Mathematical analysis and numerical methods for a PDE model governing a ratchet-cap pricing in the Libor Market Model

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In this paper we present a PDE formulation for the ratchet cap pricing problem. The underlying LIBOR interest rates are assumed to follow the LIBOR market model. For this PDE problem the existence and uniqueness of solution are obtained in the classical framework of uniformly parabolic PDEs in terms of a sequence of nested Cauchy problems. Moreover, this approach allows to obtain a new numerical method based on the approximation by computable fundamental solutions of constant coefficient operators. This method is compared with classical Monte Carlo simulation and a proposed characteristics Crank-Nicolson time discretization combined with finite elements strategy.

Keywords: interest rate derivatives; ratchet caps; Libor market model; Black-Scholes equations; fundamental solutions; numerical methods

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1. Introduction

The payoff of an interest rate derivative depends on the level of certain interest rates. Generally speaking, the study of interest rate derivatives is more difficult than equity derivatives because the evolution of interest rates is more complex to describe than that of stocks and typically leads to multi-dimensional problems. Moreover, technical issues arise due to the fact that interest rates are also involved in the pricing formulas as discount factors of the payoff. Among a variety of interest rates types, the LIBOR (London Interbank Offer Rate) is the rate at which large international banks lend money to each other. Moreover, forward rates are rates which are valid for future periods, which in turn are implicit in certain spot rates and therefore can be computed from them.

Since the seminal papers by Brace, Gatarek and Musiela[4], Jashmidian[9] and Miltersen, Sandmann and Sondermann[10], the LIBOR Market Model (LMM) has become one of the more popular interest rate market models due to its agreement with market pricing formulas for caps. Also it is referred as Log-normal Forward LIBOR Model (LFM) in the book of Brigo and Mercurio[5]. More precisely, it models the dynamics of LIBOR forward rates so that pricing caps and floors is consistent with Black formulas used in the market. Moreover, its parameters can be calibrated with market data and liquid products. Notice that LMM is not compatible with the Swap Market Model (also referred in Brigo and Mercurio[5] as Lognormal forward Swap Model (LSM)), as forward swap rates cannot be log-normal under their own measure in the LMM.

From the numerical point of view, in the LMM framework most of the pricing is carried out by means of Monte Carlo simulation taking advantage of its general applicability to almost all interest rate financial derivatives (see Brigo and Mercurio[5] or Pelsser[12], for example). However, the main limitation comes from the long computational times, specially when a lot of prices are required. Alternative numerical techniques in LMM setting turn out from the formulation of the pricing problem in terms of partial differential equations (PDE). This approach is more classically addressed in option pricing (see, for instance, Pascucci[11] and Wilmott[17]).

In the present paper we introduce the appropriate PDE model for pricing the interest rate derivative known as ratchet cap (compounded of ratchet caplets), which is described, for example, in Brigo and Mercurio[5]. The ratchet caplet payoff depends on a variable strike in terms of the reset value of all previous forward LIBOR rates. The number of involved rates increases as the time interval approaches to ratchet cap maturity. In the work of Pelsser[12], the particular choice of some parameter reduces this dependence to only the last two previous LIBOR rates. In this particular setting a parabolic PDE on two spatial dimensions (the two LIBOR rates) is obtained and a comparison between Monte Carlo simulation and explicit finite differences for the PDE model is presented in Pietersz[13] . More recently, in Taboada and Vázquez[15] a Crank-Nicholson-characteristics method has been proposed for the same problem.

In this paper we mainly address the general case where the strike depends on all previous LIBOR rates so that the dimension of the PDE domain increases with the index of the forward LIBOR rate. In this more general setting, we obtain the existence and uniqueness of a classical solution for the PDE problem. Moreover, this solution is expressed in terms of the fundamental solutions of the associated operators and provides a numerical algorithm to compute the solution. Then, for the particular case considered by Pelsser[12], the results obtained with a new numerical method based on an analytical approximation by using the fundamental solutions associated to constant coefficient operators, the Crank-Nicholson-characteristics method combined with finite elements and Monte Carlo simulation are compared.

The paper is organized as follows. Section 2 is devoted to the presentation of some basic ideas, the main notation concerning LMM and the description of the ratchet cap and ratchet caplet contracts. Section 3 contains the rigorous statement of the PDE models for ratchet caplets pricing and the prove of existence and uniqueness of solution. Section 4 describes the two proposed numerical methods for solving the PDE problem. Section 5 shows some numerical examples to illustrate the comparison of the numerical methods and Monte Carlo simulation. Section 6 contains the conclusions.

2. Financial product

As it has been mentioned in the introduction, a ratchet cap depends on one or more LIBOR forward rates. So, we first introduce some notation related to LMM and then we describe the ratchet caps.

2.1. *Some basics on LMM for forward rates*

We assume there are N forward LIBOR rates associated to a tenor structure $\{T_0, T_1, ..., T_N\}$, with $0 < T_0 < T_1 < \cdots < T_N$. The i-th forward rate accounts for the period $[T_{i-1}, T_i]$ and we denote by $\left(L_t^i\right)_{t\leq T_{i-1}}$ the value process of the $i-$th forward LIBOR rate. If we consider the bond $B^i(t)$ that matures at time T_i as numeraire, then the no arbitrage hypotheses guarantees the existence of a martingale measure Q^i associated with the numeraire B_i, such that the process $\left(L_t^i\right)$ is a martingale under Q^i.

In the standard LIBOR market model, the dynamics of the forward rates under the martingale probability Q^i, are given by the stochastic differential equation

$$dL_t^i = L_t^i \sigma^i(t) d\mathcal{B}_t^i,$$

where

- $\mathcal{B} = \left(\mathcal{B}^1, \ldots, \mathcal{B}^N\right)$ is a N-dimensional correlated Brownian motion with covariance matrix ρ (i.e. $d\mathcal{B}_t^i d\mathcal{B}_t^j = \rho_{i,j} dt$);
- σ^i is the deterministic volatility of the i-th LIBOR forward rate;
- $\delta_i = T_i - T_{i-1}$ the i-th accrual factor.

By a change of numeraire technique, for $j < i$ we also have

$$dL_t^j = -L_t^j \sigma^j(t) \sum_{h=j+1}^{i} \frac{\rho_{j,h} \delta_h \sigma^h(t) L_t^h}{1 + \delta_h L_t^h} dt + L_t^j \sigma^j(t) d\mathcal{B}_t^j.$$

2.2. *The ratchet cap and ratchet caplet contracts*

A ratchet cap is a contract that can be decomposed into simpler contracts, called ratchet caplets. The payments associated to each ratchet caplet are similar to caplet payments where the variable strike depends on earlier LIBOR resets (see Pelsser[12], for example). For convenience, we denote $\bar{L}^i = L^i_{T_{i-1}}$ for $i = 1, \dots, N$. Then, the ratchet caplet payoff, paid at time T_i, is given by $(\bar{L}^i - K_i)^+$, where the strike K_i is recursively defined as follows:

$$\begin{cases} K_1 & \text{is given,} \\ K_{j+1} = \left(a\bar{L}^j + bK_j + c\right)^+ & \text{for } 1 < j < i. \end{cases} \tag{2.1}$$

with a, b and c real parameters. In particular, we remark that K_{j+1} is a function of $\bar{L}^1, \dots, \bar{L}^j$.

3. PDE models for ratchet caplets pricing

In this section the PDE model and the main results concerning the existence and uniqueness of solution to the PDE model governing a ratchet caplet price are presented.

First, by usual no-arbitrage arguments, the *discounted* price of the i-th ratchet caplet is given by

$$\Pi^i_t = E^{Q^i}\left[(\bar{L}^i - K_i)^+ \mid \mathcal{F}_t\right], \qquad t \le T_{i-1},$$

and the *absolute* price is equal to $\Pi^i_t B^i_t$. Since we are in a Markovian framework, we have a representation of the price in terms of solutions of a sequence of Cauchy problems. In the following theorem, we denote by L_i the real variable corresponding to the i-th forward LIBOR rate for $i = 1, \dots, N$ and we set $T_{-1} = 0$ by convention.

Theorem 3.1. *For a fixed index $i \in \{1, \dots, N\}$, assume that for $j = 1, \dots, i$ the matrix $(\rho_{h,k}\sigma^h(t)\sigma^k(t))_{h,k=j,\dots,i}$ is bounded and uniformly positive definite. Then we have*

$$\Pi^i_t = u^{i,j}(t, L^j_t, L^{j+1}_t, \dots, L^i_t; K_j), \qquad t \in [T_{j-2}, T_{j-1}],\ j = 1, \dots, i, \tag{3.1}$$

where $K_j = K_j\left(\bar{L}^1, \dots, \bar{L}^{j-1}\right)$ is defined in (2.1) and the function

$$u^{i,j} = u^{i,j}(t, L_j, L_{j+1}, \dots, L_i; K), \qquad t \in [T_{j-2}, T_{j-1}],\ L_j, L_{j+1}, \dots, L_i > 0,\ K \ge 0,$$

is uniquely defined by the following backward recursion starting from $j = i$:

- *$u^{i,i}$ is the unique non-negative solution to the Cauchy problem*

$$\begin{cases} \mathcal{L}^{i,i} u^{i,i} = 0, & \text{in }]T_{i-2}, T_{i-1}[\times\mathbb{R}_+, \\ u^{i,i}(T_{i-1}, L_i; K) = (L_i - K)^+, & \text{in } \mathbb{R}_+ \end{cases} \tag{3.2}$$

where $\mathcal{L}^{i,i}$ is the two-dimensional operator

$$\mathcal{L}^{i,i} = \frac{\left(\sigma^i(t)L_i\right)^2}{2}\partial_{L_i L_i} + \partial_t; \tag{3.3}$$

- $u^{i,j}$, *with* $j < i$, *is the unique non-negative solution to the Cauchy problem*

$$\begin{cases} \mathcal{L}^{i,j} u^{i,j} = 0, & \text{in }]T_{j-2}, T_{j-1}[\times \mathbb{R}_+^{i-j+1}, \\ u^{i,j}(T_{j-1}, L_j, L_{j+1}, \dots, L_i; K) = & \\ \qquad = u^{i,j+1}(T_{j-1}, L_{j+1}, L_{j+2}, \dots, L_i; (aL_j + bK + c)^+), & \text{in } \mathbb{R}_+^{i-j+1}, \end{cases} \tag{3.4}$$

where $\mathcal{L}^{i,j}$ *is the following* $(i-j+2)$*-dimensional operator acting in the variables* $t, L_j, L_{j+1}, \dots, L_i$*:*

$$\mathcal{L}^{i,j} = \frac{1}{2} \sum_{h,k=j}^{i} \rho_{h,k} \sigma^h(t) \sigma^k(t) L_h L_k \partial_{L_h L_k} - \\ - \sum_{h=j}^{i-1} \sum_{k=h+1}^{i} \rho_{h,k} \sigma^h(t) \sigma^k(t) \frac{\delta_k L_k}{1 + \delta_k L_k} L_h \partial_{L_h} + \partial_t. \tag{3.5}$$

Proof. First step. We show that the functions $\{u^{i,j}\}_{j=1,\dots,i}$ are well defined recursively as the solutions of problems (3.2)-(3.5).

First, in the case $i = j$, by the change of variable $x_i = \log L_i$, i.e. setting

$$\bar{u}^{i,i}(t, x_i; K) = u^{i,i}(t, e^{x_i}; K), \qquad x_i \in \mathbb{R}, \tag{3.6}$$

problem (3.2) becomes

$$\begin{cases} \bar{\mathcal{L}}^{i,i} \bar{u}^{i,i} = 0, & \text{in }]T_{i-2}, T_{i-1}[\times \mathbb{R}, \\ \bar{u}^{i,i}(T_{i-1}, x_i; K) = (e^{x_i} - K)^+, & \text{in } \mathbb{R}, \end{cases} \tag{3.7}$$

where $\bar{\mathcal{L}}^{i,i}$ is the backward heat operator

$$\bar{\mathcal{L}}^{i,i} = \frac{(\sigma^i(t))^2}{2} (\partial_{x_i x_i} - \partial_{x_i}) + \partial_t. \tag{3.8}$$

Since by assumption $(\sigma^i)^2$ is bounded from above and below by positive constants, standard results of the theory of parabolic PDEs (see, for instance, Chap.6 in Pascucci [11]) ensure that problem (3.7) has a unique non-negative classical solution given by

$$\bar{u}^{i,i}(t, x_i; K) = \int_{\mathbb{R}} \bar{\Gamma}^{i,i}(t, x_i; T_{i-1}, y_i) \left(e^{y_i} - K\right)^+ dy_i, \tag{3.9}$$

where $\bar{\Gamma}^{i,i}$ denotes the Gaussian fundamental solution of $\bar{\mathcal{L}}^{i,i}$:

$$\bar{\Gamma}^{i,i}(t, x; T, y) = \frac{1}{\bar{\sigma}^i(t,T)\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{y - x + \frac{1}{2} \left(\bar{\sigma}^i(t,T)\right)^2}{\bar{\sigma}^i(t,T)} \right)^2 \right), \tag{3.10}$$

for $x, y \in \mathbb{R}$ and $t < T$, with

$$\bar{\sigma}^i(t,T) = \sqrt{\int_t^T \left(\sigma^i(s)\right)^2 ds}\,. \tag{3.11}$$

From the representation formula (3.9) we also get the following estimate:

$$\left|\bar{u}^{i,i}(t,x_i;K)\right| \le Ce^{C|x_i|^2}, \qquad (t,x_i)\in[T_{i-2},T_{i-1}]\times\mathbb{R},\ K\ge 0, \tag{3.12}$$

for some positive constant C (see, for instance, Pascucci[11] Chap. 8).

Next, by backward induction we show existence and uniqueness of $u^{i,j}$ for $j<i$. First, by the change of variables $x_j=\log L_j$, that is, by setting

$$\bar{u}^{i,j}\left(t,x_j,x_{j+1},\dots,x_i;K\right)=u^{i,j}(t,e^{x_j},e^{x_{j+1}},\dots,e^{x_i};K), \qquad x_j,x_{j+1},\dots,x_i\in\mathbb{R},$$

problem (3.4) can be rewritten as follows:

$$\begin{cases}\bar{\mathcal{L}}^{i,j}\bar{u}^{i,j}=0, & \text{in }]T_{j-2},T_{j-1}[\times\mathbb{R}^{i-j+1},\\ \bar{u}^{i,j}(T_{j-1},x_j,x_{j+1},\dots,x_i;K)= & \\ \qquad =\bar{u}^{i,j+1}\left(T_{j-1},x_{j+1},x_{j+2},\dots,x_i;(ae^{x_j}+bK+c)^+\right), & \text{in } \mathbb{R}^{i-j+1}\end{cases} \tag{3.13}$$

where

$$\bar{\mathcal{L}}^{i,j}=\frac{1}{2}\sum_{h,k=j}^{i}\rho_{h,k}\sigma^h(t)\sigma^k(t)\partial_{x_hx_k}-\frac{1}{2}\sum_{k=j}^{i}(\sigma^k(t))^2\partial_{x_k}- -\sum_{h=j}^{i-1}\sum_{k=h+1}^{i}\rho_{h,k}\sigma^h(t)\sigma^k(t)\frac{\delta_ke^{x_k}}{1+\delta_ke^{x_k}}\partial_{x_h}+\partial_t$$

is a second order differential operator that, by assumption, is uniformly parabolic and has bounded coefficients (note in particular, that $\frac{\delta_ke^{x_k}}{1+\delta_ke^{x_k}}\in]0,1[$ for $x_k\in\mathbb{R}$). Then we recall that $\bar{\mathcal{L}}^{i,j}$ has a fundamental solution $\bar{\Gamma}^{i,j}$ satisfying the following Gaussian upper bound:

$$\bar{\Gamma}^{i,j}(t,x_j,x_{j+1},\dots,x_i;T,y_j,y_{j+1},\dots,y_i)\le \le C\Gamma^{i,j}_{\text{heat}}(t,x_j,x_{j+1},\dots,x_i;T,y_j,y_{j+1},\dots,y_i) \tag{3.14}$$

for $x_j,y_j,\dots,x_i,y_i\in\mathbb{R}$, where C is a positive constant only dependent on $T-t$ and $\Gamma^{i,j}_{\text{heat}}$ is the Gaussian fundamental solution of a suitable parabolic operator with constant coefficients (cf., for instance, Chap. 8 in Pascucci [11]).

Now let us assume that $\bar{u}^{i,j+1}$ is a continuous and non-negative function satisfying the growth condition

$$\left|\bar{u}^{i,j+1}\left(t,x_{j+1},\dots,x_i;K\right)\right|\le Ce^{C|(x_{j+1},\dots,x_i)|^2} \tag{3.15}$$

for $(t,x_{j+1},\dots,x_i)\in[T_{j-1},T_j]\times\mathbb{R}^{i-j}$ and $K\ge0$, with C some positive constant. Then problem (3.13) has unique non-negative classical solution given by

$$\bar{u}^{i,j}(t,x_j,\dots,x_i;K):=\int_{\mathbb{R}^{i-j+1}}\bar{\Gamma}^{i,j}(t,x_j,\dots,x_i;T,y_j,\dots,y_i)\cdot \cdot\ \bar{u}^{i,j+1}\left(T_{j-1},y_{j+1},\dots,y_i;(ae^{y_j}+bK+c)^+\right)dy_j\cdots dy_i \tag{3.16}$$

for $K \geq 0$ and $(t, x_j, \dots, x_i) \in [T_{j-2}, T_{j-1}] \times \mathbb{R}^{i-j+1}$. Moreover, combining estimates (3.14) and (3.15) with formula (3.16), we also deduce

$$\left|\bar{u}^{i,j}(t, x_j, \dots, x_i; K)\right| \leq Ce^{C|(x_j,\dots,x_i)|^2},$$

for $(t, x_j, \dots, x_i) \in [T_{j-2}, T_{j-1}] \times \mathbb{R}^{i-j+1}$ and $K \geq 0$, with some positive constant C. From the above results, a simple backward inductive argument shows that the functions $\{u^{i,j}\}_{j=1,\dots,i}$ are well defined as in the statement, in a unique way.

Second step. We prove formula (3.1) by backward induction on j. Since $u^{i,i}(t, L_i; K)$ is a classical solution to problem (3.2), by Feynman-Kač theorem we have

$$u^{i,i}(t, L_i; K_i) = E^{Q^i}\left[(L^i_{T_{i-1}} - K_i)^+ \mid \mathcal{F}_t\right] = \Pi^i_t, \qquad t \in [T_{i-2}, T_{i-1}],$$

and this proves the thesis for $j = i$.

Now we assume that (3.1) is valid for a generic $j+1$ and we prove it for j. Hence we assume that

$$\Pi^i_t = u^{i,j+1}(t, L^{j+1}_t, L^{j+2}_t, \dots, L^i_t; K_{j+1}), \qquad t \in [T_{j-1}, T_j].$$

Since the process Π^i is a Q^i-martingale, we have for $t \in [T_{j-2}, T_{j-1}]$

$$\begin{aligned}
\Pi^i_t &= E^{Q^i}\left[\Pi^i_{T_{j-1}} \mid \mathcal{F}_t\right] = \\
&= E^{Q^i}\left[u^{i,j+1}(T_{j-1}, L^{j+1}_{T_{j-1}}, L^{j+2}_{T_{j-1}}, \dots, L^i_{T_{j-1}}; K_{j+1}) \mid \mathcal{F}_t\right] = \\
&= E^{Q^i}\left[u^{i,j+1}\left(T_{j-1}, L^{j+1}_{T_{j-1}}, L^{j+2}_{T_{j-1}}, \dots, L^i_{T_{j-1}}; \left(aL^j_{T_{j-1}} + bK_j + c\right)^+\right) \mid \mathcal{F}_t\right] = \\
&= u^{i,j}(t, L^j_t, L^{j+1}_t, \dots, L^i_t; K_j),
\end{aligned}$$

where we have sequentially used the induction hypothesis, the expression of K_{j+1} and the Feynman-Kač theorem (since $u^{i,j}$ is the unique non-negative classical solution to problem (3.4)). □

Remark 3.1. Theorem 3.1 provides an algorithm for the computation of the ratchet price via PDEs techniques. Indeed, according to formula (3.1), Π^i can be computed by solving recursively the Cauchy problems (3.2)-(3.4) starting from the last period $[T_{i-2}, T_{i-1}]$ back to the first period $[0, T_0]$. Notice that at each step the dimension of the Cauchy problems increases by one due to the dependence of an additional forward rate. We emphasize that the problem (3.2)-(3.4) depends on the parameter K. Moreover, at each step it must be solved for *any* value of K since the solution $u^{i,j+1}(T_j, L_{j+1}, L_{j+2}, \dots, L_i; K)$ enters as final condition of the subsequent Cauchy problem (posed on $]T_{j-2}, T_{j-1}[\times \mathbb{R}^{i-j+1}_+$) as a function of $L_{j+1}, L_{j+2}, \dots, L_i$ and K. This fact puts severe restrictions on the applicability of standard numerical techniques for PDEs.

Remark 3.2. By formula (3.1) the discounted price Π^i_t of the i-th ratchet cap in the last period $[T_{i-2}, T_{i-1}]$ is given in terms of the solution $u^{i,i}$ of problem (3.7) and this corresponds to the price of a standard caplet with strike K_i. More precisely, by the Black formula, we have

$$\Pi_t^i = u^{i,i}(t, L_t^i; K_i) = L_t^i \mathcal{N}(d^+(t, L_t^i)) - K_i \mathcal{N}(d^-(t, L_t^i)) \qquad t \in [T_{i-2}, T_{i-1}], \tag{3.17}$$

where $\mathcal{N}$ denotes the normal cumulative distribution function and

$$d^{\pm}(t, L_i) = \frac{\log(\frac{L_i}{K_i}) \pm \frac{1}{2}\left(\bar{\sigma}^i(t, T_{i-1})\right)^2}{\bar{\sigma}^i(t, T_{i-1})}$$

with $\bar{\sigma}^i(t, T_{i-1})$ as in (3.11).

Remark 3.3. The pricing problem gets definitely easier when the parameter b is null since in this case the strike K_i depends only on the forward rate L^{i-1} and not on the previous rates. More precisely, in this case the payoff of the i-th ratchet caplet is equal to

$$(\bar{L}^i - K_i)^+ \quad \text{with} \quad K_i = (a\bar{L}^{i-1} + c)^+.$$

Then the discounted price Π_t^i is given by the Black formula (3.17) for $t \in [T_{i-2}, T_{i-1}]$ and by

$$\Pi_t^i = E^{Q^i}\left[\left(L_{T_{i-1}}^i - (aL_{T_{i-2}}^{i-1} + c)^+\right)^+ \mid \mathcal{F}_t\right] = u^{i,i-1}(t, L_t^{i-1}, L_t^i),\ t \in [0, T_{i-2}], \tag{3.18}$$

where $u^{i,i-1}$ is the non-negative solution of the Cauchy problem

$$\begin{cases} \mathcal{L}^{i,i-1} u^{i,i-1} = 0, & \text{in }]0, T_{i-2}[\times \mathbb{R}_+^2, \\ u^{i,i-1}(T_{i-2}, L_{i-1}, L_i) = u^{i,i}(T_{i-2}, L_i; (aL_{i-1} + c)^+), & \text{in } \mathbb{R}_+^2, \end{cases} \tag{3.19}$$

with $u^{i,i}$ as in (3.17) and

$$\begin{aligned} \mathcal{L}^{i,i-1} &= \frac{1}{2}\left(\sigma^{i-1}(t) L_{i-1}\right)^2 \partial_{L_{i-1}L_{i-1}} + \rho_{i-1,i}\sigma^{i-1}(t)\sigma^i(t) L_{i-1} L_i \partial_{L_{i-1}L_i} + \\ &+ \frac{1}{2}\left(\sigma^i(t) L_i\right)^2 \partial_{L_i L_i} - \rho_{i-1,i}\sigma^{i-1}(t)\sigma^i(t)\frac{\delta_i L_i}{1 + \delta_i L_i} L_{i-1}\partial_{L_{i-1}} + \partial_t. \end{aligned} \tag{3.20}$$

Notice that in the case $b = 0$, as the strike only depends on L_{i-1}, the definition of Π_t^i in (3.1) written in terms of $u^{i,j}$ in $[T_{j-2}, T_{j-1}]$, actually does not depend on j. This is taken in account in (3.18) where notation $u^{i,i-1}$ is used for the large interval $[0, T_{i-2}]$. Thus, by the change of variables

$$\bar{u}^{i,i-1}(t, x_{i-1}, x_i; K) = u^{i,i-1}(t, e^{x_{i-1}}, e^{x_i}; K), \qquad x_{i-1}, x_i \in \mathbb{R}, t < T_{i-2},$$

problem (3.4) can be rewritten as follows:

$$\begin{cases} \bar{\mathcal{L}}^{i,i-1}\bar{u}^{i,i-1} = 0, & \text{in }]0, T_{i-2}[\times \mathbb{R}^2, \\ \bar{u}^{i,i-1}(T_{i-2}, x_{i-1}, x_i; K) = u^{i,i}\left(T_{i-2}, e^{x_i}; (ae^{x_{i-1}} + c)^+\right), & \text{in } \mathbb{R}^2, \end{cases} \tag{3.21}$$

where

$$\begin{aligned}\bar{\mathcal{L}}^{i,i-1} =& \frac{\left(\sigma^{i-1}(t)\right)^2}{2}\left(\partial_{x_{i-1}x_{i-1}} - \partial_{x_{i-1}}\right) + \frac{\left(\sigma^{i}(t)\right)^2}{2}\left(\partial_{x_i x_i} - \partial_{x_i}\right) + \\ &+ \rho_{i-1,i}\sigma^{i-1}(t)\sigma^i(t)\partial_{x_{i-1}x_i} - \rho_{i-1,i}\sigma^{i-1}(t)\sigma^i(t)\frac{\delta_i e^{x_i}}{1+\delta_i e^{x_i}}\partial_{x_{i-1}} + \partial_t .\end{aligned} \tag{3.22}$$

In terms of the representation formulas (3.9)-(3.16), we have

$$\begin{aligned}&\bar{u}^{i,i-1}(t, x_{i-1}, x_i; K) = \\ &= \int_{\mathbb{R}^2} \bar{\Gamma}^{i,i-1}(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, y_i) \cdot u^{i,i}\left(T_{i-2}, e^{y_i}; (ae^{y_{i-1}} + c)^+\right) dy_i dy_{i-1} = \\ &= \int_{\mathbb{R}^2} \bar{\Gamma}^{i,i-1}(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, y_i) \cdot \\ &\qquad \cdot \int_{\mathbb{R}} \bar{\Gamma}^{i,i}(T_{i-2}, y_i; T_{i-1}, \eta_i)\left(e^{\eta_i} - (ae^{y_{i-1}} + c)^+\right)^+ d\eta_i dy_i dy_{i-1} ,\end{aligned} \tag{3.23}$$

where $\bar{\Gamma}^{i,i}$ is the Gaussian fundamental solution of $\bar{\mathcal{L}}^{i,i}$, whose explicit expression is given in (3.10) and $\bar{\Gamma}^{i-1,i}$ is the (unknown) fundamental solution of $\bar{\mathcal{L}}^{i,i-1}$.

4. Numerical methods

In this section, we consider analytical and numerical approximations of the ratchet price in the case the parameter b is equal to zero in the expression (2.1). Throughout this section, for sake of simplicity, we consider volatilities constant in time. By Remark 3.3, the ratchet caplet price is given in terms of the solution $\bar{u}^{i,i-1}$ to the Cauchy problem (3.21).

4.1. *Analytical approximation*

We get an analytical approximation of the ratchet caplet price by starting from the integral representation (3.23) for $\bar{u}^{i,i-1}$. We first recall the expression of the Gaussian fundamental solution $\bar{\Gamma}^{i,i}$ of the one dimensional heat operator

$$\bar{\mathcal{L}}^{i,i} = \frac{(\sigma^i)^2}{2}(\partial_{y_i y_i} - \partial_{y_i}) + \partial_t,$$

given by (3.10) with $\bar{\sigma}^i(t,T) = \sigma^i\sqrt{T_{i-1} - t}$ (cf. (3.11)), so that

$$\bar{\Gamma}^{i,i}(T_{i-2}, y_i; T_{i-1}, \eta_i) = \frac{1}{\sigma^i\sqrt{2\pi\delta_{i-1}}}\exp\left[-\frac{1}{2}\left(\frac{2(\eta_i - y_i) + (\sigma^i)^2\delta_{i-1}}{2\sigma^i\sqrt{\delta_{i-1}}}\right)^2\right], \tag{4.1}$$

for $y_i, \eta_i \in \mathbb{R}$ and $T_{i-2} < T_{i-1}$ ($\delta_{i-1} = T_{i-1} - T_{i-2}$).

Next we approximate the fundamental solution $\bar{\Gamma}^{i,i-1}$ by means of the fundamental solution $\widetilde{\Gamma}^{i,i-1}$ of the constant coefficients operator

$$\begin{aligned}\widetilde{\mathcal{L}}^{i,i-1} :=& \frac{\left(\sigma^{i-1}\right)^2}{2}\left(\partial_{x_{i-1}x_{i-1}} - \partial_{x_{i-1}}\right) + \frac{\left(\sigma^{i}\right)^2}{2}\left(\partial_{x_i x_i} - \partial_{x_i}\right) + \\ &+ \rho_{i-1,i}\ \sigma^{i-1}\sigma^{i}\ \partial_{x_{i-1}x_i} - \bar{c}_i\,\rho_{i-1,i}\ \sigma^{i-1}\sigma^{i}\partial_{x_{i-1}} + \partial_t,\end{aligned}$$

which is obtained by freezing the variable coefficient $\delta_i e^{x_i}(1+\delta_i e^{x_i})^{-1}$ appearing in (3.22) to the value defined by the spot, i.e:

$$\bar{c}_i = \frac{\delta_i L_i^0}{1+\delta_i L_i^0}. \tag{4.2}$$

So, its fundamental solution $\widetilde{\Gamma}^{i,i-1}$ is given by

$$\widetilde{\Gamma}^{i,i-1}(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, y_i) = \frac{\exp(F(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, y_i))}{2\pi\sigma^i\sigma^{i-1}(T_{i-2}-t)\sqrt{1-\rho_{i-1,i}^2}}, \tag{4.3}$$

for $x_{i-1}, x_i, y_{i-1}, y_i \in \mathbb{R}$, $t < T_{i-2}$, where:

$$\begin{aligned} &F(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, y_i) = \\ &= \frac{1}{8\left(1-\rho_{i-1,i}^2\right)}\Big[(\sigma^{i-1})^2(t-T_{i-2}) + 4(x_{i-1}+x_i-y_i-y_{i-1}) + \\ &+ 8\bar{c}_i\ \rho_{i-1,i}^2\ (y_{i-1}-x_{i-1}) + (\sigma^i)^2(t-T_{i-2})\left(1+4(-1+\bar{c}_i)\ \bar{c}_i\ \rho_{i-1,i}^2\right) + \\ &+ \frac{4(x_i-y_i)^2}{(\sigma^{i-1})^2(t-T_{i-2})} + \frac{4(x_{i-1}-y_{i-1})^2}{(\sigma^{i})^2(t-T_{i-2})} + \\ &+ \frac{2\sigma^i(-1+2\bar{c}_i)\left((\sigma^{i-1})^2(t-T_{i-2}) + 2(x_i-y_i)\right)\rho_{i-1,i}}{\sigma^{i-1}} - \\ &- \frac{4\left((\sigma^{i-1})^2(t-T_{i-2}) + 2(x_i-y_i)\right)(x_{i-1}-y_{i-1})\rho_{i-1,i}}{\sigma^i\sigma^{i-1}(t-T_{i-2})}\Big] \end{aligned} \tag{4.4}$$

Thus we get the following analytical approximation formula:

$$\begin{aligned}\bar{u}^{i,i-1}(t, x_{i-1}, x_i; K) \approx &\int_{\mathbb{R}^2} \widetilde{\Gamma}^{i,i-1}(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, y_i)\cdot \\ &\cdot \int_{\mathbb{R}} \bar{\Gamma}^{i,i}(T_{i-2}, y_i; T_{i-1}, \eta_i)\left(e^{\eta_i} - (a e^{y_{i-1}} + c)^+\right)^+ d\eta_i dy_i dy_{i-1}\end{aligned} \tag{4.5}$$

Notice that formula (4.5) involves a triple integral but two of the integrals can be computed analytically. We defer all the explicit formulas to the Appendix.

Finally, the approximation of the price is given by

$$\Pi_t^i \equiv u^{i,i-1}(t, L_{i-1}, L_i; K) = \bar{u}^{i,i-1}(t, \log\, L_{i-1}, \log\, L_i; K).$$

4.2. *Finite elements*

In this section we study the numerical solution to the ratchet pricing problem (3.19) by a Crank-Nicholson-characteristics method combined with finite elements. Some difficulties in the numerical solution are due to the fact that the spatial domain is unbounded in both forwards directions: therefore, domain truncature and boundary conditions are proposed as a previous step to perform a finite element numerical approximation of the solution. On the other hand, the diffusion matrix degenerates at the axis (i.e. convection dominates near the axes) so we propose a higher order Lagrange-Galerkin methods. Thus, we use a combination of the Crank-Nicholson characteristics method for the time discretization and piecewise quadratic finite elements method for the spatial discretization. In the literature we can find different applications of the classical method of characteristics of first order (introduced in [14]) for the resolution of financial problems (see D'Halluin, Forsyth and Labahn[6] or Vázquez[16], for example). Other alternative finite differences numerical for Kolmogorov equations appearing in a stochastic volatility model are proposed in Di Francesco and Pascucci[7], Di Francesco, Foschi and Pascucci[8]. Recently, the here used Crank-Nicholson characteristic methods of order two for general convection-diffusion-reaction equations (eventually degenerated) have been proposed and analyzed numerically in Bermúdez, Nogueiras and Vázquez[1,2]. Furthermore, they have been applied to price Asian options in Bermúdez, Nogueiras and Vázquez[3].

4.2.1. *Divergence form and truncated domain*

In order to rewrite the problem (3.21) as an initial value problem in divergence form, we introduce the new time variable $\tau = T_{i-2} - t$ and pose the equivalent problem:

$$\partial_\tau u^{i,i-1} + \vec{v} \cdot \nabla u^{i,i-1} - Div(A \nabla u^{i,i-1}) = 0 \quad \text{in} \quad]0, T_{i-2}[\times \mathbb{R}^2 \,, \tag{4.6}$$

$$u^{i,i-1}(0, L_{i-1}, L_i) = u^{i,i}\left(T_{i-2}, L_i; (aL_{i-1} + c)^+\right) \quad \text{in} \quad \mathbb{R}^2 \,, \tag{4.7}$$

where:

$$A(L_{i-1}, L_i) = \begin{pmatrix} \frac{1}{2}(\sigma^{i-1})^2 L_{i-1}^2 & \frac{1}{2}\rho_{i-1,i}\sigma_{i-1}\sigma_i L_{i-1} L_i \\ \frac{1}{2}\rho_i \sigma^{i-1}\sigma^i L_{i-1} L_i & \frac{1}{2}(\sigma^i)^2 L_i^2 \end{pmatrix} , \tag{4.8}$$

$$\vec{v}(L_{i-1}, L_i) = \begin{pmatrix} \frac{\delta_i L_{i-1} L_i}{1+\delta_i L_i}\sigma^{i-1}\sigma^i + (\sigma^{i-1})^2 L_{i-1} + \frac{1}{2}\sigma^{i-1}\sigma^i L_{i-1} \\ \frac{1}{2}\sigma^{i-1}\sigma^i L_i + (\sigma^i)^2 L_i \end{pmatrix} . \tag{4.9}$$

Numerical discretization using finite differences, finite volumes or finite elements makes necessary truncating the spatial unbounded domain and introducing appropriate boundary conditions on the boundaries of the bounded domain. In general, these conditions are obtained with financial and/or mathematical arguments and are taking into account in the weak formulation of the problem.

The process of aproximating a problem posed on an unbounded domain by a problem posed on a bounded domain is called localization process. For this purpose, let us consider both L_{i-1}^{∞} and L_i^{∞} large enough real numbers suitably chosen and let the bounded domain $\Omega =]0, L_{i-1}^{\infty}[\times]0, L_i^{\infty}[$ with Lipschitz boundary Γ, such that $\Gamma = \Gamma_1^+ \bigcup \Gamma_2^+ \bigcup \Gamma_1^- \bigcup \Gamma_2^-$. Then, problem (4.6)-(4.7) is replaced by

Find $u^{i,i-1} : [0, T_{i-2}] \times \Omega \to \mathbb{R}$ such that

$$\partial_\tau u^{i,i-1} + \vec{v} \cdot \nabla u^{i,i-1} - Div(A \nabla u^{i,i-1}) = 0 \;\text{ in }]0, T_{i-2}] \times \Omega\,, \tag{4.10}$$

$$u(0, L_{i-1}, L_i) = u^{i,i}\left(T_{i-2}, L_i; (aL_{i-1} + c)^+\right) \;\text{ in } \Omega\,, \tag{4.11}$$

where A and $\vec{v}$ are defined in (4.8)-(4.9). Moreover, as the coefficients of second order terms vanish at the boundaries Γ_1^- and Γ_2^-, we just consider the additional boundary conditions

$$u^{i,i-1}(t, L_{i-1}, L_i) = 0 \;\text{ on } [0, T_{i-2}] \times \Gamma_1^+\,, \tag{4.12}$$

$$u^{i,i-1}(t, L_{i-1}, L_i) = M\delta_i(L_i - (aL_{i-1} + c)_+)_+ \;\text{ on } [0, T_{i-2}] \times \Gamma_2^+\,, \tag{4.13}$$

which are based on financial arguments.

4.2.2. *Crank-Nicholson characteristics discretization*

The method of characteristics is used for the time discretization and it is included in the more general setting of upwinding methods, which take in account the local direction of the flux. More precisely, it is based on a finite differences scheme for the discretization of the material derivative, i. e., the time derivative along the characteristic lines of the convective part of the equation. In this section we will also introduce the variational formulation for the time discretized problem.

First, we define the characteristics curve through $L = (L_{i-1}, L_i)$ at time $\bar{t}$, $X_e(L, \bar{t}; \tau)$, which verifies:

$$\partial_\tau X_e(L, \bar{t}; \tau) = \vec{v}(X_e(L, \bar{t}; \tau)), \quad X_e(L, \bar{t}; \bar{t}) = L\,. \tag{4.14}$$

The final value problem (4.14) can be exactly solved and we obtain:

$$X_e^1(L, \bar{t}; \tau) = L_{i-1} \exp\left(\frac{1}{2}\sigma^{i-1}\sigma^i + (\sigma^{i-1})^2 + \frac{\delta_i L_i \sigma^{i-1}\sigma^i}{1 + \delta_i L_i}(\bar{t} - \tau)\right)$$

$$X_e^2(L, \bar{t}; \tau) = L_i \exp\left((\frac{1}{2}\sigma^{i-1}\sigma^i + (\sigma^i)^2)(\bar{t} - \tau)\right)$$

Now, for $i = 1, ..., N$, let us consider the time step $\Delta t = \frac{T_{i-1}}{N}$ and the time meshpoints $t_n = n\Delta t$, $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N$. The material derivative approximation by characteristics method is given by:

$$\frac{Du^{i,i-1}}{Dt} \approx \frac{(u^{i,i-1})^{n+1} - (u^{i,i-1})^n \circ X_e^n}{\Delta t}$$

where $X_e^n(L) := X_e(L, t^{n+1}; t^n)$, the components of which are given by

$$X_e^{n,1}(L) = L_{i-1}\exp\left(\frac{1}{2}\sigma^{i-1}\sigma^i + (\sigma^{i-1})^2 + \frac{\delta_i L_i \sigma^{i-1}\sigma^i}{1+\delta_i L_i}\Delta t\right)$$

$$X_e^{n,2}(L) = L_i\exp\left((\frac{1}{2}\sigma^{i-1}\sigma^i + (\sigma^i)^2)\Delta t\right)$$

Next, we consider a Crank-Nicholson scheme around $(X_e(\mathbf{x}, t_{n+1};\tau),\tau)$ for $\tau = t_{n+\frac{1}{2}}$. So, for n=0,...,N-1, the time discretized equation can be written as:

Find $(u^{i,i-1})^{n+1}$ such that:

$$\begin{aligned}\frac{(u^{i,i-1})^{n+1}(L) - (u^{i,i-1})^n(X_e^n(L))}{\Delta t} - \frac{1}{2}Div(A\nabla(u^{i,i-1})^{n+1})(L) - \\ -\frac{1}{2}Div(A\nabla(u^{i,i-1})^{n})(X_e^n(L)) = 0\,.\end{aligned} \tag{4.15}$$

Now, multiplying equation (4.15) by a suitable test function ψ and integrating in Ω, we have:

$$\begin{aligned}\int_\Omega \frac{(u^{i,i-1})^{n+1} - \Pi_i^n \circ X_e^n}{\Delta t}\psi dL - \frac{1}{2}\int_\Omega Div(A\nabla(u^{i,i-1})^{n+1})\psi dL - \\ -\frac{1}{2}\int_\Omega Div(A\nabla(u^{i,i-1})^{n})(X_e^n(L))\psi dL = 0\,.\end{aligned} \tag{4.16}$$

Applying Lemma 3.4 that appears in Bermúdez, Nogueiras and Vázquez[3] and the usual Green's formula, equation (4.16) is equivalent to:

$$\begin{aligned}\int_\Omega \frac{(u^{i,i-1})^{n+1} - (u^{i,i-1})^n \circ X_e^n}{\Delta t}\psi dL + \frac{1}{2}\int_\Omega A\nabla(u^{i,i-1})^{n+1}\nabla\psi dL + \\ + \frac{1}{2}\int_\Omega (F_e^n)^{-1}(A\nabla(u^{i,i-1})^n)(X_e^n(L))\nabla\psi dL + \\ + \frac{1}{2}\int_\Omega Div(F_e^n)^{-t}(A\nabla(u^{i,i-1})^n)(X_e^n(L))\psi dL = \\ = \frac{1}{2}\int_\Gamma \vec{n}\cdot A\nabla(u^{i,i-1})^{n+1}\psi dA_L + \\ + \frac{1}{2}\int_\Gamma (F_e^n)^{-t}\vec{n}\cdot(A\nabla(u^{i,i-1})^n)(X_e^n(L))\psi dA_L \quad .\end{aligned} \tag{4.17}$$

Notice that the tensor $(F_e^n)^{-t}$ can be easily computed by

$$(\mathbf{F}_e^n)^{-t}(L) = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22}\end{pmatrix},$$

with

$$b_{11}(L) = \exp\left(\frac{1}{2}\sigma^{i-1}\sigma^{i} + (\sigma^{i-1})^2 + \frac{\delta_i L_i \sigma^{i-1}\sigma^{i}}{1+\delta_i L_i}\Delta t\right)$$
$$b_{21}(L) = L_{i-1}\Delta t\left(\frac{\delta_i \sigma^{i-1}\sigma^{i} 1 + \delta_i L_i - \delta_i^2 \sigma^{i-1}\sigma^{i} L_i}{1+\delta_i {L_i}^2}\right)\exp\left((\frac{1}{2}\sigma^{i-1}\sigma^{i} + (\sigma^{i})^2)\Delta t\right)$$
$$b_{22}(L) = \exp\left((\frac{1}{2}\sigma^{i-1}\sigma^{i} + (\sigma^{i})^2)\Delta t\right).$$

Next, let us precise the boundary integrals appearing in formulation (4.17). First, notice that we have $\vec{n}\cdot A\nabla(u^{i,i-1})^{n+1} = 0$ on $\Gamma_1^- \cup \Gamma_2^-$ and $\psi = 0$ on $\Gamma_1^+ \cup \Gamma_2^+$. Therefore, the first boundary integral on the right hand side of equation (4.17) vanishes. Moreover, for the second integral, we have

$$\int_{\Gamma} (F_e^n)^{-t}\vec{n}\cdot(A\nabla(u^{i,i-1})^n)(X_e^n(L))\psi dA_L = \int_{\Gamma_1^- \cup \Gamma_2^-} g^n \psi dA_L, \tag{4.18}$$

where $g^n :]0,\infty)\times]0,\infty) \to \mathbb{R}$ is given by,

$$\mathrm{g}^n(L) = \begin{cases} -\frac{1}{2}((F_e^n)^{-t})_{21}(\sigma^i)^2 L_i^2 \frac{\partial (u^{i,i-1})^n}{\partial L_i}(X_e^n(L)) & on \quad \Gamma_1^- \\ -\frac{1}{2}((F_e^n)^{-t})_{12}(\sigma^{i-1})^2 L_{i-1}^2 \frac{\partial (u^{i,i-1})^n}{\partial L_{i-1}}(X_e^n(L)) & on \quad \Gamma_2^- \end{cases}$$

Therefore, equation (4.17) becomes

$$\begin{aligned} \int_\Omega \frac{(u^{i,i-1})^{n+1} - (u^{i,i-1})^n \circ X_e^n}{\Delta t}\psi dL + \frac{1}{2}\int_\Omega A\nabla(u^{i,i-1})^{n+1}\nabla\psi dL + \\ + \frac{1}{2}\int_\Omega (F_e^n)^{-1}(A\nabla(u^{i,i-1})^n)(X_e^n(L))\nabla\psi dL + \\ + \frac{1}{2}\int_\Omega Div(F_e^n)^{-t}(A\nabla(u^{i,i-1})^n)(X_e^n(L))\psi dL = \\ = \frac{1}{2}\int_{\Gamma_1^-\cup\Gamma_2^-} g^n(L)\psi dA_L, \end{aligned} \tag{4.19}$$

for all $\psi \in H^1_{0,\Gamma_D}(\Omega)$, where the involved functional sets are,

$$H^1_{\Gamma_D}(\Omega) = \{\psi \in H^1(\Omega)/\psi_{|_{\Gamma_1^+}} = 0, \psi_{|_{\Gamma_2^+}} = M\delta_i(L_i - aL_{i-1} - c)_+\},$$
$$H^1_{0,\Gamma_D}(\Omega) = \{\psi \in H^1(\Omega)/\psi_{|_{\Gamma_D}} = 0\}.$$

4.2.3. *Finite elements discretization*

As we mention at the beginning of the section, we use the characteristics-Crank-Nicholson method for the time discretization jointly with finite elements for spatial discretization. For this purpose, we consider $\{\tau_h\}$ a quadrangular mesh of the domain Ω. Let $(T, \mathcal{Q}_2, \Sigma_T)$ be a family of quadratic Lagrangian finite elements, where $\mathcal{Q}_2$ is the space of polynomials defined in $T \in \tau_h$ with degree less or equal than two in each spatial variable and Σ_T the subset of nodes of the element T. Now, let us define the subset of finite elements V_h and the space of test functions V_{h,Γ_D}:

$$V_h = \{\varphi_h \in \mathcal{C}^0(\bar{\Omega}) : \varphi_{h_T} \in \mathcal{Q}_2, \forall T \in \tau_h\},$$
$$V_{h,\Gamma_D} = \{\varphi_h \in V_h : \quad \varphi_h = 0, \quad on \quad \Gamma_D\},$$

where $\mathcal{C}^0(\bar{\Omega})$ is the space of continuous functions on $\bar{\Omega}$. Further details about the application of this characteristics-Crank-Nicholson method to the ratchet caplet pricing problem can be found in Taboada and Vázquez[15]. The numerical analysis of the method for a more general equation can be found in Bermúdez, Nogueiras and Vázquez[1,2].

5. Numerical results

In this section we show some numerical tests to illustrate the performance of the analytical approximation of Section 4.1 and the finite elements solution of Section 4.2, in comparison with standard Monte Carlo simulation.

FORTRAN scientific computing language has been chosen for the implementation of the Lagrange-Galerkin numerical methods, MATLAB for Monte Carlo and MATHEMATICA for the computations related to the approximation by means of fundamental solutions. As indicated in the previous section, we only consider the case b equal to zero in the expression (2.1).

Moreover, concernig the data volatilities and correlations have been taken constant in time. In Table 1 and Table 2 we show the data for the different tests. More precisely, Table 1 shows the constant volatilities, correlations and accrual jointly with the constants appearing in the strike definition (2.1) while Table 2 shows the forward LIBOR spot values for the different tests.

Concerning to standard Monte Carlo, we determine the 99% confidence intervals related to 500.000 simulations. The computational time for each price is approximately 26 minutes on a Intel(R) Core(TM)2 Duo CPU T8100 2.10 GHz.

In the finite elements method, the spatial quadrangular meshes used are structured, uniform and with edges parallel to the axis with 4096 elements and 16641 nodes. Pricing a ratchet caplet using finite elements takes about 26 minutes. Notice that this method provides simultaneously the prices for the 16641 mesh nodes, prices for any LIBOR values can be easily obtained from them by interpolation.

The analytical approximation method results to be very fast as it gives one ratchet caplet price in about 0.031 seconds, while keeping a good level of precision, especially for not too long maturities.

Finally, comparison of the computed numerical results are shown in Table 3, Table 4 and Table 5 for different spot values of the forward LIBOR rates.

6. Conclusions

In this paper different approaches for the pricing of a particular interest rate derivative, the ratchet cap, in the framework of LMM for forward LIBOR interest rates have been considered. Besides classical Monte Carlo simulation, two methods based on an appropriate PDE formulation of the problem have been developed.

The PDE approach leads to a sequence of nested Cauchy problems for which existence and uniqueness results have been proved. This theoretical approach allows to approximate the analytical solution by freezing a variable coefficient and using the fundamental solutions associated to the resulting constant coefficient operators.

On the other hand, the departure point for the finite element approximation is the domain truncation to obtain an initial-boundary value problem and the consideration of suitable boundary conditions at the artificial boundaries. In this formulation setting, a Crank-Nicholson-characteristics time discretization combined with piecewise quadratic elements is the proposed numerical method.

The obtained results confirm that the computed prices for different data sets are very close each other for the three methods. The main pros of the finite elements approach are the precision and the fact that many prices can be computed in a single run at a moderate computational cost. The pros of the proposed analytical approximation method are the precision and the very fast computing.

Index frequency	Semi Annual
δ_i	0.5
$\rho_{i-1,i}$	0.8
σ^i	0.2
σ^{i-1}	0.2
a, b, c	0.9, 0.0, 0.01
t	0

Table 1. Numerical data (I).

	At-the-money	**In-the-money**	**Out-of-the-money**
L_0^{i-1}	0.05	0.03	0.06
L_0^i	0.05	0.05	0.05

Table 2. Numerical data (II).

T_{i-2}	T_{i-1}	**Monte Carlo**	**Anal. Appr.**	**Fin. Elem.**
0.5	1.0	[0.0014551, 0.0015599]	0.00150373	0.00151720
1.5	2.0	[0.0021582, 0.0023049]	0.00220159	0.00224088
2.5	3.0	[0.0027734, 0.0029599]	0.00280634	0.00287529
3.5	4.0	[0.0033225, 0.0035454]	0.00334588	0.00344715
4.5	5.0	[0.0038577, 0.0041172]	0.00383673	0.00397233

Table 3. Tests results for ratchet caplet i At-the-money ($L_0^{i-1} = 0.05$).

T_{i-2}	T_{i-1}	**Monte Carlo**	**Anal. Appr.**	**Fin. Elem.**
0.5	1.0	[0.012981, 0.013218]	0.0130957	0.0131012
1.5	2.0	[0.013164, 0.013462]	0.0132759	0.0132989
2.5	3.0	[0.013371, 0.001372]	0.0135010	0.0135448
3.5	4.0	[0.013606, 0.014004]	0.0137462	0.0138108
4.5	5.0	[0.013902, 0.014346]	0.0139994	0.0140839

Table 4. Tests results for ratchet caplet i In-the-money ($L_0^{i-1} = 0.03$).

T_{i-2}	T_{i-1}	**Monte Carlo**	**Anal. Appr.**	**Fin. Elem.**
0.5	1.0	[0.0002542, 0.0002981]	0.0002741	0.0002807
1.5	2.0	[0.0005890, 0.0006765]	0.0006273	0.0006430
2.5	3.0	[0.0009783, 0.0010905]	0.0010120	0.0010458
3.5	4.0	[0.0013795, 0.0015256]	0.0014000	0.0014557
4.5	5.0	[0.0017733, 0.0019517]	0.0017812	0.0018621

Table 5. Tests results for ratchet caplet i Out-of-the-money ($L_0^{i-1} = 0.06$).

7. Appendix: Computations related to the analytical approximation

In this appendix we detail the intermediate computations to obtain the ratchet caplet price via fundamental solutions approach.

First, we can compute explicitly

$$G_3(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, T_{i-1}, \eta_i) = \int_{\mathbb{R}} \widetilde{\Gamma}^{i,i-1}(t, x_{i-1}, x_i; T_{i-2}, y_{i-1}, y_i) \bar{\Gamma}^{i,i}(T_{i-2}, y_i; T_{i-1}, \eta_i) dy_i,$$

obtaining the following expression:

$$G_3(t,x_{i-1},x_i;T_{i-2},y_{i-1},T_{i-1},\eta_i) = \frac{\exp(M_i)}{(2\pi)^{3/2}(\sigma^i)^2\sigma^{i-1}\tau\sqrt{\delta_{i-1}(1-\rho_{i-1,i}^2)}}\,, \quad (7.1)$$

where we recall that $\tau = T_{i-2} - t$ and M_i is given by

$$\begin{aligned} M_i = &-\frac{\left(\frac{1}{2}(\sigma^i)^2\delta_{i-1}+\eta_i-y_i\right)^2}{2(\sigma^i)^2\delta_{i-1}} - \\ &-\frac{(\sigma^{i-1})^2\tau - 4(x_{i-1}+x_i-y_i-y_{i-1}) + \frac{4(x_i-y_i)^2}{(\sigma^{i-1})^2\tau} + \frac{4(x_{i-1}-y_{i-1})^2}{(\sigma^i)^2\tau}}{8\left(1-\rho_{i-1,i}^2\right)} + \\ &+\frac{\sigma^i(2\bar{c}_i-1)\left((\sigma^{i-1})^2\tau-(x_i-y_i)\right)\rho_{i-1,i}}{4\sigma^{i-1}\left(1-\rho_{i-1,i}^2\right)} - \\ &-\frac{\left((\sigma^{i-1})^2\tau+2(x_i-y_i)\right)(x_{i-1}-y_{i-1})\rho_{i-1,i}}{2\tau\sigma^i\sigma^{i-1}\left(1-\rho_{i-1,i}^2\right)} + \\ &+\frac{8\bar{c}_i\rho_{i-1,i}^2(-x_{i-1}+y_{i-1}) - (\sigma^i)^2\tau\left(1+4(-1+\bar{c}_i)\bar{c}_i(\rho_{i-1,i})^2\right)}{8\left(1-\rho_{i-1,i}^2\right)} \end{aligned} \quad (7.2)$$

Once G_3 has been obtained, we compute

$$\begin{aligned} &G_4(t,x_{i-1},x_i;T_{i-2},y_{i-1},T_{i-1}) = \\ &= \int_{\log(ae^{y_{i-1}}+c)}^{\infty} G_3(t,x_{i-1},x_i;T_{i-2},y_{i-1},T_{i-1},\eta_i)(e^{\eta_i}-(ae^{y_{i-1}}+c))d\eta_i\,, \end{aligned} \quad (7.3)$$

the value of which is given by

$$\begin{aligned} G_4(t,x_{i-1},x_i;T_{i-2},y_{i-1},T_{i-1}) &= \frac{\exp(H_i)}{4\sigma^i\sqrt{-\pi B_i\tau((\sigma^i)^2\delta_{i-1}+(\sigma^{i-1})^2\tau(1-\rho_{i-1,i}^2))}}\cdot \\ &\cdot\left(-2+K\exp\left(\frac{1+S_i^1}{4B_i}\right)\cdot\left(1+\mathcal{N}\left(\frac{F_i}{2\sqrt{B_i}}\right)\right)+\mathcal{N}\left(\frac{1+F_i}{2\sqrt{B_i}}\right)\right), \end{aligned} \quad (7.4)$$

where $\mathcal{N}$ denotes the standard normal distribution and

$$\begin{aligned} H_i(t,x_{i-1},x_i;T_{i-2},y_{i-1},T_{i-1}) = &\frac{S_i^2}{8} + \frac{\bar{c}_i(x_{i-1}-y_{i-1})\rho_{i-1,i}^2}{1-\rho_{i-1,i}^2} + \frac{S_i^3}{2\tau(1-\rho_{i-1,i}^2)} + \\ &+\frac{S_i^4}{2(1-\rho_{i-1,i}^2)} + \frac{S_i^5}{\tau(1-\rho_{i-1,i}^2)^2A_i} + \frac{S_i^6}{(1-\rho_{i-1,i}^2)^2A_i} + \frac{S_i^7}{\tau^2(1-\rho_{i-1,i}^2)^2A_i} \end{aligned} \quad (7.5)$$

$$
\begin{aligned}
A_i =& -\frac{1}{2(\sigma^i)^2\delta_{i-1}} - \frac{1}{2(\sigma^{i-1})^2\tau(1-\rho_{i-1,i}^2)} \\
B_i =& \frac{1}{2(\sigma^i)^2\delta_{i-1}} + \frac{1}{2(\sigma^i)^4\delta_{i-1}^2A_i} \\
E_i =& a\exp(y_{i-1}) + c \\
F_i =& -\frac{1}{2} + \frac{(\rho_{i-1,i})^2}{4(\sigma^i)^2\delta_{i-1}(1-\rho_{i-1,i}^2)A_i} - \frac{x_i}{2(\sigma^i)^2(\sigma^{i-1})^2\delta_{i-1}\tau(1-\rho_{i-1,i}^2)A_i} + \\
&+ \frac{C\rho_{i-1,i}}{4\sigma^i\sigma^{i-1}\delta_{i-1}(1-\rho_{i-1,i}^2)A_i} + \frac{(x_{i-1}-y_{i-1})\rho_{i-1,i}}{2(\sigma^i)^3(\sigma^{i-1})\delta_{i-1}\tau(1-\rho_{i-1,i}^2)A_i} - \\
&- 2B_i\,\log(E_i)
\end{aligned}
\tag{7.6}
$$

$$
\begin{aligned}
S_i^1 =& -1 + \frac{\rho_{i-1,i}}{2(\sigma^i)^2\delta_{i-1}(1-\rho_{i-1,i}^2)A_i} - \frac{x_i}{(\sigma^i)^2(\sigma^{i-1})^2\delta_{i-1}\tau(1-\rho_{i-1,i}^2)A_i} - \\
&- \frac{(2\bar{c}_i-1)\rho_{i-1,i}}{2\sigma^i\sigma^{i-1}\delta_{i-1}(1-\rho_{i-1,i}^2)A_i} + \frac{(x_{i-1}-y_{i-1})\rho_{i-1,i}}{(\sigma^i)^3s^{i-1}\delta_{i-1}\tau(1-\rho_{i-1,i}^2)A_i} \\
S_i^2 =& (\sigma^i)^2\delta_{i-1} + \frac{(\sigma^{i-1})^2\tau}{(1-\rho_{i-1,i}^2)} - \frac{1}{(1-\rho_{i-1,i}^2)A_i} + \frac{1}{2A_i} \\
S_i^3 =& -\tau(x_{i-1}+x_i-y_{i-1}) + \frac{x_i^2}{(\sigma^{i-1})^2} + \frac{(x_{i-1}-y_{i-1})^2}{(\sigma^i)^2} - \\
&- \frac{2x_i(x_{i-1}-y_{i-1})\rho_{i-1,i}}{\sigma^i\sigma^{i-1}} + \frac{(x_{i-1}-y_{i-1})\rho_{i-1,i}}{2\sigma^i\sigma^{i-1}A_i} \\
S_i^4 =& \frac{\sigma^i\sigma^{i-1}(2\bar{c}_i-1)\tau\rho_{i-1,i}}{2} - \frac{\sigma^i(2\bar{c}_i-1)x_i\rho_{i-1,i}}{\sigma^{i-1}} + \\
&+ \frac{\sigma^{i-1}(x_{i-1}-y_{i-1})\rho_{i-1,i}}{\sigma^i} + \frac{(\sigma^i)^2\tau\left(1+4(-1+\bar{c}_i)\bar{c}_i\rho_{i-1,i}^2\right)}{4} \\
S_i^5 =& -\frac{x_i + (2\bar{c}_i-1)(x_{i-1}-y_{i-1})\rho_{i-1,i}^2 - \tau^{-1}x_i^2 - (1-\rho_{i-1,i}^2)^{-1}x_i}{4(\sigma^{i-1})^2} - \\
&- \frac{\sigma^i(2\bar{c}_i-1)x_i\rho_{i-1,i}}{4(\sigma^{i-1})^3} + \frac{(x_{i-1}-y_{i-1})\rho_{i-1,i}}{4\sigma^i\sigma^{i-1}} \\
S_i^6 =& \frac{(\sigma^i)^2(2\bar{c}_i-1)^2\rho_{i-1,1}^2}{16(\sigma^{i-1})^2} + \frac{\sigma^i(2\bar{c}_i-1)\rho_{i-1,i}}{8\sigma^{i-1}} - \\
&- \frac{(1-\rho_{i-1,i}^2)\sigma^i(2\bar{c}_i-1)\rho_{i-1,i}}{8\sigma^{i-1}} + \frac{1}{16} \\
S_i^7 =& -\frac{x_i(x_{i-1}-y_{i-1})\rho_{i-1,i}}{2\sigma^i(\sigma^{i-1})^3} + \frac{(x_{i-1}-y_{i-1})^2\rho_{i-1,i}^2}{4(\sigma^i)^2(\sigma^{i-1})^2}
\end{aligned}
\tag{7.7}
$$

Finally, we use numerical integration with MATHEMATICA to compute

$$\bar{u}^{i,i-1}(t,x_{i-1},x_i;K) = \int_{\mathbb{R}} G_4(t,x_{i-1},x_i;T_{i-2},y_{i-1},T_{i-1})dy_{i-1}\,.$$

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