance. Our technique develops the classical *parametrix method* that was introduced by Levi in [5] to prove a theoretical result, namely the existence of a fundamental solution of an elliptic equation. We aim to show how this method, suitably adapted and modified, allows to derive sharp analytical approximations with explicit error bounds.

To enlighten the main ideas, we consider a one dimensional financial model. Note however that our method is designed for high dimensional problems as well. We consider a so-called local volatility model where the dynamic of an asset price is given by the stochastic differential equation (SDE)

$$dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t. \tag{1}$$

In (1), W is a one-dimensional Brownian motion and μ and σ satisfy usual hypotheses. Assuming a constant riskless interest rate r, standard arbitrage arguments show that the price $V(t, S)$ of an European option with maturity T and payoff $\varphi(S_T)$ is given by $V(t, S) = e^{rt}u(t, -rt+\log S)$, where u is the solution to the PDE

$$Lu(t,x) := \frac{a^2(t,x)}{2}(\partial_{xx}u(t,x) - \partial_x u(t,x)) + \partial_t u(t,x) = 0, \qquad x \in \mathbb{R},\ t \in]0,T[, \tag{2}$$

with $a(t, x) = \sigma\left(t, e^{x+rt}\right)$, subject to the final condition $u(T, x) = e^{-rT}\varphi\left(e^{x+rT}\right)$, $x \in \mathbb{R}$. By the Feynman-Kac theorem, u has the following representation

$$u(t,x) = e^{-rT}\int_{\mathbb{R}} \varphi\left(e^{\xi+rT}\right)\Gamma\left(t,x;T,\xi\right)d\xi = e^{-rT}\mathbb{E}_{t,x}\left[\varphi\left(e^{X_T+rT}\right)\right] \tag{3}$$

where X is solution to the SDE

$$dX_t = -\frac{a^2(t,X_t)}{2}dt + a(t,X_t)dW_t, \tag{4}$$

that gives the risk neutral dynamic of the discounted log-price. In (4), $\Gamma = \Gamma(t, x; T, \xi)$ denotes the fundamental solution of the PDE (2) or, equivalently, the transition density of the process X starting from x at time t and ending at time T. In applications, transition densities play a central role in several issues: pricing and hedging of financial derivatives, parameter estimation and calibration, study of model risk and of the robustness of a model to perturbation in its structure.

2. Parametrix approximation

Hereafter we use the notation $z = (t, x)$ and $\zeta = (\tau, \xi)$. Under the assumption that L in (2) is a uniformly parabolic operator with bounded and Hölder continuous coefficients, the classical Levi parametrix method gives a proof of the existence of the fundamental solution Γ of L. The main idea is that of constructing Γ, evaluated at (z, ζ), by using the fundamental solution Z, evaluated at (z, ζ), to the constant coefficients (heat) equation obtained by freezing the coefficients of L at ζ. The Gaussian function Z is commonly called the *parametrix* for Γ. This choice seems appropriate for the theoretical proof of the existence of Γ; however, from the practical point of view, $Z = Z(z, \zeta)$ has the disadvantage of being a Gaussian function only as a function of z (for fixed ζ) but not as a function of ζ (for fixed z) since, by definition, the frozen coefficients vary with ζ. Consequently, noting that the integration in (3) is performed with respect to the variable ξ, we have that $Z(z, \zeta)$ cannot be interpreted as the transition density of a stochastic process related to the original log-price process X in (4). To recover this relationship, our idea is to construct a new parametrix by using of the adjoint operator $\widetilde{L}$ of L. To avoid confusion, when necessary, we write $\widetilde{L}^{(\zeta)}$ instead of $\widetilde{L}$

in order to indicate that the operator is acting in the variable ζ. Under standard regularity assumptions on the coefficients, it is well known that the fundamental solution $\widetilde{\Gamma}$ of $\widetilde{L}^{(\zeta)}$ is given by $\widetilde{\Gamma}(\zeta;z)=\Gamma(z;\zeta)$. Next, for fixed $w\in\mathbb{R}^2$, we consider the frozen adjoint operator $\widetilde{L}_w^{(\zeta)} := \frac{a^2(w)}{2}(\partial_{\xi\xi}+\partial_\xi)-\partial_\tau$, whose fundamental solution is explicitly given by

$$\widetilde{\Gamma}_w(\zeta;z)=\frac{1}{a(w)\sqrt{2\pi(\tau-t)}}\exp\left(-\frac{(\xi-x)^2}{2a^2(w)(\tau-t)}-\frac{\xi-x}{2}-\frac{a^2(w)}{8}(\tau-t)\right),\qquad \tau>t. \tag{5}$$

and we define the *backward parametrix* for L as $P(z;\zeta)=\widetilde{\Gamma}_z(\zeta;z)$. Then we proceed as in the classical case (cf. for instance, [4]) and impose that Γ is of the form $\Gamma(z;\zeta)=P(z;\zeta)+\int_t^\tau\int_{\mathbb{R}}P(w;\zeta)\Phi(z;w)dw$. The unknown function Φ can be determined by applying $\widetilde{L}^{(\zeta)}$ to both sides of the previous equation and using the fact that $\widetilde{L}^{(\zeta)}\Gamma(z;\zeta)=0$. More precisely, we get an equation that can be recursively solved and leads to the following *backward parametrix expansion of* Γ*:*

$$\Gamma(z;\zeta)=\sum_{n=0}^{\infty}P_n(z;\zeta),\qquad P_0(z;\zeta)=P(z;\zeta),\qquad P_n(z;\zeta)=\int_t^\tau\int_{\mathbb{R}}P(w;\zeta)A_n(z;w)dw, \tag{6}$$

where, for $w=(s,y)$, $A_1(z;w)=\widetilde{L}^{(w)}P(z;w)$ and $A_{n+1}(z;w)=\int_t^s\int_{\mathbb{R}}\widetilde{L}^{(w)}P(\bar{z};w)A_n(z;\bar{z})d\bar{z}$. Explicit error estimates for the truncated expansion of any order are proved in [1]: for any $T>0$ there exist two (explicit) positive constants C,M such that

$$\left|\Gamma(z;\zeta)-\sum_{k=0}^{n}P_k(z;\zeta)\right|\le C\frac{(\tau-t)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!}\Gamma_M(z;\zeta), \tag{7}$$

for $n\ge 0$ and any $z,\zeta\in\mathbb{R}^2$ such that $0<\tau-t<T$, where Γ_M is the fundamental solution of the heat operator $M\partial_{xx}+\partial_t$. This estimate shows the extremely fast rate of convergence of the approximation for $\tau-t<1$ that is, in financial terms, for short times to maturity.

3. Analytical parametrix approximation and application to Monte Carlo simulation

The first term of the parametrix expansion (6) is given explicitly by (5). It turns out that the subsequent terms can be analytically approximated as well. By using a simple trapezoidal method in [1] and [3], the following backward parametrix expansion of order two is derived:

$$\Gamma(z;\zeta)\simeq P(z;\zeta)+\frac{\tau-t}{2}\widetilde{L}^{(\zeta)}P(z;\zeta)=P(z;\zeta)+\frac{\tau-t}{2}\left(\widetilde{L}^{(\zeta)}-\widetilde{L}_z^{(\zeta)}\right)P(z;\zeta). \tag{8}$$

We explicitly remark that the results of the previous sections can be straightforwardly generalized to dimension $N>1$: in particular an explicit error estimate analogous to (7) is valid. Moreover the closed form approximation (8) remains formally unchanged for $N>1$ and can be used in combination with Monte Carlo methods. More precisely, let us consider a financial model with a high number N of underlying assets, for instance $N\ge 50$, whose risk neutral dynamics for the vector X of discounted log-prices is given by a system of SDEs analogous to (4). Due to the high dimension, it is unlikely that the numerical solution of the valuation formula (3) can be performed by deterministic approximations (finite differences or finite elements schemes), so we have to rely upon Monte Carlo techniques. This presumes the ability of generating a sequence $X_T^{(1)},X_T^{(2)},\dots$ of independent realizations of X_T, that is usually done by an Euler type discretization of the SDEs. However this translates in a high numerical burden which can be

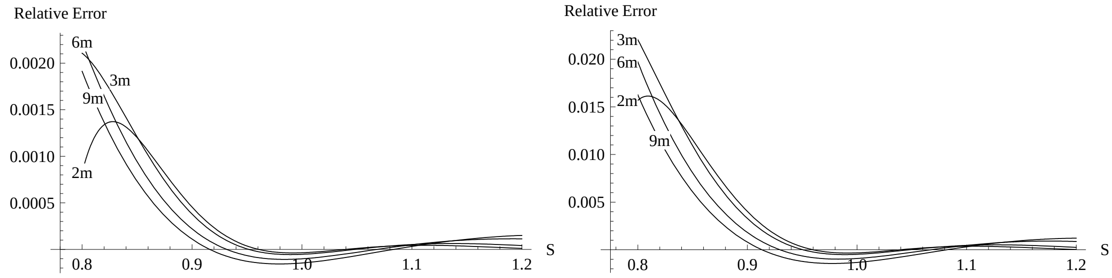

Figure 1. Relative errors of the 2-terms parametrix approximation for $T = 2, 3, 6, 9$ months and $\alpha = \frac{1}{4}$ (left), $\alpha = \frac{3}{4}$ (right), $\sigma_0 = 30\%$, $r = 5\%$ and $K = 1$

excessive even for N moderately large. The backward parametrix method offers an alternative by directly providing an explicit approximation of the density of X_T in terms of formulas like (8). Then Monte Carlo simulations can be performed avoiding the preliminary and computationally expensive discretization of the system of SDEs.

4. A numerical test: analytical pricing formulas in the CEV model

The constant elasticity of variance (CEV) is a classical specification of the local volatility model (1) where $\sigma(t, S) = \sigma_0 S^{-\alpha}$ with $\sigma_0 > 0$ and $\alpha \in]0, 1[$. For this model analytic approximation formulas for European Call options are available (cf. [2]) so that we have reference numbers to test numerically and evaluate the performance of the backward parametrix approximation. By inserting the second order expansion (8) in the valuation formulas (3), we obtain the following *Black&Scholes type formula* for the price $V(t, S)$ of a Call option at time t with strike K and maturity T:

$$V(t,S) = V_{\mathrm{BS}}(t, S; \sigma_0 S^{-\alpha}) + e^{-r(T-t)} \frac{(T-t)K}{4} \left(K^{-2\alpha} - S^{-2\alpha}\right) P(r(T-t) + \log S, T-t; \log K, 0), \quad (9)$$

where $V_{\mathrm{BS}}(t, S; \sigma)$ denotes the Black&Scholes price with (constant) volatility equal to σ, and $P(z; \zeta)$ is the backward parametrix in (5). Figure 1 reports the *relative errors* of the 2-terms parametrix approximation for $\alpha = \frac{1}{4}$ (left) and $\alpha = \frac{3}{4}$ (right) as a function of $S \in [0.8, 1.2]$, for different maturities and typical values of the parameters K, r, σ_0. For $S > 0.9$, relative errors are less that 0.05% for $\alpha = \frac{1}{4}$ and less that 0.5% for $\alpha = \frac{3}{4}$. For deep out-of-the-money options, the price is very close to zero and also the CEV-expansion in [2] seems to be not reliable. We refer to [1] for a more detailed analysis of the numerical results.

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