Intrinsic Taylor formula for Kolmogorov-type homogeneous groups

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Abstract

We consider a class of ultra-parabolic Kolmogorov-type operators satisfying the Hörmander's condition. We prove an intrinsic Taylor formula with global and local bounds for the remainder given in terms of the norm in the homogeneous Lie group naturally associated to the differential operator.

Keywords: Kolmogorov operators, hypoelliptic operators, Hörmander's condition, intrinsic Taylor formula

1 Introduction

We consider a class of Kolmogorov operators of the form

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{p_0} \partial_{x_i x_i} + \sum_{i,j=1}^{d} b_{ij} x_j \partial_{x_i} + \partial_t, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{1.1}$$

where $1 \le p_0 \le d$ and $B = (b_{ij})$ is a constant $d \times d$ matrix. If $p_0 = d$ then $\mathcal{L}$ is a parabolic operator while in general, for $p_0 < d$, $\mathcal{L}$ is degenerate and not uniformly parabolic. Some structural assumptions on B implying that $\mathcal{L}$ is a hypoelliptic operator will be introduced and discussed below.

Operators of the form (1.1) appear in several applications in physics, biology and mathematical finance. We recall that $\mathcal{L}$ is the linearized prototype of the Fokker-Planck operator arising in fluidodynamics (cf. Chandresekhar (1943)). Moreover $\mathcal{L}$ was extensively studied by Kolmogorov (1991) as the infinitesimal generator of the linear stochastic equation in $\mathbb{R}^d$

$$\mathrm{d}X_t = BX_t \mathrm{d}t + \sigma \mathrm{d}W_t, \tag{1.2}$$

where W is a p_0-dimensional standard Brownian motion and σ is a $d \times p_0$ matrix such that

$$\sigma\sigma^T = \begin{pmatrix} I_{p_0} & 0 \\ 0 & 0 \end{pmatrix},$$

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with I_{p_0} being the $p_0 \times p_0$ identity matrix. A particular case of (1.2) is the well-known Langevin equation from kinetic theory, which in simplified form reads

$$\begin{cases} \mathrm{d}X_t^1 = \mathrm{d}W_t \\ \mathrm{d}X_t^2 = X_t^1 \mathrm{d}t, \end{cases}$$

where W is a real Brownian motion, and whose generator is the Kolmogorov operator

$$\frac{1}{2}\partial_{x_1 x_1} + x_1 \partial_{x_2} + \partial_t, \qquad (t, x_1, x_2) \in \mathbb{R}^3. \tag{1.3}$$

We also refer to Bossy et al. (2011) for a recent study of Navier-Stokes equations involving more general Kolmogorov-type operators.

In mathematical finance, Kolmogorov equations arise in models incorporating some sort of dependence on the past: typical examples are Asian options (see, for instance, Ingersoll (1987), Barucci et al. (2001), Pascucci (2008), Frentz et al. (2010)) and some volatility models (see, for instance, Hobson and Rogers (1998) and Foschi and Pascucci (2008)).

It is natural to place operator $\mathcal{L}$ in the framework of Hörmander's theory; indeed, let us set

$$X_j = \partial_{x_j}, \quad j = 1, \dots, p_0, \quad \text{and} \quad Y = \langle Bx, \nabla \rangle + \partial_t, \tag{1.4}$$

where $\langle \cdot, \cdot \rangle$ and $\nabla = (\partial_{x_1}, \dots, \partial_{x_d})$ denote the inner product and the gradient in $\mathbb{R}^d$ respectively. Then $\mathcal{L}$ can be written as a sum of vector fields:

$$\mathcal{L} = \frac{1}{2} \sum_{j=1}^{p_0} X_j^2 + Y.$$

Under the Hörmander's condition

$$\operatorname{rank}\left(\operatorname{Lie}(X_1, \dots, X_{p_0}, Y)\right) = d + 1, \tag{1.5}$$

operator $\mathcal{L}$ is hypoelliptic and Kolmogorov (1934) and Hörmander (1967) constructed an explicit fundamental solution of $\mathcal{L}u = 0$, which is the transition density of X in (1.2). We remark that X is a Gaussian process and condition (1.5) turns out to be equivalent to the non-degeneracy of the covariance matrix of X_t for any positive t (see, for instance, Karatzas and Shreve (1991) and Pascucci (2011)).

Operator $\mathcal{L}$ in (1.1) is the prototype of the more general class of *Kolmogorov operators with variable coefficients.* The study of general Kolmogorov operators has been successfully carried out by several authors in the framework of the theory of homogeneous groups: Folland (1975), Folland and Stein (1982), Varopoulos et al. (1992) and Bonfiglioli et al. (2007) serve as a reference for the analysis of homogeneous groups. We recall that Lanconelli and Polidoro (1994) first studied the non-Euclidean intrinsic geometry induced by Kolmogorov operators and Polidoro (1994), Di Francesco and Pascucci (2005) proved the existence of a fundamental solution under optimal regularity assumptions on the coefficients; in particular, Polidoro (1994) generalized and greatly improved the classical results by Weber (1951), Il′in (1964), Sonin (1967) and Gencev (1963) where unnecessary Euclidean-type regularity was required.

The intrinsic Lie group structure modeled on the vector fields $X_1, \dots, X_{p_0}, Y$ and the related non-Euclidean functional analysis (Hölder and Sobolev spaces) were studied by several authors, among others

Polidoro and Ragusa (1998), Di Francesco and Polidoro (2006), Bramanti et al. (1996), Manfredini (1997), Lunardi (1997), Kunze et al. (2010), Nyström et al. (2010), Priola (2009) and Menozzi (2011). When dealing with intrinsic Hölder spaces, Taylor-type formulas (and the related estimates for the remainder) form one of the cornerstones for the development of the theory. Classical results about intrinsic Taylor polynomials on homogeneous groups were proved in great generality by Folland and Stein (1982). Recently, Bonfiglioli (2009) derived explicit formulas for Taylor polynomials on homogeneous groups and the corresponding remainders by adapting the classical Taylor formula with integral remainder.

The main result of this paper is a new and more explicit representation of the intrinsic Taylor polynomials for Kolmogorov-type homogeneous groups. The distinguished features of our formulas are as follows:

i) in Folland and Stein (1982) and Bonfiglioli (2009), Taylor polynomials of order n are defined for functions that are differentiable up to order n in the Euclidean sense; the constants in the error estimates for the remainders (that is, the differences between the function and its Taylor polynomials) depend on the norms of the function in the Euclidean Hölder spaces. Conversely, in this paper we define n-th order Taylor polynomials for functions that are regular *in the intrinsic sense* and the constants appearing in the error estimates *depend only on the norms of the intrinsic derivatives up to order n.* At the best of our knowledge, a similar result under such intrinsic regularity assumptions only appeared in Arena et al. (2010), but limited to the particular case of the Heisenberg group. Moreover, the fact that we assume intrinsic regularity on the function, as opposed to Euclidean one, allows us to yield some global error bounds for the remainders when the function belongs to the *intrinsic global* Hölder spaces. This represents another key difference with respect to the existing literature, where such bounds are only local.

ii) since the vector fields $X_1, \dots, X_{p_0}$ do not commute with Y, there are different representations for the Taylor polynomials depending on the order of the derivatives: specifically, the representation in Folland and Stein (1982) and Bonfiglioli (2009) is given as a sum over all possible permutations of the derivatives. Thus, computing explicitly the n-th order Taylor polynomials can be very lengthy since the number of terms involved grows proportionally to d^n. On the contrary, even though our Taylor polynomials are algebraically equivalent to those given by Folland and Stein (1982) and Bonfiglioli (2009), in Theorem 2.10 we determine a privileged way to order the vector fields so that we are able to get compact Taylor polynomials with a number of terms increasing linearly with respect to the order of the polynomial itself (see (2.14) below); this is quite relevant for practical computations, as we will show through a simple example in Section 3.1.

iii) besides the theoretical interest, our result might be useful for diverse applications. For instance, in the recent works by Lorig et al. (2014) and Pagliarani and Pascucci (2014) the authors have developed a perturbative technique to analytically approximate the solution of a parabolic Cauchy problem with variable coefficients. The Taylor polynomials of the coefficients and of the terminal datum play an important role in this technique. For instance, the short-time precision of the approximation turns out to be dependent on the regularity of the terminal datum. Within this prospective, an intrinsic Taylor formula represents a crucial ingredient in order to extend such results to the case of ultra-parabolic (i.e. $p_0 < d$) Kolmogorov operators with variable coefficients. In particular, the intrinsic regularity of the coefficients and of the terminal datum can be exploited to improve the accuracy of the approximate solutions. We refer to Section 3.4 for further details.

The paper is organized as follows: in the next section we state the structural hypothesis on the matrix B, we give the definition of *intrinsic Hölder spaces* and we state our main result. In Section 3 we review and compare with the previous literature (Sections 3.1 and 3.2) and present examples and applications (Section 3.4). In Section 4 we prove some results that are preliminary to the proof of the main theorem, which will be eventually proved in Section 5.

2 Hölder spaces and Taylor expansions

As first observed by Lanconelli and Polidoro (1994), operator $\mathcal{L}$ in (1.1) has the remarkable property of being invariant with respect to left translations in the group $\big(\mathbb{R}\times\mathbb{R}^d,\circ\big)$, where the non-commutative group law "$\circ$" is defined by

$$(t,x)\circ(s,\xi)=\big(t+s,e^{tB}x+\xi\big)\,,\qquad (t,x),(s,\xi)\in\mathbb{R}\times\mathbb{R}^d. \tag{2.6}$$

Precisely, we have

$$\big(\mathcal{L}u^{(s,\xi)}\big)(t,x)=(\mathcal{L}u)\big((s,\xi)\circ(t,x)\big),\qquad (t,x),(s,\xi)\in\mathbb{R}\times\mathbb{R}^d, \tag{2.7}$$

where

$$u^{(s,\xi)}(t,x)=u((s,\xi)\circ(t,x)).$$

Notice that $(\mathbb{R}\times\mathbb{R}^d,\circ)$ is a group with the identity element $\mathrm{Id}=(0,0)$ and inverse $(t,x)^{-1}=\big(-t,e^{-tB}x\big)$.

Lanconelli and Polidoro (1994) proved that the Hörmander's condition (1.5) is equivalent to the following one: for a certain basis on $\mathbb{R}^d$, the matrix B takes the form

$$B=\begin{pmatrix} * & * & \cdots & * & * \\ B_1 & * & \cdots & * & * \\ 0 & B_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_r & * \end{pmatrix} \tag{2.8}$$

where each B_j is a $p_j\times p_{j-1}$ matrix of rank p_j with

$$p_0\geq p_1\geq\cdots\geq p_r\geq 1,\qquad \sum_{j=0}^{r}p_j=d,$$

and the $*$-blocks are arbitrary. Moreover, if (and only if) the $*$-blocks in (2.8) are null then $\mathcal{L}$ *is homogeneous of degree two* with respect the dilations $(D(\lambda))_{\lambda>0}$ on $\mathbb{R}\times\mathbb{R}^d$ given by

$$D(\lambda)=\mathrm{diag}\big(\lambda^2,\lambda I_{p_0},\lambda^3 I_{p_1},\cdots,\lambda^{2r+1}I_{p_r}\big),$$

where I_{p_j} are $p_j \times p_j$ identity matrices: specifically, we have

$$\big(\mathcal{L}u^{(\lambda)}\big)(t,x) = \lambda^2 (\mathcal{L}u)\big(D(\lambda)(t,x)\big), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d,\ \lambda > 0, \tag{2.9}$$

where

$$u^{(\lambda)}(t,x) = u(D(\lambda)(t,x)).$$

Throughout this paper we assume the following standing

Assumption 2.1. *B is a $d \times d$ constant matrix as in* (2.8)*, where each block B_j has rank p_j and each $*$-block is null.*

Remark 2.2. Under Assumption 2.1, the matrix B uniquely identifies the *homogeneous Lie group* (in the sense of Folland and Stein (1982))

$$\mathcal{G}_B := \left(\mathbb{R} \times \mathbb{R}^d, \circ, D(\lambda)\right).$$

We define the $D(\lambda)$-homogeneous norm on $\mathcal{G}_B$ as follows:

$$\|(t,x)\|_B = |t|^{1/2} + |x|_B, \qquad |x|_B = \sum_{j=1}^{d} |x_j|^{1/q_j},$$

where $(q_j)_{1 \le j \le d}$ are the integers such that

$$D(\lambda) = \mathrm{diag}\big(\lambda^2, \lambda^{q_1}, \cdots, \lambda^{q_d}\big).$$

For any $\zeta \in \mathbb{R} \times \mathbb{R}^d$, we denote by

$$D_B(\zeta, r) = \{z \in \mathbb{R} \times \mathbb{R}^d \mid \left\|\zeta^{-1} \circ z\right\|_B < r\}$$

the open ball of radius r, centered at ζ, in the homogeneous group $\mathcal{G}_B$.

Remark 2.3. There exist two constants $C_1 \ge 1$ and $C_2 > 0$, both depending only on B, such that

$$\begin{aligned}
&\|\zeta \circ z\|_B \le C_1 \left(\|\zeta \circ \eta\|_B + \left\|\eta^{-1} \circ z\right\|_B\right), && z, \zeta, \eta \in \mathbb{R} \times \mathbb{R}^d, \\
&\frac{1}{C_2}|z - \zeta| \le \left\|\zeta^{-1} \circ z\right\|_B \le C_2 |z - \zeta|^{2r+1}, && \text{for } |z - \zeta|, \left\|\zeta^{-1} \circ z\right\|_B \le 1.
\end{aligned}$$

The first inequality implies that $\|\cdot\|_B$ is a quasi-norm, while the second formula shows that the intrinsic distance is locally equivalent to the Euclidean one. For a proof we refer to Manfredini (1997), Proposition 2.1.

Next we introduce the notions of B-intrinsic regularity and B-Hölder space. Let X be a Lipschitz vector field on $\mathbb{R} \times \mathbb{R}^d$. For any $z \in \mathbb{R} \times \mathbb{R}^d$, we denote by $\delta \mapsto e^{\delta X}(z)$ the integral curve of X defined as the unique solution of

$$\begin{cases} \frac{d}{d\delta} e^{\delta X}(z) = X\left(e^{\delta X}(z)\right), & \delta \in \mathbb{R}, \\ e^{\delta X}(z)|_{\delta=0} = z. \end{cases}$$

Explicitly, if $X \in \{X_1, \cdots, X_{p_0}, Y\}$ is one of the vector fields in (1.4), we have

$$e^{\delta X_i}(t,x) = (t, x + \delta e_i), \quad i = 1, \cdots, p_0, \qquad e^{\delta Y}(t,x) = (t + \delta, e^{\delta B} x), \tag{2.10}$$

for any $(t,x) \in \mathbb{R} \times \mathbb{R}^d$.

In order to define a *gradation* for the homogeneous group $\mathcal{G}_B$ (see Section 2.4 in Bonfiglioli (2009)), we associate a *formal degree* $m_X \in \mathbb{R}_+$ to each $X \in \{X_1, \cdots, X_{p_0}, Y\}$ in the following canonical way:

Assumption 2.4. *The formal degrees of the vector fields $X_1, \cdots, X_{p_0}$ and Y are set as $m_{X_j} = 1$ for $1 \le j \le p_0$ and $m_Y = 2$.*

Next we recall the general notion of Lie differentiability and Hölder regularity.

Definition 2.5. Let X be a Lipschitz vector field and u be a real-valued function defined in a neighborhood of $z \in \mathbb{R} \times \mathbb{R}^d$. We say that u is *X-differentiable* in z if the function $\delta \mapsto u\left(e^{\delta X}(z)\right)$ is differentiable in 0. We will refer to the function $z \mapsto \frac{d}{d\delta} u\left(e^{\delta X}(z)\right)\big|_{\delta=0}$ as *X-Lie derivative of u*, or simply *Lie derivative of u* when the dependence on the field X is clear from the context.

Definition 2.6. Let X be a Lipschitz vector field on $\mathbb{R} \times \mathbb{R}^d$ with formal degree $m_X > 0$. For $\alpha \in]0, m_X]$, we say that $u \in C_X^\alpha$ if the semi-norm

$$\|u\|_{C_X^\alpha} := \sup_{\substack{z \in \mathbb{R} \times \mathbb{R}^d \\ \delta \in \mathbb{R} \setminus \{0\}}} \frac{\left|u\left(e^{\delta X}(z)\right) - u(z)\right|}{|\delta|^{\frac{\alpha}{m_X}}}$$

is finite.

Now, let Ω be a domain in $\mathbb{R} \times \mathbb{R}^d$. For any $z \in \Omega$ we set

$$\delta_z = \sup\left\{\bar{\delta} \in]0,1] \mid e^{\delta X}(z) \in \Omega \text{ for any } \delta \in [-\bar{\delta}, \bar{\delta}]\right\}.$$

If Ω_0 is a bounded domain with $\overline{\Omega}_0 \subseteq \Omega$, we set

$$\delta_{\Omega_0} = \min_{z \in \overline{\Omega}_0} \delta_z.$$

Note that $\delta_{\Omega_0} \in]0,1]$.

Definition 2.7. For $\alpha \in]0, m_X]$, we say that $u \in C^\alpha_{X,\mathrm{loc}}(\Omega)$ if for any bounded domain Ω_0 with $\overline{\Omega}_0 \subseteq \Omega$, the semi-norm

$$\|u\|_{C_X^\alpha(\Omega_0)} := \sup_{\substack{z \in \Omega_0 \\ 0 < |\delta| < \delta_{\Omega_0}}} \frac{\left|u\left(e^{\delta X}(z)\right) - u(z)\right|}{|\delta|^{\frac{\alpha}{m_X}}}$$

is finite.

Now we define the intrinsic Hölder spaces on the homogeneous group $\mathcal{G}_B$.

Definition 2.8. Let $\alpha \in]0,1]$, then:

i) $u \in C_B^{0,\alpha}$ if $u \in C_Y^\alpha$ and $u \in C_{\partial_{x_i}}^\alpha$ for any $i = 1, \dots, p_0$. For any $u \in C_B^{0,\alpha}$ we define the semi-norm

$$\|u\|_{C_B^{0,\alpha}} := \|u\|_{C_Y^\alpha} + \sum_{i=1}^{p_0} \|u\|_{C_{\partial_{x_i}}^\alpha} .$$

ii) $u \in C_B^{1,\alpha}$ if $u \in C_Y^{1+\alpha}$ and $\partial_{x_i} u \in C_B^{0,\alpha}$ for any $i = 1, \dots, p_0$. For any $u \in C_B^{1,\alpha}$ we define the semi-norm

$$\|u\|_{C_B^{1,\alpha}} := \|u\|_{C_Y^{\alpha+1}} + \sum_{i=1}^{p_0} \|\partial_{x_i} u\|_{C_B^{0,\alpha}} .$$

iii) For $k \in \mathbb{N}$ with $k \geq 2$, $u \in C_B^{k,\alpha}$ if $Yu \in C_B^{k-2,\alpha}$ and $\partial_{x_i} u \in C_B^{k-1,\alpha}$ for any $i = 1, \dots, p_0$. For any $u \in C_B^{k,\alpha}$ we define the semi-norm

$$\|u\|_{C_B^{k,\alpha}} := \|Yu\|_{C_B^{k-2,\alpha}} + \sum_{i=1}^{p_0} \|\partial_{x_i} u\|_{C_B^{k-1,\alpha}} .$$

Similarly, according to Definition 2.7, we define the spaces $C_{B,\text{loc}}^{k,\alpha}(\Omega)$ of *locally* Hölder continuous functions on a domain Ω of $\mathbb{R} \times \mathbb{R}^d$, and the related semi-norms $\|\cdot\|_{C_B^{k,\alpha}(\Omega_0)}$ on bounded domains Ω_0 with $\overline{\Omega}_0 \subseteq \Omega$.

Remark 2.9. The following inclusion holds: $C_{B,\text{loc}}^{k,\alpha} \subseteq C_{B,\text{loc}}^{k',\alpha'}$ for $0 \leq k' \leq k$ and $0 < \alpha' \leq \alpha \leq 1$. Moreover we have $C_B^{k,\alpha} \subseteq C_{B,\text{loc}}^{k,\alpha}$ for $k \geq 0$.

In the sequel, $\beta = (\beta_1, \cdots, \beta_d) \in \mathbb{N}_0^d$ will denote a multi-index. As usual

$$|\beta| := \sum_{j=1}^{d} \beta_j \quad \text{ and } \quad \beta! := \prod_{j=1}^{d} (\beta_j!)$$

are called the length and the factorial of β respectively. Moreover, for any $x \in \mathbb{R}^d$, we set

$$x^\beta = x_1^{\beta_1} \cdots x_d^{\beta_d} \quad \text{ and } \quad \partial^\beta = \partial_x^\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_d}^{\beta_d}.$$

We also introduce the B-length of β defined as

$$|\beta|_B := \sum_{i=0}^{r} (2i+1) \left|\beta^{[i]}\right|$$

where $\beta^{[i]} \in \mathbb{N}_0^d$ is the multi-index

$$\beta_k^{[i]} := \begin{cases} \beta_k & \text{for } \bar{p}_{i-1} < k \leq \bar{p}_i, \\ 0 & \text{otherwise,} \end{cases} \tag{2.11}$$

with

$$\bar{p}_i = p_0 + p_1 + \cdots + p_i, \qquad 0 \leq i \leq r, \tag{2.12}$$

and $\bar{p}_{-1} \equiv 0$. We are now in position to state our main result.

Theorem 2.10. *Let Ω be a domain of $\mathbb{R}\times\mathbb{R}^d$, $\alpha\in]0,1]$ and $n\in\mathbb{N}_0$. If $u\in C^{n,\alpha}_{B,\mathrm{loc}}(\Omega)$ then we have:*

1) there exist the derivatives

$$Y^k\partial_x^\beta u\in C^{n-2k-|\beta|_B,\alpha}_{B,\mathrm{loc}}(\Omega),\qquad 0\le 2k+|\beta|_B\le n;$$

2) for any $\zeta\in\Omega$ there exist $r_\zeta,R_\zeta>0$ such that $\overline{D_B(\zeta,R_\zeta)}\subseteq\Omega$ and

$$|u(z)-T_nu(\zeta,z)|\le c_{B,\zeta}\|u\|_{C^{n,\alpha}_{B,\mathrm{loc}}(D_B(\zeta,R_\zeta))}\|\zeta^{-1}\circ z\|_B^{n+\alpha},\qquad z\in D_B(\zeta,r_\zeta),\tag{2.13}$$

*where $c_{B,\zeta}$ is a constant that depends on B and ζ, while $T_nu(\zeta,\cdot)$ is the n-*th order *B*-Taylor polynomial of *u* around *$\zeta=(s,\xi)$ defined as*

$$T_nu(\zeta,z):=\sum_{0\le 2k+|\beta|_B\le n}\frac{1}{k!\,\beta!}\big(Y^k\partial_\xi^\beta u(s,\xi)\big)(t-s)^k\big(x-e^{(t-s)B}\xi\big)^\beta,\qquad z=(t,x)\in\mathbb{R}\times\mathbb{R}^d;\tag{2.14}$$

3) if $u\in C^{n,\alpha}_B$ then we have

$$Y^k\partial_x^\beta u\in C^{n-2k-|\beta|_B,\alpha}_B\qquad \textit{for}\ \ 0\le 2k+|\beta|_B\le n,\tag{2.15}$$

and

$$\big|u(z)-T_nu(\zeta,z)\big|\le c_B\|u\|_{C^{n,\alpha}_B}\|\zeta^{-1}\circ z\|_B^{n+\alpha},\qquad z,\zeta\in\mathbb{R}\times\mathbb{R}^d,\tag{2.16}$$

where c_B is a positive constant that only depends on B.

A direct consequence of estimate (2.16) in the particular case $n=0$ is the following

Corollary 2.11. *A function $u\in C^{0,\alpha}_B$ if and only if there exists a positive constant c such that*

$$|u(z)-u(\zeta)|\le c\left\|\zeta^{-1}\circ z\right\|_B^\alpha,\qquad z,\zeta\in\mathbb{R}\times\mathbb{R}^d,$$

i.e. u is B-Hölder continuous in the sense of Definition 1.2 in Polidoro (1994).

For a comparison between intrinsic and Euclidean Hölder continuity we refer to Proposition 2.1 in Polidoro (1994).

Corollary 2.12. *If $u\in C^{2r+1,\alpha}_{B,\mathrm{loc}}(\Omega)$, then there exists $\partial_t u\in C^{0,\alpha}_{B,\mathrm{loc}}(\Omega)$. Moreover, we have*

$$\partial_t u(t,x)=Yu(t,x)-\langle Bx,\nabla u(t,x)\rangle.\tag{2.17}$$

Proof. In Theorem 2.10 we take $\zeta=(t,x)$, $z=(t+\delta,x)$ and note that, in this case, the spatial increments become

$$x-e^{\delta B}x=-\delta Bx+O(\delta^2)\qquad\text{as }\delta\to 0.$$

Now, by Theorem 2.10 all the spatial first-order derivatives exist and

$$u(z)-T_{2r+1}u(\zeta,z)=u(t+\delta,x)-u(t,x)-\delta Yu(t,x)+\delta\sum_{i=1}^d\partial_{x_i}u(t,x)(Bx)_i+O(\delta^2),\qquad\text{as }\delta\to 0.$$

Since

$$\|\zeta^{-1} \circ z\|_B^{2r+1+\alpha} = \|(\delta, x - e^{\delta B} x)\|_B^{2r+1+\alpha} = O(|\delta|^{1+\frac{\alpha}{2r+1}}), \qquad \text{as } \delta \to 0,$$

we get

$$\frac{u(t+\delta, x) - u(t,x)}{\delta} - Yu(t,x) + \sum_{i=1}^{d} (Bx)_i \partial_{x_i} u(t,x) = O(|\delta|^{\frac{\alpha}{2r+1}}) \qquad \text{as } \delta \to 0.$$

This implies that the time-derivative exists and formula (2.17) holds. Now, it also easily follows that $\partial_t u \in C^{0,\alpha}_{B,\mathrm{loc}}(\Omega)$ since all the derivatives appearing in the right-hand side of (2.17) are in $C^{0,\alpha}_{B,\mathrm{loc}}(\Omega)$. $\square$

3 Comparison with known results, examples and applications

3.1 Taylor formulas for homogeneous Lie groups

Our results can be seen within the more general setting of *homogeneous Lie groups* (cf. Folland and Stein (1982)). A Lie group $\mathcal{G} = (\mathbb{R}^N, *)$ is said to be homogeneous if there exists a family of group automorphisms of $\mathcal{G}$, $(D_\lambda)_{\lambda>0}$, called dilations, of the form

$$D_\lambda(x_1, \dots, x_N) = (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_N} x_N), \qquad \lambda > 0,$$

for some $1 \le \sigma_1 \le \sigma_2 \le \cdots \le \sigma_N$. The existence of such dilations implies that the exponential map Exp between the Lie algebra $\mathfrak{g}$ and $\mathcal{G}$ is a global diffeomorphism whose inverse is denoted by Log. Moreover, we have a privileged basis on $\mathfrak{g}$, the Jacobian one, whose elements are the left-invariant vector fields Z_i uniquely defined by

$$Z_i|_{x=0} \equiv \partial_{x_i} \qquad i = 1, \dots, N.$$

In this framework it is natural to define the intrinsic degree of Z_i as σ_i and the D_λ-homogeneous norm

$$|x|_{\mathbb{G}} = \sum_{i=1}^{N} |x_i|^{\frac{1}{\sigma_i}}.$$

Following Bonfiglioli (2009), the n-th order intrinsic Taylor polynomial $P_n f(x_0, \cdot)$ of a function f around the point x_0, can be defined as the unique polynomial function such that

$$f(x) - P_n f(x_0, x) = O(|x_0^{-1} * x|_{\mathbb{G}}^{n+\varepsilon}) \qquad \text{as } |x_0^{-1} * x|_{\mathbb{G}} \to 0,$$

for some $\varepsilon > 0$. For $f \in C^{n+1}$ existence and uniqueness of $P_n f$ was proved in Folland and Stein (1982); under the same hypothesis, a more explicit expression and a better estimate of the remainder was given in Bonfiglioli (2009). Precisely, in the latter the author proved that

$$P_n f(x_0, x) = f(x_0) + \sum_{k=1}^{n} \sum_{\substack{1 \le i_1, \dots, i_k \le N \\ I = (i_1, \dots, i_k),\ \sigma(I) \le n}} \frac{Z_I f(x_0)}{k!} \mathrm{Log}_{i_1}(x_0^{-1} * x) \cdots \mathrm{Log}_{i_k}(x_0^{-1} * x). \tag{3.18}$$

Here $\sigma(I) := i_1 \sigma_{i_1} + \cdots + i_k \sigma_{i_k}$ denotes the intrinsic order of the operator $Z_I := Z_{i_1} \cdots Z_{i_k}$ and Log_i is the i-th component of the Log map in the basis $\{Z_1, \dots, Z_N\}$.

Note that, in general, operators Z_i do not commute. Therefore, formula (3.18) typically involves a large number of terms. In the special case of a Kolmogorov-type group, the Taylor polynomial (2.14) is much more compact that (3.18) because we can exploit the fact that all but one of the Z_i coincide with Euclidean derivatives and thus commute with each other; moreover, our increments along the integral curves of the vector fields are different from those in (3.18). We illustrate this fact in the following example.

Let us consider the simplest Kolmogorov group, namely the one induced by the operator defined in (1.3). This case corresponds to the matrix

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.19}$$

and the dilations $D(\lambda)$ take the following explicit form:

$$D(\lambda)(t, x_1, x_2) = (\lambda^2 t, \lambda x_1, \lambda^3 x_2), \qquad (t, x_1, x_2) \in \mathbb{R} \times \mathbb{R}^2.$$

Moreover, if $z = (t, x_1, x_2), \zeta = (s, \xi_1, \xi_2)$, then we also have

$$\zeta \circ z = (s + t, x_1 + \xi_1, x_2 + \xi_2 + t\xi_1), \qquad \zeta^{-1} \circ z = (t - s, x_1 - \xi_1, x_2 - \xi_2 - (t - s)\xi_1).$$

The components of left-hand side vector in the previous formula are exactly the increments appearing in (2.14). With regard to formula (3.18), we have

$$Z_0 = Y, \qquad Z_1 = \partial_{x_1}, \qquad Z_2 = \partial_{x_2},$$

while the corresponding components of the Log map are

$$\mathrm{Log}_0(\zeta^{-1} \circ z) = t - s, \quad \mathrm{Log}_1(\zeta^{-1} \circ z) = x_1 - \xi_1, \quad \mathrm{Log}_2(\zeta^{-1} \circ z) = x_2 - \xi_2 - (t - s)\xi_1 - \frac{(t - s)(x_1 - \xi_1)}{2}.$$

Note that the first two components coincide with the increments mentioned above while the third one is different. It follows that, up to order two, the two versions of the Taylor polynomial coincide. On the other hand, according to our definition, the third and fourth polynomials are given by

$$\begin{aligned}
T_3 u(\zeta, z) &= T_2 u(\zeta, z) + \frac{1}{3!}\partial_{x_1}^3 u(\zeta)(x_1 - \xi_1)^3 + Y\partial_{x_1} u(\zeta)(x_1 - \xi_1)(t - s) + \partial_{x_2} u(\zeta)(x_2 - \xi_2 - (t - s)\xi_1), \\
T_4 u(\zeta, z) &= T_3 u(\zeta, z) + \frac{1}{4!}\partial_{x_1}^4 u(\zeta)(x_1 - \xi_1)^4 + \frac{1}{2!} Y\partial_{x_1}^2 u(\zeta)(x_1 - \xi_1)^2(t - s) \\
&\quad + \frac{1}{2!} Y^2 u(\zeta)(t - s)^2 + \partial_{x_2}\partial_{x_1} u(\zeta)(x_1 - \xi_1)(x_2 - \xi_2 - (t - s)\xi_1),
\end{aligned}$$

while, according to formula (3.18), we have

$$\begin{aligned}
T_3 u(\zeta, z) &= T_2 u(\zeta, z) + \frac{1}{2!}(Y\partial_{x_1} + \partial_{x_1} Y) u(\zeta)(x_1 - \xi_1)(t - s) + \frac{1}{3!}\partial_{x_1}^3 u(\zeta)(x_1 - \xi_1)^3 \\
&\quad + \partial_{x_2} u(\zeta)\Big(x_2 - \xi_2 - (t - s)\xi_1 - \frac{(t - s)(x_1 - \xi_1)}{2}\Big), \\
T_4 u(\zeta, z) &= T_3 u(\zeta, z) + \frac{1}{2!} Y^2 u(\zeta)(t - s)^2 + \frac{1}{4!}\partial_{x_1}^4 u(\zeta)(x_1 - \xi_1)^4 \\
&\quad + \frac{1}{3!}(Y\partial_{x_1}^2 + \partial_{x_1} Y\partial_{x_1} + \partial_{x_1}^2 Y) u(\zeta)(x_1 - \xi_1)^2(t - s) \\
&\quad + \partial_{x_2}\partial_{x_1} u(\zeta)(x_1 - \xi_1)\Big(x_2 - \xi_2 - (t - s)\xi_1 - \frac{(t - s)(x_1 - \xi_1)}{2}\Big).
\end{aligned}$$

Notice that the above expressions of the Taylor polynomials can be proved to be algebraically equivalent by using the identity $\partial_{x_1} Y = Y\partial_{x_1} + \partial_{x_2}$.

3.2 Intrinsic Hölder spaces in the literature

Intrinsic Hölder spaces play a central role in the study of the existence and the regularity properties of solutions to Kolmogorov operators with variables coefficients. In order to prove Schauder-type estimates, different notions of Hölder spaces have been proposed by several authors (see, for instance, Manfredini (1997), Lunardi (1997), Pascucci (2003), Di Francesco and Polidoro (2006) and Frentz et al. (2010)): we note that some authors introduce only the definition of $C_B^{0,\alpha}$ and $C_B^{2,\alpha}$. Indeed, the definition of $C_B^{1,\alpha}$ is technically more elaborate because it involves derivatives of fractional (in the intrinsic sense) order and therefore is sometimes omitted.

In Manfredini (1997) and Di Francesco and Polidoro (2006), $C_B^{0,\alpha}$ is defined as the space of functions that are bounded and Hölder continuous with respect to the homogeneous group structure: precisely, a function $u \in C_B^{0,\alpha}$ on a domain Ω of $\mathbb{R} \times \mathbb{R}^d$ if

$$|u|_{\alpha,B,\Omega} := \sup_{z\in\Omega} |u(z)| + \sup_{\substack{z,\zeta\in\Omega \\ z\neq\zeta}} \frac{|u(z)-u(\zeta)|}{\|\zeta^{-1}\circ z\|_B^\alpha} < \infty. \tag{3.20}$$

Note that by adopting this definition, the estimate for the remainder of the 0th order Taylor polynomial trivially follows. Corollary 2.11 shows that definition (3.20) is basically equivalent to Definition 2.8-i). Similarly, Frentz et al. (2010) define the following norm in the space $C_B^{1,\alpha}$:

$$|u|_{1+\alpha,B,\Omega} := |u|_{\alpha,B,\Omega} + \sum_{i=1}^{p_0} |\partial_{x_i} u|_{\alpha,B,\Omega} + \sup_{\substack{z,\zeta\in\Omega \\ z\neq\zeta}} \frac{|u(z)-T_1u(\zeta,z)|}{\|\zeta^{-1}\circ z\|_B^{1+\alpha}}.$$

Various definitions of the space $C_B^{2,\alpha}(\Omega)$ are used in the literature. Manfredini (1997) requires bounded and Hölder continuous second order derivatives, while Di Francesco and Polidoro (2006) and Frentz et al. (2010) also require the function u and its first p_0 spatial derivatives to be Hölder continuous. Precisely, Manfredini (1997) introduces the norm

$$|u|^{(M)}_{2+\alpha,B,\Omega} := \sup_{\Omega} |u| + \sum_{i=1}^{p_0} \sup_{\Omega} |\partial_{x_i} u| + \sum_{i,j=1}^{p_0} |\partial_{x_i,x_j} u|_{\alpha,B,\Omega} + |Yu|_{\alpha,B,\Omega},$$

while Di Francesco and Polidoro (2006) and Frentz et al. (2010) define

$$|u|_{2+\alpha,B,\Omega} := |u|_{\alpha,B,\Omega} + \sum_{i=1}^{p_0} |\partial_{x_i} u|_{\alpha,B,\Omega} + \sum_{i,j=1}^{p_0} |\partial_{x_i,x_j} u|_{\alpha,B,\Omega} + |Yu|_{\alpha,B,\Omega}.$$

3.3 Examples of functions in $C^{n,\alpha}_{B,\mathrm{loc}}$

For comparison, we give some examples of functions with different intrinsic and Euclidean regularity. We set $d = 2$ and B as in (3.19) corresponding to the *prototype* Kolmogorov operator in (1.3).

Example 3.13. Consider the function $u : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $u(t, x_1, x_2) = |x_2 - c|$, with $c \in \mathbb{R}$. This function is particularly relevant for financial applications since it is often related to the payoff of so-called Asian-style derivatives. Clearly u is Lipschitz continuous in the Euclidean sense, but intrinsically we have $u \in C^{1,1}_{B,\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^2)$ because $\partial_{x_1} u \in C^{0,1}_{B,\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^2)$ and $u \in C^2_{Y,\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^2)$. Note that $u \notin C^{2,\alpha}_{B,\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^2)$

because u is not Y-differentiable in $x_2 = c$: nevertheless a (2.16)-like estimate for $n = 2$ and $\alpha = 1$ holds for two points $z, \zeta \in \mathbb{R} \times \mathbb{R}^2$ sharing the same time-component, i.e.

$$|u(z) - u(\zeta)| \le |x_2 - \xi_2| \le \|\zeta^{-1} \circ z\|_B^3, \qquad z = (t, x),\ \zeta = (t, \xi) \in \mathbb{R} \times \mathbb{R}^2.$$

Example 3.14. As a variant of the previous example let us consider the function $u : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $u(t, x_1, x_2) = |x_2 - c|^{\frac{3}{2}}$, with $c \in \mathbb{R}$. This time $u \in C^{1,1/2}$, that is differentiable with Hölder continuous derivatives in the Euclidean sense, but intrinsically we have $u \in C^{2,1}_{B,\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^2)$ because $\partial_{x_1} u \equiv 0$ and

$$Yu(t, x_1, x_2) = \frac{3}{2} x_1 \, |x_2 - c|^{\frac{1}{2}} \operatorname{sgn}(x_2 - c) \in C^{0,1}_{B,\mathrm{loc}}.$$

Also in the present example the function shows higher intrinsic regularity than the Euclidean one.

Example 3.15. It is easy to check that any function of the form $u(t, x_1, x_2) = f(x_2 - t x_1)$ is constant along the integral curves $e^{\delta Y}(z) = (t + \delta, x_1, x_2 + \delta x_1)$ for any $z \in \Omega$. Therefore, we have $Y^n u \equiv 0$ for any $n \in \mathbb{N}$. In this particular case, we have that $u \in C^{n,\alpha}_{B,\mathrm{loc}}$ if and only if $u \in C^{n,\alpha}_{\mathrm{loc}}$ in the Euclidean sense.

Example 3.16. The following function belongs to $C^{2,\alpha}_{B,\mathrm{loc}}$ but only to $C^{0,\alpha}_{\mathrm{loc}}$:

$$u(t, x_1, x_2) = \begin{cases} \frac{1}{\sqrt{2\pi x_1^4}} \int_{\mathbb{R}} \exp\left(-\frac{(y - x_2)^2}{2 x_1^4}\right) |y| \, \mathrm{d}y & \text{if } x_1 \neq 0, \\ |x_2| & \text{if } x_1 = 0. \end{cases}$$

Indeed u is continuous and smooth on $\{x_1 \neq 0\}$; in particular, $u \in C^{2,1}_{\mathrm{loc}}(\{x_1 \neq 0\})$ and $u \in C^{2,1}_{B,\mathrm{loc}}(\{x_1 \neq 0\})$. On the plane $\{x_1 = 0\}$ the Euclidean derivative $\partial_{x_2} u$ does not exist in $x_2 = 0$ for any t and thus $u \notin C^{2,\alpha}_{\mathrm{loc}}$ for any $\alpha \in (0, 1]$. On the other hand, $\partial_{x_1} u$, $\partial_{x_1 x_1} u$ and Yu exist on $\{x_1 = 0\}$ and they are all equal to 0. In particular, we have $\partial_{x_1 x_1} u, Yu \in C^1_{Y,\mathrm{loc}}$ and $\partial_{x_1} u \in C^2_{Y,\mathrm{loc}}$. Moreover, one can directly prove that $\partial_{x_1 x_1} u, Yu \in C^1_{\partial_{x_1},\mathrm{loc}}$ and thus, $u \in C^{2,1}_{B,\mathrm{loc}}$.

3.4 Asymptotic expansions for ultra-parabolic operators and application to mathematical finance.

We briefly discuss the possibility of exploiting our result in order to obtain asymptotic expansions for variable-coefficients ultra-parabolic operators of the type

$$\mathcal{L} = \frac{1}{2} \sum_{j=1}^{p_0} a_{ij}(t, x) \partial_{x_j x_j} + \sum_{i=1}^{p_0} a_i(t, x) \partial_{x_i} + \langle Bx, \nabla \rangle + \partial_t, \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d, \tag{3.21}$$

where $(a_{ij}(t, x))_{1 \le i,j \le p_0}$ is a positive definite $p_0 \times p_0$ matrix for any $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ and B is as in (2.8). In the elliptic case, i.e. $p_0 = d$, two of the authors had previously proposed a *Gaussian perturbative method* to carry out a closed-from approximation of the solution of the backward Cauchy problem

$$\begin{cases} \mathcal{L} u(t, x) = 0, & (t, x) \in [0, T[\, \times \mathbb{R}^d, \\ u(t, x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.22}$$

for a given $T > 0$ and a given terminal datum φ. For a recent and thorough description of such approach the reader can refer to Lorig et al. (2014) for the uniformly parabolic case and to Pagliarani and Pascucci (2014) for the locally-parabolic case. Roughly speaking, under mild assumptions on the coefficients a_{ij} and a_i, the authors proved a small-time asymptotic expansion of the type

$$u(t,x) = u_0(t,x) + \sum_{n=1}^{N} \mathcal{G}_n u_0(t,x) + R_N(t,x), \qquad N \in \mathbb{N}, \tag{3.23}$$

with

$$R_N(t,x) = \mathrm{O}\left((T-t)^{\frac{m_{a,N}+m_\varphi}{2}}\right) \qquad \text{as } T-t \to 0^+. \tag{3.24}$$

Here u_0 is the solution of the *heat-type* operator $\mathcal{L}_0$ obtained by freezing the coefficients of $\mathcal{L}$ and $(\mathcal{G}_n)_{n\geq 1}$ is a family of differential operators acting on x, polynomial in $(T-t)$, and dependent on the Taylor coefficients of the functions a_{ij} and a_i. The positive exponents $m_{a,N}$ and m_φ, determining the asymptotic rate of convergence of the expansion, depend on the regularity of the coefficients a_{ij}, a_i and of the terminal datum φ respectively. Typically, we have $m_{a,N} = N+1$ and $m_\varphi = k+1$ if $a_{ij}, a_i \in C^{N,1}$ and $\varphi \in C^{k,1}$ respectively, in the classical Euclidean meaning.

In light of the invariance properties (2.7)-(2.9) it seems reasonable that, when trying to extend such results to the ultra-parabolic framework, intrinsic regularity should be considered as opposed to classical one. Precisely, we could perform our analysis by assuming the coefficients $a_{ij}, a_i \in C_B^{N,1}$, the terminal datum $\varphi \in C_B^{k,1}$, and make use of the intrinsic Taylor formula of Theorem 2.10 to carry out an asymptotic expansion similar to that in (3.23)-(3.24). Thus we could obtain accurate closed-form approximate solutions to the Cauchy problem (3.22). At the best of our knowledge, such a general result for ultra-parabolic operators is not available in the literature: clearly, such an extension could be also performed by considering Euclidean regularity for the coefficients and the terminal datum. However, the benefit in exploiting the intrinsic regularity is twofold:

i) first, since the operators $\mathcal{G}_n$ in (3.23) depend on the Taylor coefficients of the functions a_{ij} and a_i, it is convenient to use the intrinsic Taylor polynomial, being the latter typically a projection of the Euclidean one. In this way we avoid taking up terms in the expansion that do not improve the quality of the approximation;

ii) secondly, since the bound in (3.24) also depends on the regularity of the datum φ, we can prove a higher accuracy for the approximation (3.23) since the intrinsic regularity of φ is typically greater than the Euclidean regularity: this is the case, for instance, in financial applications (see (3.27)).

The interest for seeking approximate solutions for degenerate (3.21)-like operators is justified by their connection with *stochastic differential equations* and by their vast impact in numerous applications. As a matter of example, we could consider the problem arising in mathematical finance of pricing path-dependent options of Asian style. To fix the ideas, let us denote by X^1 a risky asset following the stochastic differential equation

$$\mathrm{d}X_t^1 = \sigma(X_t^1) X_t^1 \mathrm{d}W_t,$$

where W is a one-dimensional standard real Brownian motion under the risk-neutral measure $\mathbb{Q}$. The averaging prices for an *arithmetic Asian option* are described by the additional $\mathbb{R}^+$-valued state process X^2 satisfying

$$\mathrm{d}X_t^2 = X_t^1 \mathrm{d}t.$$

By usual arbitrage arguments and by standard Feynman-Kac representation formula, the price of a European Asian option with terminal payoff $\varphi(X_T^1, X_T^2)$ is given by

$$A_t = \mathbb{E}[\varphi(X_T^1, X_T^2)|\mathcal{F}_t] = u(t, X_t^1, X_t^2), \tag{3.25}$$

where u solves the Cauchy problem (3.22) with

$$\mathcal{L} = \frac{\sigma^2(x_1)x_1^2}{2}\,\partial_{x_1x_1} + x_1\partial_{x_2} + \partial_t. \tag{3.26}$$

Typical payoff functions are given by

$$\begin{aligned} \varphi_{\mathrm{fix}}(x_1,x_2) &= \left(\frac{x_2}{T} - K\right)^+ \qquad \text{(fixed strike arithmetic Call)},\\ \varphi_{\mathrm{flo}}(x_1,x_2) &= \left(x_1 - \frac{x_2}{T}\right)^+ \qquad \text{(floating strike arithmetic Call)}. \end{aligned} \tag{3.27}$$

Under suitable regularity and growth conditions, existence and uniqueness of the solution to the Cauchy problem (3.26) were proved in Barucci et al. (2001).

Even in the standard Black&Scholes model (i.e. $\sigma(\cdot) \equiv \sigma$), the arithmetic average X_t^2 is not log-normally distributed and its distribution is not trivial to analytically characterize. An integral representation was obtained by Yor (1992). However, the latter is not very relevant for the practical computation of the expectation in (3.25). Therefore, several authors proposed diverse alternative approaches to efficiently compute the prices of arithmetic Asian options. A short and incomplete list includes Linetsky (2004), Dewynne and Shaw (2008), Gobet and Miri (2014) and Foschi et al. (2013). In the latter, the authors obtained an explicit third order approximation of the type (3.23) for the fundamental solution of $\mathcal{L}$ and thus for the joint distribution of the couple (X_t^1, X_t^2), but did not prove any bound for the remainder R_3. By means of the Taylor expansion of Theorem 2.10, it seems possible to derive a general asymptotic expansion of type (3.23)-(3.24) and prove rigorous error bounds at any order N: we plan to pursue this research direction in a forthcoming paper.

We would like to emphasize that, in particular, the accuracy would depend on the intrinsic regularity of the payoff function. For instance, $\varphi_{\mathrm{fix}} \in C^{1,1}_{B,\mathrm{loc}}$ (see Example 3.13) and the short-time asymptotic convergence would be of order $(N+3)/2$. This is an interesting point because the same approximating technique would return a slower asymptotic convergence, namely $(N+2)/2$, if we would only take into account the Euclidean regularity of the payoff, i.e. $\varphi_{\mathrm{fix}} \in C^{0,1}_{\mathrm{loc}}$. Furthermore, it is also interesting to observe that, if we assumed the coefficient σ to be also dependent on x_2, the partial derivatives w.r.t. x_2 would start appearing in the intrinsic Taylor polynomial, and thus in the operators $\mathcal{G}_n$, only from $n = 2$. Performing the expansion by means of the classical Taylor polynomial would thus yield some additional terms that have no impact on the asymptotic convergence.

4 Preliminaries

In this section we collect several results that are preliminary to the proof of Theorem 2.10. We first recall that under the standing Assumption 2.1, the matrix B takes the form

$$B = \begin{pmatrix} 0_{p_0\times p_0} & 0_{p_0\times p_1} & \cdots & 0_{p_0\times p_{r-1}} & 0_{p_0\times p_r} \\ B_1 & 0_{p_1\times p_1} & \cdots & 0_{p_1\times p_{r-1}} & 0_{p_1\times p_r} \\ 0_{p_2\times p_0} & B_2 & \cdots & 0_{p_2\times p_{r-1}} & 0_{p_2\times p_r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{p_r\times p_0} & 0_{p_r\times p_1} & \cdots & B_r & 0_{p_r\times p_r} \end{pmatrix}, \tag{4.28}$$

where $0_{p_i\times p_j}$ is a $p_i \times p_j$ null block. We also recall notation (2.12) for $\bar{p}_k$ with $k = 0,\ldots,r$. As a direct consequence of (4.28), we have that for any $n \le r$

$$B^n = \begin{pmatrix} 0_{\bar{p}_{n-1}\times p_0} & 0_{\bar{p}_{n-1}\times p_1} & \cdots & 0_{\bar{p}_{n-1}\times p_{r-n}} & 0_{\bar{p}_{n-1}\times(\bar{p}_r-\bar{p}_{r-n})} \\ \prod\limits_{j=1}^{n} B_j & 0_{p_n\times p_1} & \cdots & 0_{p_n\times p_{r-n}} & 0_{p_n\times(\bar{p}_r-\bar{p}_{r-n})} \\ 0_{p_{n+1}\times p_0} & \prod\limits_{j=2}^{n+1} B_j & \cdots & 0_{p_{n+1}\times p_{r-n}} & 0_{p_{n+1}\times(\bar{p}_r-\bar{p}_{r-n})} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{p_r\times p_0} & 0_{p_r\times p_1} & \cdots & \prod\limits_{j=r-n+1}^{r} B_j & 0_{p_r\times(\bar{p}_r-\bar{p}_{r-n})} \end{pmatrix}, \tag{4.29}$$

where

$$\prod_{j=1}^{n} B_j = B_n B_{n-1}\cdots B_1.$$

Moreover $B^n = 0$ for $n > r$, so that

$$e^{\delta B} = I_d + \sum_{h=1}^{r} \frac{B^h}{h!}\delta^h, \tag{4.30}$$

where I_d is the $d\times d$ identity matrix.

We also recall the notation (2.11): for any $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$ and $n = 0,\ldots,r$, we denote by $x^{[n]} \in \mathbb{R}^d$ the projection of x on $\{0\}^{\bar{p}_{n-1}} \times \mathbb{R}^{p_n} \times \{0\}^{d-\bar{p}_n}$. Then $\mathbb{R}^d$ can be represented as a direct sum:

$$\mathbb{R}^d = \bigoplus_{n=0}^{r} V_n, \qquad V_n := \{x^{[n]} \mid x \in \mathbb{R}^d\}, \quad n = 0,\ldots,r.$$

Remark 4.17. By (4.29) it is clear that

$$B^n v \in \bigoplus_{k=n}^{r} V_k, \qquad v \in \mathbb{R}^d, \tag{4.31}$$

and if $v \in V_0$ then

$$B^n v \in V_n, \qquad n = 0,\ldots,r. \tag{4.32}$$

More precisely, let us set

$$\bar{B}_n = \begin{pmatrix} 0_{\bar{p}_{n-1}\times p_0} & 0_{\bar{p}_{n-1}\times(r-p_0)} \\ \prod\limits_{j=1}^{n} B_j & 0_{p_n\times(r-p_0)} \\ 0_{(\bar{p}_r-\bar{p}_n)\times p_0} & 0_{(\bar{p}_r-\bar{p}_n)\times(r-p_0)} \end{pmatrix},$$

where the $p_n \times p_0$ matrix $\prod\limits_{j=1}^{n} B_j$ has full rank. Then we have

$$B^n v = \bar{B}_n v, \qquad v \in V_0,$$

and the linear application $\bar{B}_n : V_0 \to V_n$ is surjective but, in general, not injective. For this reason, for any $n = 1, \cdots, r$, we define the subspaces $V_{0,n} \subseteq V_0$ as

$$V_{0,n} = \{x \in V_0 | x_j = 0 \ \forall j \notin \Pi_{B,n}\},$$

with $\Pi_{B,n}$ being the set of the indexes corresponding to the first p_n linear independent columns of $\prod\limits_{j=1}^{n} B_j$. It is now trivial that the linear map

$$\bar{B}_n : V_{0,n} \to V_n$$

is also injective. Notice that

$$V_{0,r} \subseteq V_{0,r-1} \subseteq \cdots \subseteq V_{0,1} \subseteq V_{0,0} := V_0. \tag{4.33}$$

4.1 Commutators and integral paths

In this section we construct approximations of the integral paths of the commutators of the vector fields $X_1, \ldots, X_{p_0}$ and Y in (1.4). In the sequel we shall use the following notations: for any $v \in \mathbb{R}^d$ we set

$$Y_v^{(0)} = \sum_{i=1}^{d} v_i \partial_{x_i}.$$

Hereafter we will always consider $v \in V_0$. In such way $Y_v^{(0)}$ will be actually a linear combination of $X_1, \ldots, X_{p_0}$. Moreover we define recursively

$$Y_v^{(n)} = [Y_v^{(n-1)}, Y] = Y_v^{(n-1)} Y - Y Y_v^{(n-1)}, \qquad n \in \mathbb{N}. \tag{4.34}$$

Remark 4.18. By induction it is straightforward to show that for any $u \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$, we have

$$Y_v^{(n)} u = \langle B^n v, \nabla u \rangle, \qquad n \in \mathbb{N},$$

with $B^n v \in V_n$ by (4.32).

When applied to functions in $C^{n,\alpha}_{B,\mathrm{loc}}$, operator $Y_v^{(n)}$ can be interpreted as a composition of Lie derivatives. Indeed we have the following.

Lemma 4.19. *Let $n \in \mathbb{N}$ and $u \in C^{n,\alpha}_{B,loc}$. Then, for any $v \in V_0$ and $k \in \mathbb{N} \cup \{0\}$ with $2k+1 \le n$, we have $Y_v^{(k)} u \in C^{n-2k-1,\alpha}_{B,loc}$.*

Proof. If $k = 0$, the thesis is obvious since, by assumption, $\partial_{x_i} u \in C^{n-1,\alpha}_{B,\mathrm{loc}}$ for $i = 1, \dots, p_0$. To prove the general case we proceed by induction on n. If $n \le 2$ there is nothing to prove because we only have to consider the case $k = 0$. Fix now $n \ge 2$. We assume the thesis to hold for any $m \le n$ and prove it true for $n + 1$. We proceed by induction on k. We have already shown the case $k = 0$. Thus, we assume the statement to hold for $k \in \mathbb{N} \cup \{0\}$ with $2(k + 1) + 1 \le n + 1$ and we prove it true for $k + 1$. Note that, by definition (4.34) we clearly have

$$Y_v^{(k+1)} u = Y_v^{(k)} Y u - Y\, Y_v^{(k)} u,$$

with $v \in V_0$. Then the thesis follows by inductive hypothesis and since, by definition of intrinsic Hölder space, $Y u \in C^{n-1,\alpha}_{B,\mathrm{loc}}$. □

Next we show how to approximate the integral curves of the commutators $Y_v^{(k)}$ by using a rather classical technique from control theory. For any $n \in \{0, \dots, r\}$, $z = (t, x) \in \mathbb{R} \times \mathbb{R}^d$, $\delta \in \mathbb{R}$ and $v \in V_0$, we define iteratively the family of trajectories $\left(\gamma^{(n,k)}_{v,\delta}(z)\right)_{k=n,\dots,r}$ as

$$\gamma^{(n,n)}_{v,\delta}(z) = e^{\delta^{2n+1} Y_v^{(n)}}(z) = \big(t, x + \delta^{2n+1} B^n v\big), \tag{4.35}$$

$$\gamma^{(n,k+1)}_{v,\delta}(z) = e^{-\delta^2 Y}\left(\gamma^{(n,k)}_{v,-\delta}\left(e^{\delta^2 Y}\left(\gamma^{(n,k)}_{v,\delta}(z)\right)\right)\right), \qquad n \le k \le r-1. \tag{4.36}$$

We also set

$$\gamma^{(-1,k)}_{v,\delta}(z) = \gamma^{(0,k)}_{v,\delta}(z), \qquad 0 \le k \le r. \tag{4.37}$$

Lemma 4.20. *For any $n \in \{0, \cdots, r\}$, $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, $\delta \in \mathbb{R}$ and $v \in V_0$ we have*

$$\gamma^{(n,k)}_{v,\delta}(t, x) = (t, x + S_{n,k}(\delta) v)\,, \qquad k = n, \dots, r, \tag{4.38}$$

where

$$S_{n,n}(\delta) = \delta^{2n+1} B^n v \quad \text{and} \quad S_{n,k}(\delta) = (-1)^{k-n} \delta^{2n+1} B^n \sum_{\substack{h \in \mathbb{N}^{k-n} \\ |h| \le r}} \frac{(-B)^{|h|}}{h!} \delta^{2|h|}, \qquad k = n+1, \dots, r, \tag{4.39}$$

with $|h| = h_1 + \cdots + h_k$.

Proof. Fix $n = 0$ and proceed by induction on k. The case $k = n$ is trivial. Now, assuming (4.38)-(4.39) as inductive hypothesis and noting that $S_k(-\delta) = -S_k(\delta)$, we have

$$\begin{aligned}
\gamma^{(k+1)}_{v,\delta}(t, x) &= e^{-\delta^2 Y}\left(\gamma^{(n,k)}_{v,-\delta}\left(e^{\delta^2 Y}\left(\gamma^{(n,k)}_{v,\delta}(t, x)\right)\right)\right) = e^{-\delta^2 Y}\left(\gamma^{(n,k)}_{v,-\delta}\left(e^{\delta^2 Y}\left(t, x + S_{n,k}(\delta) v\right)\right)\right) \\
&= e^{-\delta^2 Y}\left(\gamma^{(n,k)}_{v,-\delta}\left(t + \delta^2, e^{\delta^2 B}\left(x + S_{n,k}(\delta) v\right)\right)\right) = e^{-\delta^2 Y}\left(t + \delta^2, e^{\delta^2 B}\big(x + S_{n,k}(\delta) v\big) - S_{n,k}(\delta) v\right) \\
&= \left(t, e^{-\delta^2 B}\left(e^{\delta^2 B}\big(x + S_{n,k}(\delta) v\big) - S_{n,k}(\delta) v\right)\right) = \left(t, x + S_{n,k}(\delta) v - e^{-\delta^2 B} S_{n,k}(\delta) v\right).
\end{aligned}$$

On the other hand, by (4.30) we have

$$\begin{aligned}
x + S_{n,k}(\delta) v - e^{-\delta^2 B} S_{n,k}(\delta) v &= x + S_{n,k}(\delta) v - \left(I_d + \sum_{j=1}^{r} \frac{(-B)^j}{j!} \delta^{2j}\right) S_{n,k}(\delta) v \\
&= x - \left(\sum_{j=1}^{r} \frac{(-B)^j}{j!} \delta^{2j}\right) S_{n,k}(\delta) v = x + S_{n,k+1}(\delta) v,
\end{aligned}$$

and this concludes the proof. □

Remark 4.21. Note that

$$S_{n,k}(\delta) = \delta^{2k+1}B^k + \widetilde{S}_{n,k}(\delta), \qquad n \le k \le r,$$

with

$$\widetilde{S}_{n,n}(\delta) := 0 \quad \text{ and } \quad \widetilde{S}_{n,k}(\delta) := (-1)^{k-n}\delta^{2n+1}B^n \sum_{\substack{h\in\mathbb{N}^{k-n}\\ k-n<|h|\le r}} \frac{(-B)^{|h|}}{h!}\delta^{2|h|}, \qquad k = n+1,\dots,r.$$

Then we deduce from (4.38) that

$$\gamma_{v,\delta}^{(n,k)}(z) = \big(t, x + \delta^{2k+1}B^k v\big) + \big(0, \widetilde{S}_{n,k}(\delta)v\big), \qquad n \le k. \tag{4.40}$$

It is important to remark that $\widetilde{S}_{n,r}(\delta) = 0$ and, by (4.31), we have

$$\widetilde{S}_{n,k}(\delta)v \in \bigoplus_{j=k+1}^{r} V_j, \qquad k = n,\dots,r; \tag{4.41}$$

since $v \in V_0$, then by (4.32) we have

$$\gamma_{v,\delta}^{(n,n)}(z) = (t,x) + (0, \delta^{2n+1}B^n v), \quad \text{with } B^n v \in V_n.$$

Thus, by using notation (2.11), for any $k = n,\dots,r$ we have

$$\Big|\big(\widetilde{S}_{n,k}(\delta)v\big)^{[j]}\Big| \le c_B|\delta|^{2j+1}|v|, \qquad j = k+1,\dots,r, \quad \delta \in \mathbb{R}, \tag{4.42}$$

where the constant c_B *depends only on the matrix* B. If $|v| = 1$, (4.42) also implies

$$\big\|\big(\gamma_{v,\delta}^{(n,k)}(z)\big)^{-1} \circ z\big\|_B = \big\|z^{-1} \circ \gamma_{v,\delta}^{(n,k)}(z)\big\|_B = \Big\|\big((t, x + \delta^{2k+1}B^k v) + (0, \widetilde{S}_{n,k}(\delta)v)\big)^{-1} \circ (t,x)\Big\|_B$$

$$= \Big\|\big(0, -\delta^{2k+1}B^k v - \widetilde{S}_{n,k}(\delta)v\big)\Big\|_B = |-\delta^{2k+1}B^k v - \widetilde{S}_{n,k}(\delta)v|_B \le c_B|\delta|. \tag{4.43}$$

Next we show how to connect two points in $\mathbb{R}\times\mathbb{R}^d$ that only differ w.r.t. the spatial components by only moving along the the integral curves $\gamma^{(n,k)}$ previously defined.

Lemma 4.22. *Let* $n \in \{0,\cdots,r\}$, $\zeta = (t,\xi) \in \mathbb{R}\times\mathbb{R}^d$, $y \in \bigoplus_{k=n}^{r} V_k$ *and the points* $\zeta_k = (t,\xi_k)$, *for* $k = n-1,\cdots,r$, *defined as*

$$\zeta_{n-1} := \zeta, \qquad\qquad \zeta_k := \gamma_{v_k,\delta_k}^{(n-1,k)}(\zeta_{k-1}), \quad v_k = \frac{w_k}{|w_k|}, \qquad \delta_k = \big|w_k\big|, \quad k \ge n,$$

where w_k *is the only vector in* $V_{0,k} \subseteq V_0$ *such that* $B^k w_k = y^{[k]} + \xi^{[k]} - \xi_{k-1}^{[k]}$. *Then:*

i) for any $k \in \{n,\cdots,r\}$ *we have:*

$$\delta_k \le c_B|y|_B, \qquad\qquad \xi_k^{[j]} = \xi^{[j]} + y^{[j]}, \quad j = 0,\dots,k. \tag{4.44}$$

Note that, in particular, $\zeta_r = \zeta + (0,y)$;

ii) there exists a positive constant c_B, only dependent on the matrix B, *such that*

$$\left\|\zeta_k^{-1} \circ \zeta\right\|_B \leq c_B |y|_B, \tag{4.45}$$

for any $k = n, \cdots, r$ *and* $0 \leq \delta \leq \delta_k$.

Proof. We first prove i). The second identity in (4.44) easily stems from (4.41) and by definition of v_k and δ_k. We then focus on the first one. By Remark 4.17, it is easy to prove that

$$\delta_k \leq c_B \left|\xi^{[k]} + y^{[k]} - \xi_{k-1}^{[k]}\right|^{\frac{1}{2k+1}}. \tag{4.46}$$

Moreover, by (4.42) we get

$$\left|\xi_k^{[j]} - \xi_{k-1}^{[j]}\right| \leq c_B \left|\delta_k\right|^{2j+1}, \qquad j = k+1, \ldots, r. \tag{4.47}$$

We proceed by induction on k. For $k = n$ the thesis immediately follows by (4.46). We now fix $n \leq k \leq r-1$ and assume the estimate to hold for any $n \leq h \leq k$. By (4.46) we have

$$\delta_{k+1} \leq c_B \left|\xi^{[k+1]} + y^{[k+1]} - \xi_k^{[k+1]}\right|^{\frac{1}{2(k+1)+1}} \leq c_B \left|y^{[k+1]}\right|^{\frac{1}{2(k+1)+1}} + c_B \sum_{h=n}^{k} \left|\xi_h^{[k+1]} - \xi_{h-1}^{[k+1]}\right|^{\frac{1}{2(k+1)+1}}$$

(by (4.47))

$$\leq c_B \left|y^{[k+1]}\right|^{\frac{1}{2(k+1)+1}} + c_B \sum_{h=n}^{k} \delta_h,$$

and the thesis for $k+1$ follows by inductive hypothesis.

We now prove ii). As first step we prove that

$$\left\|\left(\gamma_{v_k,\delta}^{(n-1,k)}(\zeta_{k-1})\right)^{-1} \circ \zeta_{k-1}\right\|_B \leq c_B |y|_B.$$

By equations (4.38), (4.43) and (4.44) we get

$$\begin{aligned}
\left\|\left(\gamma_{v_k,\delta}^{(n-1,k)}(\zeta_{k-1})\right)^{-1} \circ \zeta_{k-1}\right\|_B &= \left\|(t, \xi_{k-1} + S_{n-1,k}(\delta) v_k)^{-1} \circ (t, \xi_{k-1})\right\|_B \\
&= \left\|(0, \xi_{k-1} - (\xi_{k-1} + S_{n-1,k}(\delta) v_k))\right\|_B \\
&= \left\|(0, -S_{n-1,k}(\delta) v_k)\right\|_B \leq c_B \delta_k \leq c_B |y|_B.
\end{aligned}$$

This estimate along with equations (4.43) and (4.44) allow us to conclude. Precisely, applying the quasi-triangular inequality we get

$$\left\|\zeta_k^{-1} \circ \zeta\right\|_B \leq c_B \sum_{i=n}^{k} \left\|\zeta_i^{-1} \circ \zeta_{i-1}\right\|_B \leq c_B |y|_B.$$

□

We conclude the section with the following remark that allows to control the homogeneous distance between two points along the same integral curve of Y.

Remark 4.23. By (2.10) and (2.6) we have

$$\left\|z^{-1} \circ e^{\delta Y}(z)\right\|_B = \left\|(e^{\delta Y}(z))^{-1} \circ z\right\|_B = |\delta|^{\frac{1}{2}}, \qquad z \in \mathbb{R} \times \mathbb{R}^d, \quad \delta \in \mathbb{R}, \tag{4.48}$$

5 Proof of Theorem 2.10

Theorem 2.10 will be proved by induction on n, through the following steps:

- **Step 1**: Proof for $n = 0$;
- **Step 2**: Induction from $2n$ to $2n + 1$ for any $0 \le n \le r$;
- **Step 3**: Induction from $2n + 1$ to $2(n + 1)$ for any $0 \le n \le r - 1$;
- **Step 4**: Induction from n to $n + 1$ for any $n \ge 2r + 1$.

A brief explanation is needed: the proof of Theorem 2.10 cannot be carried out by a simple induction on n, due to the qualitative differences in the Taylor polynomials of different orders. For instance, one could suppose the theorem to hold for $n = 2$ and consider a function $u \in C^{3,\alpha}_{B,\mathrm{loc}}$. By the inclusion property

$$C^{3,\alpha}_{B,\mathrm{loc}} \subseteq C^{2,\alpha}_{B,\mathrm{loc}},$$

all the derivatives of second B-order do exist, i.e.

$$Y^k \partial_x^\beta u \in C^{2-2k-|\beta|_B,\alpha}_{B,\mathrm{loc}}, \qquad 2k + |\beta|_B \le 2.$$

However, $T_3 u$ also contains the derivatives of B-order equal to 3. These are exactly

$$\partial_{x_i,x_j,x_k} u, \quad Y\partial_{x_i} u \qquad 1 \le i, j, k \le p_0,$$

whose existence is granted by definition of $C^{3,\alpha}_{B,\mathrm{loc}}$, and the Euclidean derivatives

$$\partial_{x_l} u, \qquad p_0 < l \le \bar{p}_1,$$

whose existence must be proved, as it is not trivially implied by definition of $C^{3,\alpha}_{B,\mathrm{loc}}$. In general, such problem arises every time when defining the Taylor expansion of order $2n + 1$, $n = 1, \dots, r$, i.e. when the Euclidean derivatives w.r.t. the variables of level n appear for the first time in the Taylor polynomial. This motivates the need to treat the inductive step from $2n$ to $2n + 1$ in a separate way and therefore the necessity for Step 2 and Step 3 in the proof. Eventually, Step 4 is justified by the fact that, when $n \ge 2r + 1$, the existence of the Euclidean partial derivatives w.r.t. any variable has already been proved and thus the proof goes smoothly without any further complication.

We now try to summarize the main arguments on which the proof is based. Roughly speaking, in order to prove the estimate (2.13) (and (2.16)), we shall be able to connect any pair of points $z, \zeta \in \mathbb{R} \times \mathbb{R}^d$ and to have a control of the increment of u along the connecting path. The definition of $C^{n,\alpha}_{B,\mathrm{loc}}$ (and $C^{n,\alpha}_B$) does only specify the regularity along the fields Y and $(\partial_{x_i})_{1 \le i \le p_0}$, but does not give any a priori information about the regularity along all the other Euclidean fields $(\partial_{x_i})_{p_0 < i \le d}$. It seems then clear that, when trying to connect z and ζ, we cannot simply *move* along the canonical directions $(e_i)_{1 \le i \le d}$. We shall indeed take advantage of Lemma 4.22 in order to go from ζ to z by using the integral curves $\gamma^{(n,k)}$ and then control the increment of u along the connecting paths by exploiting the estimates contained in Remark 4.21.

To simplify the exposition, for each point listed above, we will first prove the inductive step on the global version (Part 3) of Theorem 2.10, in order to keep the proof free from additional technicalities needed to prevent the possibility of the integral curves $\gamma^{(n,k)}$ to exit the domain Ω. At the end of the section we will sketch the main guidelines through which the proof of the local version (Part 1 and Part 2) will become a straightforward modification of the global one. In order to prove the main theorem we will need to state three auxiliary results, which will be proved step by step along with Theorem 2.10.

Proposition 5.24. *Let $u \in C_B^{2n+1,\alpha}$ with $\alpha \in]0,1]$ and $n \in \mathbb{N}_0$ with $n \le r$. Then, there exist the Euclidean partial derivatives $\partial_{x_i} u \in C_B^{0,\alpha}$ for any $\bar{p}_{n-1} < i \le \bar{p}_n$ and*

$$Y^{(n)}_{v_i^{(n)}} u(z) = \partial_{x_i} u(z), \qquad z \in \mathbb{R} \times \mathbb{R}^d, \tag{5.49}$$

with $\big(v_i^{(n)}\big)_{\bar{p}_{n-1} < i \le \bar{p}_n}$ being the family of vectors such that $v_i^{(n)} \in V_{0,n}$ with $B^n v_i^{(n)} = e_i$. Note that such family of vectors is univocally defined (see Remark 4.17).

Proposition 5.25. *Let $\alpha \in]0,1]$, $n \in \mathbb{N}_0$ with $n \le r$, $m \in \{0,1\}$ and $u \in C_B^{2n+m,\alpha}$. Then, for any $\max\{n-1,0\} \le k \le r$ and $v \in V_{0,k}$ with $|v| = 1$, we have:*

$$\Big|u\Big(\gamma_{v,\delta}^{(n-1,k)}(z)\Big) - T_{2n+m} u\Big(z, \gamma_{v,\delta}^{(n-1,k)}(z)\Big)\Big| \le c_B \|u\|_{C_B^{2n+m,\alpha}} |\delta|^{2n+m+\alpha}, \qquad z = (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad \delta \in \mathbb{R}, \tag{5.50}$$

where c_B is a positive constant that only depends on B.

Proposition 5.26. *Let $\alpha \in]0,1]$, $n \in \mathbb{N}_0$ with $n \le r$, $m \in \{0,1\}$ and $u \in C_B^{2n+m,\alpha}$. Then, we have:*

$$\big|u(t,x) - T_{2n+m} u\big((t,x),(t,x+\xi)\big)\big| \le c_B \|u\|_{C_B^{2n+m,\alpha}} |\xi|_B^{2n+m+\alpha}, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad \xi \in \bigoplus_{j=0}^{n-1} V_j,$$

where c_B is a positive constant that only depends on B.

Propositions 5.25 and 5.26 are particular cases of the main theorem and are preparatory to its proof. We will also make repeated use of the following.

Remark 5.27. Let $n \in \mathbb{N}_0$, $m \in \{0,1\}$ and $u \in C_B^{2n+m,\alpha}$. Then, by Definition 2.8, we have $Y^n u \in C_Y^{m+\alpha}$. Therefore, by the Euclidean mean-value theorem along the vector field Y, for any $z = (t,x) \in \mathbb{R} \times \mathbb{R}^d$ and $\delta \in \mathbb{R}$, there exists $\bar{\delta}$ with $|\bar{\delta}| \le |\delta|$ such that

$$u\big(e^{\delta Y}(z)\big) - u(z) - \sum_{i=1}^{n} \frac{\delta^i}{i!} Y^i u(z) = \delta^n \left(Y^n u\big(e^{\bar{\delta} Y}(z)\big) - Y^n u(z)\right),$$

and thus, by Definition 2.6 along with Assumption 2.4,

$$\Big|u\big(e^{\delta Y}(z)\big) - u(z) - \sum_{i=1}^{n} \frac{\delta^i}{i!} Y^i u(z)\Big| \le \|u\|_{C_B^{2n+m,\alpha}} |\delta|^{n+\frac{m+\alpha}{2}}, \qquad \delta \in \mathbb{R}, \quad z \in \mathbb{R} \times \mathbb{R}^d.$$

5.1 Step 1

Here we give the proofs for

- Proposition 5.25 for $n = 0$, $m = 0$;
- Theorem 2.10 (Part 3) for $n = 0$.

We start by recalling that:

$$T_0 u\big(z, \zeta\big) = u(z), \qquad z, \zeta \in \mathbb{R} \times \mathbb{R}^d.$$

Proof of Propostion 5.25 for $n = 0$, $m = 0$.
We prove the thesis by induction on k. For $k = 0$ the estimate (5.50) trivially follows by combining definitions (4.37) and (4.35) with the assumptions $v \in V_0$, $|v| = 1$ and $u \in C^{\alpha}_{\partial_{x_i}}$ for any $i = 1, \dots, p_0$.

We now assume the thesis to hold for $k \geq 0$ and we prove it true for $k+1$. We recall (4.36) and set

$$z_0 = z, \quad z_1 = \gamma^{(0,k)}_{v,\delta}(z_0), \quad z_2 = e^{\delta^2 Y}\left(z_1\right), \quad z_3 = \gamma^{(0,k)}_{v,-\delta}(z_2), \quad z_4 = e^{-\delta^2 Y}\left(z_3\right) = \gamma^{(0,k+1)}_{v,\delta}(z) = \gamma^{(-1,k+1)}_{v,\delta}(z).$$

Now, by triangular inequality we get

$$\left|u\big(\gamma^{(-1,k+1)}_{v,\delta}(z)\big) - u(z)\right| \leq \sum_{i=1}^{4} \left|u(z_i) - u(z_{i-1})\right|,$$

and thus, (5.50) for $k+1$ follows from the inductive hypothesis and from the assumption $u \in C^{\alpha}_{Y}$. $\square$

We are now ready to prove Part 3 of Theorem 2.10 for $n = 0$.

Proof of Theorem 2.10 (Part 3) for $n = 0$.
We first consider the particular case $z = (t, x)$, $\zeta = (t, \xi)$, with $x, \xi \in \mathbb{R}^d$. Precisely, we show that, if $u \in C^{0,\alpha}_{B}$ we have

$$|u(t,x) - u(t,\xi)| \leq c_B \|u\|_{C^{0,\alpha}_B} |x - \xi|^{\alpha}_{B}, \qquad t \in \mathbb{R}, \quad x, \xi \in \mathbb{R}^d. \tag{5.51}$$

By the triangular inequality, we obtain

$$|u(t,x) - u(t,\xi)| \leq \sum_{i=0}^{r} |u(\zeta_i) - u(\zeta_{i-1})|,$$

where the points $\zeta_k = (t, \xi_k)$, for $k = -1, 0, \cdots, r$, are defined as in Lemma 4.22 by setting $n = 0$ and $v = x - \xi$. The estimate (5.51) then stems from (5.50) with $n = 0$, combined with (4.44).

We now prove the general case. For any $z = (t,x), \zeta = (s,\xi) \in \mathbb{R} \times \mathbb{R}^d$, by triangular inequality we get

$$\begin{aligned} |u(z) - u(\zeta)| &\leq |u(z) - u(e^{(t-s)Y}(\zeta))| + |u(e^{(t-s)Y}(\zeta)) - u(\zeta)| \\ &= |u(t,x) - u(t, e^{(t-s)B}\xi)| + |u(e^{(t-s)Y}(\zeta)) - u(\zeta)|. \end{aligned} \tag{5.52}$$

Now, to prove (2.16), we use (5.51) to bound the first term in (5.52), $u \in C^{\alpha}_{Y}$ to bound the second one, and we obtain

$$|u(z) - u(\zeta)| \leq c_B \|u\|_{C^{0,\alpha}_B} \|\zeta^{-1} \circ z\|^{\alpha}_{B}, \qquad z = (t,x), \ \zeta = (s,\xi) \in \mathbb{R} \times \mathbb{R}^d,$$

which concludes the proof. $\square$

5.2 Step 2

Throughout this section we fix $\bar{n} \in \{0, \cdots, r\}$ and assume to be holding true:

- Proposition 5.24 for any $0 \le n \le \bar{n} - 1$, if $\bar{n} \ge 1$;
- Theorem 2.10 for any $0 \le n \le 2\bar{n}$.

Then we prove:

- Propositions 5.25 and 5.26 for $n = \bar{n}, m = 1$;
- Proposition 5.24 for $n = \bar{n}$;
- Theorem 2.10 (Part 3) for $n = 2\bar{n} + 1$.

This induction step has to be treated separately because we cannot assume a priori the existence of the first order Euclidean partial derivatives w.r.t. the $\bar{n}$-th level variables. Therefore, we introduce the following alternative definition of $(2\bar{n} + 1)$-th order B-Taylor polynomial of u that does not make explicit use of the derivatives $\left(\partial_{\bar{p}_{\bar{n}-1}+i}u\right)_{1 \le i \le p_{\bar{n}}}$:

$$\bar{T}_{2\bar{n}+1}u(\zeta, z) := \sum_{\substack{0 \le 2k+|\beta|_B \le 2\bar{n}+1 \\ \beta^{[\bar{n}]}=0}} \frac{1}{k!\,\beta!}\left(Y^k \partial_\xi^\beta u(\zeta)\right)(t-s)^k \left(x - e^{(t-s)B}\xi\right)^\beta + \sum_{i=\bar{p}_{\bar{n}-1}+1}^{\bar{p}_{\bar{n}}} \left(Y^{(\bar{n})}_{v_i^{(\bar{n})}} u(\zeta)\right)\left(x - e^{B(t-s)}\xi\right)_i, \qquad z = (t, x),\ \zeta = (s, \xi) \in \mathbb{R} \times \mathbb{R}^d, \tag{5.53}$$

with $\left(v_i^{(\bar{n})}\right)_{\bar{p}_{\bar{n}-1} < i \le \bar{p}_{\bar{n}}}$ being the family of vectors such that $v_i^{(\bar{n})} \in V_{0,\bar{n}}$ with $B^{\bar{n}} v_i^{(\bar{n})} = e_i$.

Remark 5.28. The Taylor polynomial $\bar{T}_{2\bar{n}+1}u$ is well-defined for any $u \in C^{2\bar{n}+1,\alpha}_{B,\mathrm{loc}}$. In fact, by Lemma 4.19 we have

$$Y^{(\bar{n})}_{v_i^{(\bar{n})}} u \in C^{0,\alpha}_{B,\mathrm{loc}}, \qquad \bar{p}_{\bar{n}-1} < i \le \bar{p}_{\bar{n}}.$$

On the other hand, by using the inclusion of the spaces $C^{n,\alpha}_{B,\mathrm{loc}}$ and the inductive hypothesis (Theorem 2.10, Part 1), the Euclidean derivatives

$$\partial_\xi^\beta u(\zeta), \qquad 0 \le |\beta|_B \le 2\bar{n} + 1, \quad \beta^{[\bar{n}]} = 0,$$

are well defined. Therefore, by combining the inductive hypothesis on Proposition 5.24 and Lemma 4.19, we have

$$Y^k \partial_\xi^\beta u(\zeta) \in C^{2\bar{n}+1-2k-|\beta|_B,\alpha}_{B,\mathrm{loc}}, \qquad 0 \le 2k + |\beta|_B \le 2\bar{n} + 1, \quad \beta^{[\bar{n}]} = 0.$$

In particular, by analogous arguments, if $u \in C^{2\bar{n}+1,\alpha}_B$ we have that

$$Y^{(\bar{n})}_{v_i^{(\bar{n})}} u \in C^{0,\alpha}_B, \qquad \bar{p}_{\bar{n}-1} < i \le \bar{p}_{\bar{n}}, \tag{5.54}$$

$$Y^k \partial_\xi^\beta u(\zeta) \in C^{2\bar{n}+1-2k-|\beta|_B,\alpha}_B, \qquad 0 \le 2k + |\beta|_B \le 2\bar{n} + 1, \quad \beta^{[\bar{n}]} = 0. \tag{5.55}$$

Remark 5.29. By simple linear algebra arguments, it is also easy to show that for a given $\alpha \in [0,1]$, $n \in \{0, \cdots, r\}$ and $u \in C_B^{2n+1,\alpha}$, we have

$$\sum_{i=\bar{p}_{n-1}+1}^{\bar{p}_n} \left(Y^{(n)}_{v_i^{(n)}} u(\zeta) \right) (B^n v)_i = Y_v^n u(\zeta), \qquad \zeta \in \mathbb{R} \times \mathbb{R}^d, \quad v \in V_{0,n}.$$

5.2.1 Proof of Propositions 5.25 and 5.26, for $\mathbf{n = \bar{n}}$ and $\mathbf{m = 1}$

We prove Propositions 5.25 and 5.26 on $\bar{T}_{2\bar{n}+1}u$, for $n = \bar{n}$ and $m = 1$. Note that, after proving Proposition 5.24 for $n = \bar{n}$, the two versions of the Taylor polynomials $\bar{T}_{2\bar{n}+1}u$ and $T_{2\bar{n}+1}u$ will turn out to be equivalent.

Proof of Proposition 5.25 for $n = \bar{n}, m = 1$.
We assume $u \in C_B^{2\bar{n}+1,\alpha}$ and we have to prove that for any $\max\{\bar{n}-1, 0\} \leq k \leq r$, $v \in V_{0,k}$ with $|v| = 1$, and $z = (t,x) \in \mathbb{R} \times \mathbb{R}^d$, we have

$$u\Big(\gamma_{v,\delta}^{(\bar{n}-1,k)}(z)\Big) = \bar{T}_{2\bar{n}+1}u\Big(z, \gamma_{v,\delta}^{(\bar{n}-1,k)}(z)\Big) + R_{v,\delta}^{(\bar{n}-1,k)}(z), \tag{5.56}$$

with

$$|R_{v,\delta}^{(\bar{n}-1,k)}(z)| \leq c_B \|u\|_{C_B^{2\bar{n}+1,\alpha}} |\delta|^{2\bar{n}+1+\alpha}, \qquad z = (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad \delta \in \mathbb{R}. \tag{5.57}$$

We prove (5.57) by induction on k.
Proof for $k = \max\{\bar{n}-1, 0\}$: because of the particular definition of $\gamma_{v,\delta}^{(n,k)}$ we have to treat separately the cases $\bar{n} = 0$, $\bar{n} = 1$ and $\bar{n} > 1$.
<u>Case $\bar{n} = 0$</u>: by (4.37) and (4.35) we have

$$\gamma_{v,\delta}^{(-1,0)}(z) = u(t, x + \delta v),$$

and thus, by (5.53), (5.56) for $k = 0$ reads as

$$u(t, x + \delta v) = u(t,x) + \delta \sum_{i=1}^{p_0} \partial_{x_i} u(t,x) v_i + R_{v,\delta}^{(-1,0)}(z).$$

Now, by the standard mean-value theorem, there exist $(\bar{v}_i)_{i=1,\dots,p_0}$ with $\bar{v}_i \in V_0$ and $|\bar{v}_i| \leq |v| \leq 1$, such that

$$u(t, x + \delta v) - u(t,x) = \delta \sum_{i=1}^{p_0} \partial_{x_i} u(t, x + \delta \bar{v}_i) v_i,$$

and thus

$$R_{v,\delta}^{(-1,0)}(z) = \delta \sum_{i=0}^{p_0} (\partial_{x_i} u(t, x + \delta \bar{v}_i) - \partial_{x_i} u(t,x)) v_i.$$

Note that $\partial_{x_i} u \in C_B^{0,\alpha}$ for any $1 \leq i \leq p_0$ because $u \in C_B^{1,\alpha}$ by assumption. Therefore estimate (5.57) stems from Part 3 of Theorem 2.10 for $n = 0$.
<u>Case $\bar{n} = 1$</u>: by (4.35) we have

$$\gamma_{v,\delta}^{(0,0)}(z) = u(t, x + \delta v),$$

and thus, by (5.53), (5.56) for $k=0$ reads as

$$u(t,x+\delta v)=u(t,x)+\delta\sum_{i=1}^{p_0}\partial_{x_i}u(t,x)v_i+\frac{\delta^2}{2!}\sum_{i,j=1}^{p_0}\partial_{x_ix_j}u(t,x)v_iv_j+\frac{\delta^3}{3!}\sum_{i,j,l=1}^{p_0}\partial_{x_ix_jx_l}u(t,x)v_iv_jv_l+R^{(0,0)}_{v,\delta}(z).$$

Now, by the mean-value theorem, there exist $(\bar v_{i,j,k})_{1\le i,j,k\le p_0}$, with $\bar v_{i,j,k}\in V_0$ and $|\bar v_{i,j,k}|\le|v|\le 1$, such that

$$u(t,x+\delta v)-u(t,x)-\delta\sum_{i=1}^{p_0}\partial_{x_i}u(t,x)v_i-\frac{\delta^2}{2!}\sum_{i,j=1}^{p_0}\partial_{x_ix_j}u(t,x)v_iv_j=\frac{\delta^3}{3!}\sum_{i,j,l=1}^{p_0}\partial_{x_ix_jx_l}u(t,x+\delta\bar v_{i,j,k})v_iv_jv_l,$$

and thus

$$R^{(0,0)}_{v,\delta}(z)=\frac{\delta^3}{3!}\sum_{i,j,l=1}^{p_0}\big(\partial_{x_i,x_j,x_l}u(t,x+\delta\bar v_{i,j,l})-\partial_{x_i,x_j,x_l}u(t,x)\big)v_iv_jv_l.$$

Note that $\partial_{x_i,x_j,x_l}u\in C^{0,\alpha}_B$ for any $1\le i,j,l\le p_0$ since, by assumption, $u\in C^{3,\alpha}_B$. Estimate (5.57) then stems from Part 3 of Theorem 2.10 for $n=0$.

<u>Case $\bar n>1$</u>: by (4.35) we have

$$\gamma^{(\bar n-1,\bar n-1)}_{v,\delta}(z)=u(t,x+\delta^{2\bar n-1}B^{\bar n-1}v),$$

and thus, by (5.53), (5.56) for $k=\bar n-1$ reads as

$$u\big(t,x+\delta^{2\bar n-1}B^{\bar n-1}v\big)=u(t,x)+\delta^{2\bar n-1}\sum_{i=\bar p_{\bar n-2}+1}^{\bar p_{\bar n-1}}\partial_{x_i}u(t,x)(B^{\bar n-1}v)_i+R^{(\bar n-1,\bar n-1)}_{v,\delta}(z).$$

Now, by the mean-value theorem, there exists a family of vectors $(\bar v_i)_{\bar p_{\bar n-2}<i\le\bar p_{\bar n-1}}$, with $\bar v_i\in V_{\bar n-1}$ and $|\bar v_i|\le|B^{\bar n-1}v|\le c_B$, such that

$$u(t,x+\delta^{2\bar n-1}B^{\bar n-1}v)-u(t,x)=\delta^{2\bar n-1}\sum_{i=\bar p_{\bar n-2}+1}^{\bar p_{\bar n-1}}\partial_{x_i}u(t,x+\delta^{2\bar n-1}\bar v_i)(B^{\bar n-1}v)_i,$$

and thus,

$$\begin{aligned}R^{(\bar n-1,\bar n-1)}_{v,\delta}(z)&=\delta^{2\bar n-1}\sum_{i=\bar p_{\bar n-2}+1}^{\bar p_{\bar n-1}}\big(\partial_{x_i}u(t,x+\delta^{2\bar n-1}\bar v_i)-\partial_{x_i}u(t,x)\big)(B^{\bar n-1}v)_i\\&=\delta^{2\bar n-1}\sum_{i=\bar p_{\bar n-2}+1}^{\bar p_{\bar n-1}}\Big(\partial_{x_i}u(t,x+\delta^{2\bar n-1}\bar v_i)-T_2(\partial_{x_i}u)\big((t,x),(t,x+\delta^{2\bar n-1}\bar v_i)\big)\Big)(B^{\bar n-1}v)_i.\end{aligned}$$

Now, by (5.55) in Remark 5.28, we have $\partial_{x_i}u\in C^{2,\alpha}_B$ for any $\bar p_{\bar n-2}<i\le\bar p_{\bar n-1}$. Therefore estimate (5.57) stems from Part 3 of Theorem 2.10 for $n=2$.

Inductive step on k: we assume the thesis to hold true for a fixed $\max\{\bar n-1,0\}\le k<r$ and prove it true for $k+1$. Consider thus $v\in V_{0,k+1}$ with $|v|=1$. Set

$$\widetilde T_{2\bar n+1}u(\zeta,z)=\bar T_{2\bar n+1}u(\zeta,z)-u(\zeta),\qquad z,\zeta\in\mathbb R\times\mathbb R^d,$$

and

$$z_0 = z, \quad z_1 = \gamma^{(\bar n-1,k)}_{v,\delta}(z_0), \quad z_2 = e^{\delta^2 Y}\left(z_1\right), \quad z_3 = \gamma^{(\bar n-1,k)}_{v,-\delta}(z_2), \quad z_4 = e^{-\delta^2 Y}\left(z_3\right) = \gamma^{(\bar n-1,k+1)}_{v,\delta}(z). \tag{5.58}$$

According to this notation we have

$$R^{(\bar n-1,k+1)}_{v,\delta}(z) = u\Big(\gamma^{(\bar n-1,k+1)}_{v,\delta}(z)\Big) - \bar T_{2\bar n+1}u\Big(z, \gamma^{(\bar n-1,k+1)}_{v,\delta}(z)\Big) = u(z_4) - \bar T_{2\bar n+1}u(z_0, z_4) = \sum_{i=1}^{6} G_i,$$

with

$$G_1 = u(z_4) - u(z_3) - \sum_{i=1}^{\bar n} \frac{(-\delta^2)^i}{i!} Y^i u(z_3), \qquad G_2 = u(z_3) - u(z_2) - \widetilde T_{2\bar n+1}u(z_2, z_3),$$

$$G_3 = \sum_{i=1}^{\bar n} \frac{(-\delta^2)^i}{i!} Y^i u(z_2) + u(z_2) - u(z_1), \qquad G_4 = \widetilde T_{2\bar n+1}u(z_1, z_0) + u(z_1) - u(z_0),$$

$$G_5 = \sum_{i=1}^{\bar n} \frac{(-\delta^2)^i}{i!} \Big(Y^i u(z_3) - Y^i u(z_2) - \widetilde T_{2(\bar n-i)+1} Y^i u(z_2, z_3)\Big),$$

$$G_6 = \widetilde T_{2\bar n+1}u(z_2, z_3) - \widetilde T_{2\bar n+1}u(z_1, z_0) - \widetilde T_{2\bar n+1}u(z_0, z_4) + \sum_{i=1}^{\bar n} \frac{(-\delta^2)^i}{i!} \widetilde T_{2(\bar n-i)+1} Y^i u(z_2, z_3).$$

Now, by applying Remark 5.27 with $n = \bar n$, $m = 1$, on G_1 and G_3, and by using the inductive hypothesis on G_2 and G_4 (note that by (4.33) $V_{0,k+1} \subseteq V_{0,k}$), we have

$$|G_1 + G_2 + G_3 + G_4| \le c_B \|u\|_{C^{2\bar n+1,\alpha}_B} |\delta|^{2\bar n+1+\alpha}, \qquad z = (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad \delta \in \mathbb{R}.$$

To bound G_5, it is enough to observe that, by Definition 2.8, $u \in C^{2\bar n+1,\alpha}_B$ implies $Y^i u \in C^{2(\bar n-i)+1,\alpha}_B$, for any $i = 1, \cdots, \bar n$. Therefore, the bound follows by applying Part 3 of Theorem 2.10 for $n = 2(\bar n - i) + 1$, combined with (4.43).

In order to estimate G_6 and conclude the proof, we need to distinguish on whether $k = \max\{\bar n - 1, 0\}$, $k = \bar n$ or $k > \bar n$.

<u>Case $k > \bar n$</u>: there is nothing to prove because, by definitions (5.53) and (5.58), we have $G_6 \equiv 0$.

<u>Case $k = \bar n$</u>: first note that, in this case, the term G_6 reduces to

$$G_6 = \widetilde T_{2\bar n+1}u(z_2, z_3) - \widetilde T_{2\bar n+1}u(z_1, z_0) = \widetilde T_{2\bar n+1}u\big(z_2, \gamma^{(\bar n-1,\bar n)}_{v,-\delta}(z_2)\big) - \widetilde T_{2\bar n+1}u\big(z_1, \gamma^{(\bar n-1,\bar n)}_{v,-\delta}(z_1)\big),$$

and by definition (5.53), along with (4.40)-(4.41), we get

$$|G_6| = \Big|\delta^{2\bar n+1} \sum_{i=\bar p_{\bar n-1}+1}^{\bar p_{\bar n}} \Big(Y^{(\bar n)}_{v^{(\bar n)}_i} u(z_1) - Y^{(\bar n)}_{v^{(\bar n)}_i} u(z_2)\Big) \left(B^{\bar n} v\right)_i\Big| =$$

(by Remark 5.29 with $n = \bar n$ and since $v \in V_{0,\bar n+1} \subseteq V_{0,\bar n}$)

$$= \Big|\delta^{2\bar n+1}\big(Y^{(\bar n)}_v u(z_1) - Y^{(\bar n)}_v u(z_2)\big)\Big| \le$$

(by hypothesis $u \in C_B^{2\bar{n}+1,\alpha}$ and thus, by Lemma 4.19, $Y_v^{(\bar{n})} u \in C_B^{0,\alpha} \subseteq C_Y^{\alpha}$)

$$\leq c_B \|u\|_{C_B^{2\bar{n}+1,\alpha}} |\delta|^{2\bar{n}+1+\alpha}.$$

<u>Case $k = \max\{\bar{n}-1, 0\}$</u>: we only need to prove the case $\bar{n} > 0$. We first consider $\bar{n} \geq 2$. We have

$$\begin{aligned} G_6 = {}& \widetilde{T}_{2\bar{n}+1} u\big(z_2, \gamma_{v,-\delta}^{(\bar{n}-1,\bar{n}-1)}(z_2)\big) - \widetilde{T}_{2\bar{n}+1} u\big(z_1, \gamma_{v,-\delta}^{(\bar{n}-1,\bar{n}-1)}(z_1)\big) - \widetilde{T}_{2\bar{n}+1} u\big(z_0, \gamma_{v,\delta}^{(\bar{n}-1,\bar{n})}(z_0)\big) \\ & + \sum_{i=1}^{\bar{n}} \frac{(-\delta^2)^i}{i!} \widetilde{T}_{2(\bar{n}-i)+1} Y^i u\big(z_2, \gamma_{v,-\delta}^{(\bar{n}-1,\bar{n}-1)}(z_2)\big). \end{aligned}$$

Now recall that, by (4.40)-(4.41),

$$\begin{aligned} \gamma_{v,-\delta}^{(\bar{n}-1,\bar{n}-1)}(z) &= \big(t, x - \delta^{2(\bar{n}-1)+1} B^{\bar{n}-1} v\big), \\ \gamma_{v,\delta}^{(\bar{n}-1,\bar{n})}(z) &= \big(t, x + \delta^{2\bar{n}+1} B^{\bar{n}} v + \widetilde{S}_{\bar{n}-1,\bar{n}}(\delta) v\big), \qquad \widetilde{S}_{\bar{n}-1,\bar{n}}(\delta) v \in \bigoplus_{j=\bar{n}+1}^{r} V_j, \end{aligned}$$

and thus, by definition (5.53), we obtain

$$G_6 = \delta^{2(\bar{n}-1)+1} \sum_{i=\bar{p}_{\bar{n}-2}+1}^{\bar{p}_{\bar{n}-1}} \big(\partial_{x_i} u(z_1) - \partial_{x_i} u(z_2) + \delta^2 \partial_{x_i} Y u(z_2)\big) (B^{\bar{n}-1} v)_i - \delta^{2\bar{n}+1} \sum_{i=\bar{p}_{\bar{n}-1}+1}^{\bar{p}_{\bar{n}}} Y_{v_i^{(\bar{n})}}^{(\bar{n})} u(z_0)\, (B^{\bar{n}} v)_i =$$

(by Proposition 5.24 for $n = \bar{n}-1$)

$$\begin{aligned} &= \delta^{2(\bar{n}-1)+1} \sum_{i=\bar{p}_{\bar{n}-2}+1}^{\bar{p}_{\bar{n}-1}} \big(Y_{v_i^{(\bar{n}-1)}}^{(\bar{n}-1)} u(z_1) - Y_{v_i^{(\bar{n}-1)}}^{(\bar{n}-1)} u(z_2) + \delta^2 Y_{v_i^{(\bar{n}-1)}}^{(\bar{n}-1)} Y u(z_2)\big) (B^{\bar{n}-1} v)_i \\ &\quad - \delta^{2\bar{n}+1} \sum_{i=\bar{p}_{\bar{n}-1}+1}^{\bar{p}_{\bar{n}}} Y_{v_i^{(\bar{n})}}^{(\bar{n})} u(z_0) (B^{\bar{n}} v)_i = \end{aligned}$$

(by applying Remark 5.29 with $n = \bar{n}-1$ and $n = \bar{n}$, and since $v \in V_{0,\bar{n}} \subseteq V_{0,\bar{n}-1}$)

$$= \delta^{2(\bar{n}-1)+1} \big(Y_v^{(\bar{n}-1)} u(z_1) - Y_v^{(\bar{n}-1)} u(z_2) + \delta^2 Y_v^{(\bar{n}-1)} Y u(z_2)\big) - \delta^{2\bar{n}+1} Y_v^{\bar{n}} u(z_0) =$$

(since, by definition (4.34), $Y_v^{\bar{n}-1} Y = Y_v^{\bar{n}} + Y Y_v^{\bar{n}-1}$)

$$= \delta^{2(\bar{n}-1)+1} \big(Y_v^{(\bar{n}-1)} u(z_1) - Y_v^{(\bar{n}-1)} u(z_2) + \delta^2 Y Y_v^{(\bar{n}-1)} u(z_2)\big) + \delta^{2\bar{n}+1} \big(Y_v^{\bar{n}} u(z_2) - Y_v^{\bar{n}} u(z_0)\big) = \sum_{i=1}^{3} F_i.$$

with

$$\begin{aligned} F_1 &= \delta^{2(\bar{n}-1)+1} \left(Y_v^{(\bar{n}-1)} u(z_1) - Y_v^{(\bar{n}-1)} u(z_2) + \delta^2 Y Y_v^{(\bar{n}-1)} u(z_2)\right), \\ F_2 &= \delta^{2\bar{n}+1} \left(Y_v^{\bar{n}} u(z_2) - Y_v^{\bar{n}} u(z_1)\right), \qquad\qquad F_3 = \delta^{2\bar{n}+1} \left(Y_v^{\bar{n}} u(z_1) - Y_v^{\bar{n}} u(z_0)\right). \end{aligned}$$

Now, to bound F_1 it is sufficient to note that, by Lemma 4.19, $Y_v^{(\bar{n}-1)} u \in C_B^{2,\alpha}$ and thus the bounds directly follow by applying Remark 5.27 with $n = 1$ and $m = 0$. To bound the terms F_2 and F_3 we use that, by

Lemma 4.19, $Y_v^{\bar n} u \in C_B^{0,\alpha}$. The estimate for F_3 then follows by Part 3 of Theorem 2.10 for $n = 0$ along with equation (4.43), whereas the one for F_2 is a consequence of the inclusion $C_B^{0,\alpha} \subseteq C_Y^{\alpha}$ and of Remark 5.27.

Finally, the case $\bar n = 1$ is analogous, but G_6 contains two more terms:

$$F_4 = \frac{\delta^2}{2!} \sum_{i,j=1}^{p_0} \big(\partial_{x_i,x_j} u(z_2) - \partial_{x_i,x_j} u(z_1)\big) v_i v_j,$$

$$F_5 = -\frac{\delta^3}{3!} \sum_{i,j,l=1}^{p_0} \big(\partial_{x_i,x_j,x_l} u(z_2) - \partial_{x_i,x_j,x_l} u(z_1)\big) v_i v_j v_l,$$

which can be estimated by using that $\partial_{x_i,x_j,x_l} u \in C_B^{0,\alpha} \subseteq C_Y^{\alpha}$ and $\partial_{x_i,x_j} u \in C_B^{1,\alpha} \subseteq C_Y^{\alpha+1}$ for any $1 \le i,j,l \le p_0$. $\square$

Proof of Proposition 5.26 for $n = \bar n$ and $m = 1$. We assume $u \in C_B^{2\bar n+1,\alpha}$ and we prove that, for any $0 \le k \le \bar n$,

$$u(t, x+\xi) = T_{2\bar n+1} u\big((t,x),(t,x+\xi)\big) + R_{\bar n}\big(t,x,\xi\big),$$

with

$$|R_{\bar n}\big(t,x,\xi\big)| \le c_B \|u\|_{C_B^{2\bar n+1,\alpha}} |\xi|_B^{2n+1+\alpha}, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \quad \xi \in \bigoplus_{j=0}^{k-1} V_j.$$

We prove the thesis by induction on k. For $k = 0$ there is nothing to prove since $R_{\bar n}\big(t,x,0\big) \equiv 0$. Now, assume $0 \le k < \bar n$, $\xi \in \bigoplus_{j=0}^{k-1} V_j$ and $v \in V_k$. Then

$$R_{\bar n}(t,x,\xi+v) = F_1 + F_2,$$

with

$$\begin{aligned} F_1 &= u(t,x+\xi+v) - T_{2\bar n+1} u\big((t,x+v),(t,x+\xi+v)\big) \\ F_2 &= T_{2\bar n+1} u\big((t,x+v),(t,x+\xi+v)\big) - T_{2\bar n+1} u\big((t,x),(t,x+\xi+v)\big). \end{aligned}$$

We can apply the inductive hypothesis on F_1 and obtain the estimate

$$|F_1| \le c_B \|u\|_{C_B^{2\bar n+1,\alpha}} |\xi|_B^{2\bar n+1+\alpha} \le c_B \|u\|_{C_B^{2\bar n+1,\alpha}} |\xi+v|_B^{2\bar n+1+\alpha}.$$

Recalling (2.14), F_2 can be written as

$$\begin{aligned} F_2 &= \sum_{\substack{0\le|\beta|_B\le 2\bar n+1\\ \beta^{[i]}=0 \text{ if } i\ge k}} \frac{1}{\beta!}\partial_x^\beta u(t,x+v)\,\xi^\beta - \sum_{\substack{0\le|\beta|_B\le 2\bar n+1\\ \beta^{[i]}=0 \text{ if } i\ge k}} \sum_{\substack{0\le|\gamma|_B\le 2\bar n+1-|\beta|_B\\ \gamma=\gamma^{[k]}}} \frac{1}{\beta!\gamma!}\,\partial_x^\gamma\partial_x^\beta u(t,x)\,\xi^\beta v^\gamma \\ &= \sum_{\substack{0\le|\beta|_B\le 2\bar n+1\\ \beta^{[i]}=0 \text{ if } i\ge k}} \frac{1}{\beta!}\Bigg(\partial_x^\beta u(t,x+v) - \sum_{\substack{0\le|\gamma|_B\le 2\bar n+1-|\beta|_B\\ \gamma=\gamma^{[k]}}} \frac{1}{\gamma!}\,\partial_x^\gamma\partial_x^\beta u(t,x)\,v^\gamma\Bigg)\xi^\beta \\ &= \sum_{\substack{0\le|\beta|_B\le 2\bar n+1\\ \beta^{[i]}=0 \text{ if } i\ge k}} \frac{1}{\beta!}\Big(\partial_x^\beta u(t,x+v) - T_{2\bar n+1-|\beta|_B}\partial_x^\beta u\big((t,x),(t,x+v)\big)\Big)\xi^\beta. \end{aligned}$$

By Remark 5.28, we get $\partial_x^\beta u \in C_B^{2\bar{n}+1-|\beta|_B,\alpha}$. Now, if $|\beta|_B \geq 1$, we can apply Part 3 of Theorem 2.10 for $n = 2\bar{n}+1-|\beta|_B$ on $\partial_x^\beta u$ and get

$$\begin{aligned}\left|\partial_x^\beta u(t,x+v) - T_{2\bar{n}+1-|\beta|_B}\partial_x^\beta u((t,x),(t,x+v))\right|\left|\xi^\beta\right| &\leq c_B\|u\|_{C_B^{2\bar{n}+1,\alpha}}|v|_B^{2\bar{n}+1-|\beta|_B+\alpha}|\xi|_B^{|\beta|_B}\\ &\leq c_B\|u\|_{C_B^{2\bar{n}+1,\alpha}}|\xi+v|_B^{2\bar{n}+1+\alpha}.\end{aligned}$$

On the other hand, if $|\beta|_B = 0$ then we have to estimate

$$u(t,x+v) - \sum_{\substack{0\leq|\gamma|_B\leq 2\bar{n}+1\\ \gamma=\gamma^{[k]}}} \frac{1}{\gamma!}\,\partial_x^\gamma u(t,x)\,v^\gamma.$$

Recall that, by definition, we have $|\gamma|_B = (2k+1)|\gamma|$ if $\gamma = \gamma^{[k]}$. Now, set

$$j := \max\{i \geq 0 \mid (2k+1)i \leq 2\bar{n}+1\}, \tag{5.59}$$

and note that $j \geq 1$ because $k < \bar{n}$. By Remark 5.28 and the mean-value theorem, there exists a family of vectors $(\bar{v}_\eta)_{\eta\in\mathcal{I}_k^j}$ where

$$\mathcal{I}_k^j = \{\eta \in \mathbb{N}_0^d \mid \eta = \eta^{[k]} \text{ and } |\eta|_B = (2k+1)j\}, \tag{5.60}$$

such that $\bar{v}_\eta \in V_k$, $|\bar{v}_\eta| \leq |v|$ and

$$u(t,x+v) - \sum_{\substack{0\leq|\gamma|_B\leq(2k+1)(j-1)\\ \gamma=\gamma^{[i]}}} \frac{v^\gamma}{\gamma!}\partial_x^\gamma u(t,x) = \sum_{\eta\in\mathcal{I}_k^j}\frac{v^\eta}{\eta!}\partial_x^\eta u(t,x+\bar{v}_\eta).$$

Therefore, we obtain

$$\left|u(t,x+v) - \sum_{\substack{0\leq|\gamma|_B\leq 2\bar{n}+1\\ \gamma=\gamma^{[i]}}} \frac{v^\gamma}{\gamma!}\partial_x^\gamma u(t,x)\right| = \Big|\sum_{\eta\in\mathcal{I}_k^j}\frac{v^\eta}{\eta!}\left(\partial_x^\eta u(t,x+\bar{v}_\eta) - \partial_x^\eta u(t,x)\right)\Big| =$$

(by (5.59))

$$= \Big|\sum_{\eta\in\mathcal{I}_k^j}\frac{1}{\eta!}\Big(\partial_x^\eta u(t,x+\bar{v}_\eta) - T_{2\bar{n}+1-(2k+1)j}\partial_x^\eta u\big((t,x),(t,x+\bar{v}_\eta)\big)\Big)v^\eta\Big| \leq$$

(by Remark 5.28, $\partial_x^\eta u \in C_B^{2\bar{n}+1-(2k+1)j,\alpha}$ and thus by Part 3 of Theorem 2.10 with $n = 2\bar{n}+1-(2k+1)j$)

$$\leq c_B\|u\|_{C_B^{2\bar{n}+1,\alpha}}\sum_{\eta\in\mathcal{I}_k^j}\frac{1}{\eta!}\,|\bar{v}_\eta|_B^{2\bar{n}+1-(2k+1)j+\alpha}\,|v|_B^{|\eta|_B} \leq$$

(since $|\bar{v}_\eta| \leq |v|$ and by (5.60))

$$\leq c_B\|u\|_{C_B^{2\bar{n}+1,\alpha}}|v|_B^{2\bar{n}+1+\alpha} \leq c_B\|u\|_{C_B^{2\bar{n}+1,\alpha}}|\xi+v|_B^{2\bar{n}+1+\alpha}$$

which concludes the proof. □

5.2.2 Proof of Proposition 5.24 for $\mathbf{n} = \bar{\mathbf{n}}$

To start we show that if $u \in C_B^{2\bar{n}+1,\alpha}$ then for any $z = (t,x)$, $\zeta = (t,\xi) \in \mathbb{R} \times \mathbb{R}^d$ we have

$$\left|u(t,x) - \bar{T}_{2\bar{n}+1} u\left((t,\xi),(t,x)\right)\right| \le c_B \|u\|_{C_B^{2\bar{n}+1,\alpha}} |x-\xi|_B^{2\bar{n}+1+\alpha}. \tag{5.61}$$

Define the point $\bar{z} = (t,\bar{x})$ with

$$\bar{x}^{[i]} = \begin{cases} x^{[i]}, & \text{if } i \ge \bar{n}, \\ \xi^{[i]}, & \text{if } i < \bar{n}. \end{cases}$$

It follows that

$$(x-\bar{x})^\beta = \begin{cases} (x-\xi)^\beta & \text{if } |\beta|_B \le 2\bar{n}+1,\ \beta^{[\bar{n}]} = 0, \\ 0, & \text{if } |\beta|_B \le 2\bar{n}+1,\ \beta^{[\bar{n}]} \ne 0, \end{cases} \tag{5.62}$$

and

$$|x-\bar{x}|_B \le |x-\xi|_B, \qquad |\bar{x}-\xi|_B \le |x-\xi|_B.$$

Then we write

$$u(t,x) - \bar{T}_{2\bar{n}+1} u\left((t,\xi),(t,x)\right) = F_1 + F_2,$$

with

$$F_1 = u(t,x) - \bar{T}_{2\bar{n}+1} u\left((t,\bar{x}),(t,x)\right), \qquad F_2 = \bar{T}_{2\bar{n}+1} u\left((t,\bar{x}),(t,x)\right) - \bar{T}_{2\bar{n}+1} u\left((t,\xi),(t,x)\right).$$

Applying Proposition 5.26 with $n = \bar{n}$ and $m = 1$, we obtain

$$|F_1| \le c_B \|u\|_{C_B^{2\bar{n}+1,\alpha}} |x-\bar{x}|_B^{2\bar{n}+1+\alpha} \le c_B \|u\|_{C_B^{2\bar{n}+1,\alpha}} |x-\xi|_B^{2\bar{n}+1+\alpha}.$$

Now, by (5.62) we have

$$F_2 = \sum_{\substack{|\beta|_B \le 2\bar{n}+1 \\ \beta^{[\bar{n}]}=0}} \frac{1}{\beta!}\Big(\partial_x^\beta u(t,\bar{x}) - \partial_x^\beta u(t,\xi)\Big)(x-\xi)^\beta - \sum_{i=\bar{p}_{\bar{n}-1}+1}^{\bar{p}_{\bar{n}}} Y_{v_i^{(\bar{n})}}^{(\bar{n})} u(t,\xi)(x-\xi)_i.$$

Moreover, by Remark 5.28 we have $\partial_x^\beta u \in C_B^{2\bar{n}+1-|\beta|_B,\alpha}$ and therefore, if $|\beta|_B > 0$, by Part 3 of Theorem 2.10 for $n = 2\bar{n}+1-|\beta|_B$, we get

$$\left|\Big(\partial_x^\beta u(t,\bar{x}) - \partial_x^\beta u(t,\xi)\Big)(x-\xi)^\beta\right| \le c_B \|u\|_{C_B^{2\bar{n}+1,\alpha}} |\bar{x}-\xi|_B^{2\bar{n}+1+\alpha-|\beta|_B} |x-\xi|^{|\beta|} \le c_B \|u\|_{C_B^{2\bar{n}+1,\alpha}} |x-\xi|_B^{2\bar{n}+1+\alpha}.$$

In order to conclude the proof of (5.61), we only have to prove

$$\left|u(t,\bar{x}) - u(t,\xi) - \sum_{j=\bar{p}_{\bar{n}-1}+1}^{\bar{p}_{\bar{n}}} Y_{v_j^{(\bar{n})}}^{(\bar{n})} u(t,\xi)(x-\xi)_j\right| \le c_B \|u\|_{C_B^{2\bar{n}+1,\alpha}} |x-\xi|_B^{2\bar{n}+1+\alpha}. \tag{5.63}$$

We set the points $\zeta_i = (t,\xi_i)$, for $i = \bar{n}-1, \cdots, r$, as defined in Lemma 4.22 for $n = \bar{n}$ and $v = \bar{x} - \xi$. By (4.44) we have

$$\bar{T}_{2\bar{n}+1} u(\zeta_{i-1},\zeta_i) = u(\zeta_{i-1}), \qquad i = \bar{n}, \ldots, r,$$

and

$$|\delta_i| \leq c_B|\bar{x}-\xi|_B \leq c_B|x-\xi|_B, \qquad i=\bar{n},\dots,r. \tag{5.64}$$

It is now clear that

$$u(t,\bar{x})-u(t,\xi)-\sum_{j=\bar{p}_{\bar{n}-1}+1}^{\bar{p}_{\bar{n}}} Y^{(\bar{n})}_{v^{(\bar{n})}_j}u(t,\xi)(x-\xi)_j = u(\zeta_r)-\bar{T}_{2\bar{n}+1}u(\zeta_{\bar{n}-1},\zeta_{\bar{n}}) = \sum_{i=\bar{n}}^{r}\Big(u(\zeta_i)-\bar{T}_{2\bar{n}+1}u(\zeta_{i-1},\zeta_i)\Big),$$

and formula (5.63) follows from Proposition 5.25 along with (5.64).

We are now ready to prove (5.49) for $n=\bar{n}$. For any $i\in\{\bar{p}_{\bar{n}-1}+1,\dots,\bar{p}_{\bar{n}}\}$ and $\delta\in\mathbb{R}$, set $x=\xi+\delta e_i$ in (5.61), where e_i is the i-th vector of the canonical basis of $\mathbb{R}^d$: we obtain

$$u(t,\xi+\delta e_i)-u(t,\xi)-\delta Y^{(\bar{n})}_{v^{(\bar{n})}_i}u(t,\xi) = O\big(|\delta|^{1+\frac{\alpha}{2\bar{n}+1}}\big), \quad \text{as } \delta\to 0.$$

This implies that $\partial_{x_i}u(t,\xi)$ exists and

$$\partial_{x_i}u(t,\xi) = Y^{(\bar{n})}_{v^{(\bar{n})}_i}u(t,\xi) \quad t\in\mathbb{R},\ \xi\in\mathbb{R}^d,\ i=\bar{p}_{\bar{n}-1}+1,\dots,\bar{p}_{\bar{n}}.$$

Finally, by Remark 5.28 we have $Y^{(\bar{n})}_{v^{(\bar{n})}_i}u\in C^{0,\alpha}_B$ and thus $\partial_{x_i}u\in C^{0,\alpha}_B$.

Remark 5.30. Incidentally we have just proved a special case of Part 3 of Theorem 2.10 for $n=2\bar{n}+1$, namely the case when there is no increment in the time variable. Precisely we have shown that, for any function $u\in C^{2\bar{n}+1}_B$, we have

$$\left|u(t,x)-T_{2\bar{n}+1}u\big((t,\xi),(t,x)\big)\right| \leq c_B\|u\|_{C^{2\bar{n}+1,\alpha}_B}|x-\xi|_B^{2\bar{n}+1+\alpha}, \qquad t\in\mathbb{R}, \quad x,\xi\in\mathbb{R}^d. \tag{5.65}$$

5.2.3 Proof of Part 3 of Theorem 2.10 for $n=2\bar{n}+1$

Relation (2.15) is a trivial consequence of Remark 5.29 (see (5.54)-(5.55)) along with Proposition 5.24 for $n=\bar{n}$. We next prove estimate (2.16): by (2.15), for any $z=(t,x)$ and $\zeta=(s,\xi)$, the B-Taylor polynomial $T_{2\bar{n}+1}u(\zeta,z)$ is well defined. Define the point $\zeta_1:=e^{(t-s)Y}(\zeta)=(t,e^{(t-s)B}\xi)$ and note that ζ_1 and z only differ in the spatial variables. Moreover, we have

$$\zeta_1^{-1}\circ z=\left(0,x-e^{(t-s)B}\xi\right), \qquad \zeta^{-1}\circ z=\left(t-s,x-e^{(t-s)B}\xi\right),$$

and therefore

$$\left\|\zeta_1^{-1}\circ z\right\|_B = \left|x-e^{(t-s)B}\xi\right|_B \leq \left\|\zeta^{-1}\circ z\right\|_B. \tag{5.66}$$

Now write

$$u(z)-T_{2\bar{n}+1}u(\zeta,z)=F_1+F_2,$$

with

$$F_1=u(z)-T_{2\bar{n}+1}u(\zeta_1,z), \qquad F_2=T_{2\bar{n}+1}u(\zeta_1,z)-T_{2\bar{n}+1}u(\zeta,z).$$

By (5.65) in Remark 5.30 along with (5.66), we obtain the estimate

$$|F_1|\leq c_B\|u\|_{C^{2\bar{n}+1,\alpha}_B}\|\zeta^{-1}\circ z\|_B^{2\bar{n}+1+\alpha}.$$

A convenient rearrangement of the terms in the Taylor polynomials allows us to estimate F_2. Precisely, we have

$$
\begin{aligned}
F_2 &= \sum_{|\beta|_B \le 2\bar{n}+1} \frac{1}{\beta!}\big(\partial_\xi^\beta u(e^{(t-s)Y}(\zeta))\big)(x-e^{(t-s)B}\xi)^\beta - \sum_{2k+|\beta|_B\le 2\bar{n}+1} \frac{Y^k\partial_\xi^\beta u(\zeta)}{\beta!k!}(x-e^{(t-s)B}\xi)^\beta (t-s)^k \\
&= \sum_{|\beta|_B \le 2\bar{n}+1} \frac{1}{\beta!}\left(\partial_\xi^\beta u(e^{(t-s)Y}(\zeta)) - \sum_{2k\le 2\bar{n}+1-|\beta|_B} \frac{(t-s)^k}{k!} Y^k \partial_\xi^\beta u(\zeta)\right)(x-e^{(t-s)B}\xi)^\beta.
\end{aligned}
$$

Now, by (2.15) we have $\partial_x^\beta u \in C_B^{2\bar{n}+1-|\beta|_B,\alpha}$ and thus, by Remark 5.27 we obtain

$$
|F_2| \le \|u\|_{C_B^{2\bar{n}+1,\alpha}} \sum_{|\beta|_B\le 2\bar{n}+1} \frac{1}{\beta!}|t-s|^{\frac{2\bar{n}+1-|\beta|_B+\alpha}{2}} \left|x-e^{(t-s)B}\xi\right|_B^{|\beta|_B} \le c_B\|u\|_{C_B^{2\bar{n}+1,\alpha}}\|\zeta^{-1}\circ z\|_B^{2\bar{n}+1+\alpha},
$$

and this concludes the proof.

5.3 Step 3

Fix $\bar{n} \in \{0, \cdots, r-1\}$. Assume to be holding true:

- Proposition 5.24 for any $0 \le n \le \bar{n}$;
- Theorem 2.10 for any $0 \le n \le 2\bar{n}+1$;

we have to prove:

- Propositions 5.25 and 5.26 for $n = \bar{n}+1$, $m = 0$;
- Part 3 of Theorem 2.10 for $n = 2\bar{n}+2$.

In this case, the proof is relatively simpler if compared to the one of Step 2. This is because we do not need to prove the existence of the Euclidean derivatives of the higher level. Hence the proofs are simpler versions of those in Step 2. We skip the details for the sake of brevity.

5.4 Step 4

Here we fix a certain $\bar{n} \ge 2r+1$, suppose Theorem 2.10 true for any $0 \le n \le \bar{n}$ and prove Part 3 of Theorem 2.10 for $n = \bar{n}+1$. To prove the claim, we will first consider the case with no increment w.r.t. the time variable, as we have done in Step 2. In that case, we used the curves $\gamma_{v,\delta}^{n,k}(z)$ in order to increment those variables w.r.t. which we had no regularity in the Euclidean sense: then we applied Proposition 5.25 to estimate the increment along such curves. This time, this will not be necessary because, since $\bar{n}+1 > 2r+1$, the existence of the Euclidean derivatives is ensured along any direction by the inductive hypothesis.

Proof of Part 3 of Theorem 2.10 for $n = \bar{n}+1$. Recall that, by hypothesis, $u \in C_B^{\bar{n}+1,\alpha}$ with $\bar{n} \ge 2r+1$. It is easy to prove that, for any $z = (t,x)$, $\zeta = (s,\xi) \in \mathbb{R}^d$, we have

$$
|u(t,x) - T_{\bar{n}+1}u((t,\xi),(t,x))| \le c_B\,\|u\|_{C_B^{\bar{n}+1,\alpha}}\,|x-\xi|_B^{\bar{n}+1+\alpha}. \tag{5.67}
$$

The proof of the latter identity is identical to that of Proposition 5.26. Precisely, under the assumption $\bar{n} \geq 2r+1$, the technical restriction made on the spatial increments in Proposition 5.26 can be dropped and the proof proceeds exactly in the same way, by making sure that the constant c_B in (5.67) is actually independent of $\bar{n}$.

The proof of Part 3 of Theorem 2.10 then follows exactly as in Step 2, by using the estimate (5.67) instead of (5.65). $\square$

5.5 Proof of the local version of Theorem 2.10 (Part 1 and Part 2)

The proof of the local version of Theorem 2.10 is based upon the same arguments used to prove its global counterpart. The main additional difficulty arising when proving Part 1 and Part 2, is to make sure that all the integral curves used in the proof to connect z to ζ do not exit the domain Ω. There comes the necessity to take z in a small ball centered at ζ with radius r. In particular, we have to check that all the connecting curves are contained in a bigger ball with radius $R > r$, compactly contained in Ω, in order to bound the remainder in (2.13) by means of the $C^{n,\alpha}_{B,\mathrm{loc}}$ norm of u on such ball.

To synthesize, the proof could be summarized as follows. First prove the Taylor estimate (2.13) for two points $z, \xi \in \mathbb{R} \times \mathbb{R}^d$ with the same time-component (estimate (5.65)); this also proves the existence of those Euclidian derivatives whose existence is not directly implied by definition of $C^{n,\alpha}_{B,\mathrm{loc}}$and thus proves Part 1 of the theorem. To do this, one can proceed as in Section 5.24: precisely, one would first apply the local version of Proposition 5.26, whose proof is exactly analogous to its global counterpart, to control the increment of u between ξ and $\bar{x}$. Secondly, one would define the points $\zeta_k = (t, \xi_k)$ for $k = \bar{n}-1, \cdots, r$, by means of the curves of Lemma 5.24 and thank to (4.44)-(4.45), obtain the bound

$$|\delta_k| + \left\|\zeta_k^{-1} \circ \zeta\right\|_B \leq c_B |x-\xi|_B \leq c_B r, \qquad k = \bar{n}, \dots, r.$$

This would ensure that each point ζ_k is inside the domain Ω, for any r suitably small, and would allow to control the increment of u between ζ_{k-1} and ζ_k by means of the local version of Proposition 5.25. The latter preliminary result can be proved analogously to the global case, by making use of the bounds (4.43) and (4.48) to control the distance of each curve $\gamma^{n,k}_{v,\delta}(z)$ from ζ.

Eventually, the proof of Part 2 for two general points $z, \zeta \in \mathbb{R} \times \mathbb{R}^d$ follows by moving along the integral curve of Y to control the increment in the time-variable and by using the bound (4.48) to check that $e^{(t-s)Y}(\zeta) \in \Omega$.

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