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Kolmogorov Equations in Physics and in Finance

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Abstract. This paper contains a survey of results about linear and nonlinear partial differential equations of Kolmogorov type arising in physics and in mathematical finance. Some recent pointwise estimates proved in collaboration with S. Polidoro are also presented.

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1. Introduction

We consider a class of the differential equations of Kolmogorov type of the form

$$Lu \equiv \sum_{i,j=1}^{p_0} a_{ij}(z)\partial_{x_i x_j} u + \sum_{i=1}^{p_0} a_i(z)\partial_{x_i} u + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u + c(z)u - \partial_t u = 0, \quad (1.1)$$

where $z = (x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $1 \le p_0 \le N$. By convenience, hereafter the term "Kolmogorov equation" will be shortened to KE. We assume the following hypotheses:

H.1 the matrix $A_0 = (a_{ij})_{i,j=1,\dots,p_0}$ is symmetric and uniformly positive definite in $\mathbb{R}^{p_0}$: there exists a positive constant μ such that

$$\frac{|\eta|^2}{\mu} \le \sum_{i,j=1}^{p_0} a_{ij}(z)\eta_i\eta_j \le \mu|\eta|^2, \qquad \forall \eta \in \mathbb{R}^{p_0},\ z \in \mathbb{R}^{N+1}; \quad (1.2)$$

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H.2 the matrix $B \equiv (b_{ij})$ has constant real entries and takes the following block from:

$$\begin{pmatrix} * & B_1 & 0 & \dots & 0 \\ * & * & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & B_r \\ * & * & * & \dots & * \end{pmatrix} \tag{1.3}$$

where B_j is a $p_{j-1} \times p_j$ matrix of rank p_j, with

$$p_0 \geq p_1 \geq \dots \geq p_r \geq 1, \qquad p_0 + p_1 + \dots + p_r = N,$$

and the $*$-blocks are arbitrary.

The prototype of (1.1) is the following equation

$$\partial_{x_1 x_1} u + x_1 \partial_{x_2} u - \partial_t u = 0, \qquad (x_1, x_2, t) \in \mathbb{R}^3, \tag{1.4}$$

whose fundamental solution was explicitly constructed by Kolmogorov [25]. In his celebrated paper [23], Hörmander generalized this result to *constant coefficients KEs*, i.e., equations of the form (1.1), with constant a_{ij} and $a_i = c \equiv 0$ for $i = 1, \dots, p_0$, satisfying the following condition:

$$\mathrm{Ker}(A) \textit{ does not contain non-trivial subspaces which are invariant for } B. \tag{1.5}$$

In (1.5), A denotes the $N \times N$ matrix

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.6}$$

Let us recall that, for constant coefficients equations, condition (1.5) is equivalent to the structural assumptions H.1–H.2 which in turn are equivalent to the classical Hörmander condition:

$$\text{rank Lie}\,(X_1, \dots, X_{p_0}, Y) = N + 1, \tag{1.7}$$

at any point of $\mathbb{R}^{N+1}$. In (1.7), $\text{Lie}\,(X_1, \dots, X_{p_0}, Y)$ denotes the Lie algebra generated by the vector fields

$$X_i = \sum_{j=1}^{p_0} a_{ij} \partial_{x_j}, \; i = 1, \dots, p_0, \qquad \text{and} \qquad Y = \langle x, BD \rangle - \partial_t, \tag{1.8}$$

where $\langle \cdot, \cdot \rangle$ and D respectively denote the inner product and the gradient in $\mathbb{R}^N$. A proof of the equivalence of these conditions is given by Kupcov in [26], Theorem 3 and by Lanconelli and Polidoro in [30], Proposition A.1.

Equation (1.4) is the lowest dimension version of the following ultraparabolic equation in $\mathbb{R}^{N+1}$ with $N = 2n$:

$$\sum_{j=1}^{n} \partial_{x_j}^2 + \sum_{j=1}^{n} x_j \partial_{x_{n+j}} - \partial_t = 0. \tag{1.9}$$

Kolmogorov introduced (1.9) in 1934 in order to describe the probability density of a system with $2n$ degree of freedom. The $2n$-dimensional space is the phase space, $(x_1, \ldots, x_n)$ is the velocity and $(x_{n+1}, \ldots, x_{2n})$ the position of the system. We also recall that (1.9) is a prototype for a family of evolution equations arising in the kinetic theory of gases that take the following general form

$$Yu = \mathcal{J}(u). \tag{1.10}$$

Here $\mathbb{R}^{2n} \ni x \longmapsto u(x,t) \in \mathbb{R}$ is the density of particles which have velocity $(x_1, \ldots, x_n)$ and position $(x_{n+1}, \ldots, x_{2n})$ at time t,

$$Yu \equiv \sum_{j=1}^{n} x_j \partial_{x_{n+j}} u + \partial_t u$$

is the so-called *total derivative of* u and $\mathcal{J}(u)$ describes some kind of collision. This last term can take different form, it may also occur in non-divergence form and its coefficients may depend on $z \in \mathbb{R}^{2n+1}$ as well as on the solution u. For instance, in the usual Fokker-Planck equation, we have

$$\mathcal{J}(u) = -\sum_{i,j=1}^{n} \partial_{x_i} \left(a_{ij} \partial_{x_j} u + b_i u \right) + \sum_{i=1}^{n} a_i \partial_{x_i} u + cu \tag{1.11}$$

where a_{ij}, a_i, b_i and c are functions of z. In the Boltzmann-Landau equation (see [9], [31] and [32])

$$\mathcal{J}(u) = \sum_{i,j=1}^{n} \partial_{x_i} \left(a_{ij}(\cdot, u) \partial_{x_j} u \right),$$

and the coefficients depend on the unknown function through some integral expressions. This kind of operator is studied as a simplified version of the Boltzmann collision operator. A description of wide classes of stochastic processes and kinetic models leading to equations of the previous type can be found in the classical monographs [10], [16] and [11].

Linear KEs also arise in mathematical finance in some generalization of the celebrated Black & Scholes model [7]. Consider a "stock" whose price S_t is given by the stochastic differential equation

$$d\,S_t = \mu_0 S_t\, dt + \sigma S_t\, dW_t, \tag{1.12}$$

where μ_0 and σ are positive constants and W_t is a Wiener process. Also consider a "bond" whose price B_t only depends on a constant interest rate r:

$$B_t = B_0 e^{t\,r}.$$

Finally, consider an "European option" which is a contract which gives the *right* (but not the *obligation*) to buy the stock at a given "exercise price" E and at a given "expiry time" T. The problem studied in [7] is to find a fair price of the option contract. Under some assumptions on the financial market, Black & Scholes show that the price of the option, as a function of the time and of the stock price $V(t, S_t)$, is the solution of the following partial differential equation

$$-rV + \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} = 0$$

in the domain $(S,t) \in \mathbb{R}^+ \times]0,T[$, with the *final condition*

$$V(T, S_T) = \max(S_T - E, 0).$$

In the last decades the Black & Scholes theory has been developed by many authors and mathematical models involving KEs have appeared in the study of the so-called path-dependent contingent claims (see, for instance, [1], [4], [5] and [48]). *Asian options* are options whose exercise price is not fixed as a given constant E, but depends on some average of the history of the stock price. In this case, the value of the option at the expiry time T is (for a a geometric average option):

$$V(S_T, M_T) = \max\left(S_T - e^{\frac{M_T}{T}}, 0\right), \quad M_t = \int_0^t \log(S_\tau)d\tau.$$

If we suppose by simplicity that the interest rate is $r = 0$, the Black & Scholes method leads to the following degenerate equation

$$S^2\partial_S^2 V + (\log S)\partial_M V + \partial_t V = 0, \qquad S, t > 0,\ M \in \mathbb{R} \tag{1.13}$$

which can be reduced to the KE (1.4) by means of an elementary change of variables (see [6], page 479). A numerical study of the solution of the Cauchy problem related to (1.13) is also proposed in [6].

A recent motivation in finance comes from the model by Hobson & Rogers [22]. In the Black & Scholes theory the hypothesis that the volatility σ in the stochastic differential equation (1.12) is constant contrasts with the empirical observations. Aiming to overcome this problem, many authors proposed different models based on a stochastic volatility (see [18] for a survey). However the presence of a second Wiener process leads some difficulties in the arbitrage argument underlying the Black & Scholes theory. The model proposed by Hobson and Rogers for European options assumes that the volatility only depends on the difference between the present stock price and the past price. This simple model seems to capture the features observed in the market and avoid the problems related to the use of many sources of randomness.

As in the study of Asian options, in the Hobson & Rogers model for European options the value of the option $V(t, S_t, M_t)$ is supposed to depend on the time t, on the price of the stock S_t, on some average M_t and must satisfy the following differential equation

$$\frac{1}{2}\sigma^2(S,M)\left(\partial_S^2 V - \partial_S V\right) + (S - M)\partial_M V + \partial_t V = 0, \tag{1.14}$$

that is a KE with Hölder continuous coefficients. In the recent paper [15] the Cauchy problem related to (1.14) has been studied numerically. In [13] the stability and the rate of convergence of different numerical methods for solving (1.14) are tested. The numerical schemes proposed in these papers rely on the approximation of the directional derivative Y by the finite difference $-\frac{u(x,y,t)-u(x,y+\delta x,t-\delta)}{\delta}$: hence this method, which is respectful of the non-Euclidean geometry of the Lie group, seems to provide a good approximation of the solution.

Finally we recall that KEs with *non linear total derivative term* of the form

$$\Delta_x u + \partial_y g(u) - \partial_t u = f, \qquad x = (x_1, \dots, x_n) \in \mathbb{R}^n, \ y, t \in \mathbb{R}, \tag{1.15}$$

have been considered for convection-diffusion models (cf. [19] and [36]), for pricing models of options with memory feedback (cf. [40]) and for mathematical models for utility functional and decision making (cf. [2], [3], [12] and [38]). The linearized equation of (1.15)

$$g'(u)\partial_y v - \partial_t v = -\Delta_x v,$$

if $g'(u)$ is different from zero and smooth enough, can be reduced to the form (1.1) with $N = n + 2$,

$$A = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

2. Constant coefficients Kolmogorov equations

We call *constant coefficients KE* any equation of the form

$$Ku \equiv \sum_{i,j=1}^{p_0} a_{ij} \partial_{x_i x_j} u + \langle x, BDu \rangle - \partial_t = 0, \tag{2.1}$$

with constant a_{ij}'s and satisfying hypotheses H.1–H.2. We set

$$\mathcal{C}(t) = \int_0^t E(s) A E^T(s) ds, \qquad t \in \mathbb{R},$$

where

$$E(t) = e^{-tB^T}. \tag{2.2}$$

It is known (see, for instance, [30]) that H.1–H.2 are equivalent to condition

$$\mathcal{C}(t) > 0, \qquad \forall t > 0. \tag{2.3}$$

If (2.3) holds then a fundamental solution to (2.1) is given by

$$\Gamma(x, t, \xi, \tau) = \Gamma(x - E(t - \tau)\xi, t - \tau), \tag{2.4}$$

where $\Gamma(x, t) = 0$ if $t \le 0$ and

$$\Gamma(x, t) = \frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det \mathcal{C}(t)}} \exp\left(-\frac{1}{4} \langle \mathcal{C}^{-1}(t) x, x \rangle - t \, \mathrm{tr}(B) \right), \qquad \text{if } t > 0. \tag{2.5}$$

Let us remark that $\Gamma(\cdot, \cdot)$ is a C^∞ function outside the diagonal of $\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$.

The denomination "constant coefficients KE" stems from the theory of parabolic PDEs. Indeed a constant coefficients parabolic equation is nothing more that a translation invariant equation on the Euclidean space. Similarly, a constant coefficients KE has the remarkable property of being invariant with respect to the *non-Euclidean* left translations in the Lie group law

$$(x,t)\circ(\xi,\tau)=(\xi+E(\tau)x,t+\tau), \qquad (x,t),(\xi,\tau)\in\mathbb{R}^N\times\mathbb{R},$$

with $E(\cdot)$ as in (2.2). The class of constant coefficients KEs contains a significant subclass of equations which are also invariant with respect to a suitable dilation group. Indeed, given B in the form (1.3), let us consider the family of dilations in $\mathbb{R}^{N+1}$:

$$\delta_\lambda=\mathrm{diag}(\lambda I_{p_0},\lambda^3 I_{p_1},\dots,\lambda^{2r+1}I_{p_r},\lambda^2),$$

where I_{p_j} denotes the $p_j\times p_j$ identity matrix. Then K is δ_λ-homogeneous of degree two, i.e.,

$$K\circ\delta_\lambda=\lambda^2\left(\delta_\lambda\circ K\right), \qquad \forall\lambda>0,$$

if and only if all the $*$-blocks in (1.3) are zero matrices. The proofs of these statements are contained in [27] and [30]. When the $*$-blocks in B are zero, the dilations $(\delta_\lambda)_{\lambda>0}$ are a group of automorphisms of the Lie group $\mathcal{G}=(\mathbb{R}^{N+1},\circ)$. Equipped with them, $\mathcal{G}$ becomes a homogeneous group with homogeneous dimension $Q+2$, where

$$Q=p_0+3p_1+\cdots+(2r+1)p_r, \tag{2.6}$$

(see [26], page 288, and [30], Remark 2.1).

As in classical theory, constant coefficients KEs serve as an essential class of prototypes and many results can be extended to the general situation of variable coefficients by perturbation arguments: in the next sections we present a survey of the main results for KEs with variable coefficients.

3. Kolmogorov equations with regular coefficients

In view of the invariance properties of constant coefficients KEs with respect to $\mathcal{G}$, it is natural to expect that the intrinsic geometry underlying L is that one determined by $\mathcal{G}$. Let $\alpha_1,\dots,\alpha_N$ be the strictly positive integers such that

$$\delta_\lambda=\mathrm{diag}\left(\lambda^{\alpha_1},\dots,\lambda^{\alpha_N},\lambda^2\right)$$

and define, for every $z\in\mathbb{R}^{N+1}\setminus\{0\}$, $\|z\|_{\mathcal{G}}=\rho$ where ρ is the unique positive solution to the equation

$$\frac{t^2}{\rho^4}+\sum_{j=1}^{N}\frac{x_j^2}{\rho^{2\alpha_j}}=1, \qquad z=(x_1,\dots,x_N,t).$$

We agree to let $\|z\|_{\mathcal{G}}=0$ if $z=0$. Then $z\longmapsto\|z\|_{\mathcal{G}}$ is a δ_λ-homogeneous function of degree one, continuous on $\mathbb{R}^{N+1}$, strictly positive and of class C^∞ in $\mathbb{R}^{N+1}\setminus\{0\}$. If we define

$$d_{\mathcal{G}}(z,\zeta)=\|\zeta^{-1}\circ z\|_{\mathcal{G}}, \qquad z,\zeta\in\mathbb{R}^{N+1},$$

then $(\mathbb{R}^{N+1},d_{\mathcal{G}})$ is a (pseudo-)metric space. We say that a function f is B-Hölder continuous of order $\alpha\in\,]0,1]$ on a domain Ω of $\mathbb{R}^{N+1}$, and we write $f\in C^\alpha_B(\Omega)$, if there exists a constant C such that

$$|f(z) - f(\zeta)| \le C d_{\mathcal{G}}(z,\zeta)^{\alpha}, \qquad \forall z, \zeta \in \Omega.$$

Assuming that the coefficients a_{ij}, $a_i, c \in C^{\alpha}_{B}(\mathbb{R}^{N+1})$, for $i, j = 1, \dots, p_0$, are bounded functions, a fundamental solution Γ for the operator L in (1.1) can be constructed by adapting the Levi's parametrix method to the Lie group and metric structures related to the matrix B (see [14] and [41] which improve and generalize the previous results by Weber [47], Il'in [24] and Sonin [46]).

The Levi's parametrix method also provides a global upper bound for Γ. Indeed let Γ^{ε} denote the fundamental solution to the constant coefficients KE

$$L^{\varepsilon} = (\mu + \varepsilon)\Delta_{\mathbb{R}^{p_0}} + Y \tag{3.1}$$

where $\varepsilon > 0$, μ is as in (1.2), $\Delta_{\mathbb{R}^{p_0}}$ denotes the Laplacian in the variables $x_1, \dots, x_{p_0}$ and Y is the vector fields in (1.8). Then for every positive ε and T, there exists a constant C, only dependent on μ, B, ε and T, such that

$$\Gamma(z,\zeta) \le C\ \Gamma^{\varepsilon}(z,\zeta) \tag{3.2}$$

for any $z, \zeta \in \mathbb{R}^{N+1}$ with $0 < t - \tau < T$. Similar estimates also hold for the derivatives of Γ (see [14] and [41]).

For operators in divergence form

$$L = \sum_{i,j=1}^{p_0} \partial_{x_i}\left(a_{ij}(z)\partial_{x_j}\right) + Y \tag{3.3}$$

with null $*$-blocks in (1.3), a lower bound for Γ analogous to (3.2) also holds. This result relies on a Harnack inequality which is invariant with respect to the translations and dilations in $\mathcal{G}$ (see [41], Theorem 1.3 which extends some Harnack inequalities for constant coefficients Kolmogorov operators first appeared in [28], [20] and [30]).

Theorem 3.1. **(Polidoro** [42]**)** *Let Γ be the fundamental solution of the divergence form operator* (3.3)*. There exists a positive constant m such that, if Γ^- denotes the fundamental solution of*

$$L^- = m^{-1}\Delta_{p_0} + \langle x, BD\rangle - \partial_t,$$

then, for every $T > 0$, there exists a positive constant C^- such that

$$C^-\Gamma^-(z,\zeta) \le \Gamma(z,\zeta) \tag{3.4}$$

for every $z = (x,t),\ \zeta = (\xi,\tau) \in \mathbb{R}^{N+1}$, $0 < t - \tau < T$.

We would like to emphasize that the functions Γ^- and Γ^{ε} appearing in (3.2) and (3.4) have the explicit form (2.4)–(2.5), with the matrix A in (1.6) replaced by $m^{-1}\mathrm{diag}(I_{p_0}, 0, \dots, 0)$ and $(\mu+\varepsilon)\ \mathrm{diag}(I_{p_0}, 0, \dots, 0)$ respectively. Theorem 3.1 was proved in [42] by using a technique which is inspired by a method of Aronson and Serrin for classical parabolic operators. The core of the method used in [42] is a kind of discretization of the connectivity Theorem of Carathéodory-Razewski-Chow.

We also recall some interior regularity results. The following Schauder type estimates proved in [34] (see also [33] and [36]) improve and generalize the previous ones contained in [21], [45] and [17]: for every bounded open set Ω_1 such that $\overline{\Omega}_1 \subseteq \Omega$ where Ω is a subset of $\mathbb{R}^{N+1}$, there exists a constant $C > 0$ such that

$$|u|_{2+\alpha,\Omega_1} \leq C \left(\sup_{\Omega} |u| + |Lu|_{\alpha,\Omega} \right),$$

for any u smooth real function defined on Ω. Here $|\cdot|_{\alpha,\Omega}$ and $|\cdot|_{2+\alpha,\Omega_1}$ denote suitable Hölder norms defined in terms of $d_{\mathcal{G}}$. In [34], the interior Schauder estimates are also used to study a first boundary value problem for L. We also quote the paper [29] in which a boundary value problem for the non-linear equation

$$\sum_{i,j=1}^{p_0} \partial_{x_i} \left(a_{ij}(z,u) \partial_{x_j} \right) + Yu = 0. \tag{3.5}$$

was studied. In [29] the a priori estimates of [34] are used as crucial tools.

The L^p regularity theory for weak solutions to equations in divergence or non-divergence form has been studied in [8], [35], [43] and [44]. In [8] and [43], interior regularity properties of strong solutions to the non-divergence form equation $Lu = f$ were studied. The main results are some L^p_{loc} estimates of the derivatives of the solution u and its Hölder continuity in terms of some L^q_{loc} norm of f. The key tools are some deep continuity results for singular integrals. The same techniques, suitably adapted, were used in [35] and in [44] in order to prove interior regularity results for weak solutions to the equation $Lu = \sum_{i=1}^{p_0} \partial_{x_i} F_i$ with L as in (3.3).

4. Kolmogorov equations with measurable coefficients

As said in the previous section, the Hölder estimates for weak solutions to (3.3) have been used for the study of nonlinear KEs. However the dependence of the Hölder constant on the regularity of the coefficients forces quite restrictive hypotheses on the nonlinearity. In order to remove such restrictions, regularity results for solutions to linear equations with merely measurable coefficients are needed. A first result in such a direction has been recently proved by the author in collaboration with S. Polidoro. In [39], the local boundedness of the weak solutions to L is proved only assuming the uniform positivity condition (1.2). The main result in [39] is the following theorem.

Theorem 4.1. *Let u be a non-negative weak solution to*

$$\sum_{i,j=1}^{p_0} \partial_{x_i} \left(a_{ij}(z) \partial_{x_j} \right) + \langle x, B\nabla u \rangle - \partial_t u = 0 \tag{4.1}$$

in a domain Ω. Let r, ρ, $0 < \frac{r}{2} \leq \rho < r$, be such that $\overline{H_r} \subseteq \Omega$ where H_r denotes a suitable cylindrical domain of radius r. Then there exists a positive constant C, only dependent on μ and on the homogeneous dimension Q (cf. (2.6)) such that, *for every* $p > 0$*, it holds*

$$\sup_{H_\rho} u^p \leq \frac{C}{(r-\rho)^{Q+2}} \int_{H_r} u^p. \tag{4.2}$$

Estimate (4.2) *also holds for every* $p < 0$ *such that* $u^p \in L^1(H_r)$.

This theorem is proved in [39] by using an iterative procedure analogous to the one introduced by Moser in the classical elliptic and parabolic cases. As it is well known, the Moser's technique is based on a combination of Caccioppoli type estimates with the classical Sobolev inequality. Actually the weak solutions to (4.1) satisfy a Caccioppoli type estimate, however this estimate only gives a L^2_{loc} bound of the first order derivatives $\partial_{x_j} u$ for $j = 1, \ldots, p_0$ and does not give any information on the others $(N - p_0)$ spatial derivatives. Thus, if $p_0 < N$, this lack of information cannot be restored by the usual Sobolev embedding theorem.

The key idea in [39] *is to prove a Sobolev type inequality for non negative sub- and super-solutions to* (4.1), good enough to be successfully combined with the previous "weak" Caccioppoli inequality. To be more specific, let us first recall the definition of weak sub- and super-solution to. We say that a function $u \in L^2_{\mathrm{loc}}(\Omega)$, Ω open subset of $\mathbb{R}^{N+1}$, is a *weak sub-solution* to (4.1) if the weak derivatives $\partial_{x_1} u, \ldots, \partial_{x_{p_0}} u$ and Yu exist, belong to $L^2_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega -\langle ADu, D\varphi\rangle + \varphi Y u \geq 0, \qquad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0.$$

If $-u$ is a weak sub-solution, we say that u is a weak super-solution. Then, the following Caccioppoli type estimate holds (cf. [39], Proposition 3.2)

Proposition 4.2. *Let* u *be a non-negative weak sub-solution of* (4.1) *in* Ω. *Let* $\rho, r > 0$, $\frac{r}{2} \leq \rho < r$, *and* $\overline{H_r} \subseteq \Omega$. *Then, there exists a constant* C*, only dependent on* μ *in* (1.2) *and on the homogeneous dimension* Q*, such that*

$$\|\partial_{x_j} u^p\|_{L^2(H_\rho)} \leq \frac{C\sqrt{1+\varepsilon}}{\varepsilon} \|u^p\|_{L^2(H_r)}, \quad \textit{where} \quad \varepsilon = \frac{|2p-1|}{4p}, \tag{4.3}$$

for every $j = 1, \ldots, p_0$ *and* $p < 0$ *or* $p \geq 1$. *The same inequality holds for non-negative weak super-solutions and* $p \in]0, 1/2[$.

The key Sobolev type inequality for weak sub- and super-solutions proved in [39] is the following.

Proposition 4.3. *Let* u *be a non-negative weak sub-solution to* (4.1) *and let* r, ρ *be as in the previous Proposition* 4.2. *Then* $u \in L^{2\kappa}_{\mathrm{loc}}(H_\rho)$, $\kappa = 1 + \frac{2}{Q}$, *and there exists a constant* C*, only dependent on* μ *and* Q*, such that*

$$\|u\|_{L^{2\kappa}(H_\rho)} \leq \frac{c}{r-\rho} \left(\|u\|_{L^2(H_r)} + \sum_{j=1}^{p_0} \|\partial_{x_j} u\|_{L^2(B_r)} \right). \tag{4.4}$$

The same inequality holds for non-negative super-solutions.

Inequalities (4.3)–(4.4) allow to start up an iterative procedure analogous to the classical Moser's one and to prove Theorem 4.1. We also recall that Theorem 4.1 has been used in [37] to obtain a pointwise global upper bound for the fundamental solution of (4.1).

Theorem 4.4. *There exists two positive constants C and ε, only dependent on μ in* (1.2) *and on B, such that*

$$\Gamma(x,t,\xi,\tau) \le C\ \Gamma^\varepsilon(x,t,\xi,\tau), \qquad \forall x,\xi \in \mathbb{R}^N,\ t>\tau,$$

where Γ^ε is the fundamental solution to (3.1).

We remark explicitly that Theorem 4.4 improves inequality (3.2) in that C is independent of the modulus of continuity of the coefficients.

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