KOLMOGOROV-FOKKER-PLANCK EQUATIONS: COMPARISON PRINCIPLES NEAR LIPSCHITZ TYPE BOUNDARIES

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Abstract. We prove several new results concerning the boundary behavior of non-negative solutions to the equation $\mathcal{K}u = 0$ where

$$\mathcal{K} := \sum_{i=1}^{m} \partial_{x_i x_i} + \sum_{i=1}^{m} x_i \partial_{y_i} - \partial_t.$$

Our results are established near the non-characteristic part of the boundary of certain local Lip_K-domains where the latter is a class of local Lipschitz type domains adapted to the geometry of $\mathcal{K}$. Generalizations to more general operators of Kolmogorov-Fokker-Planck type are also discussed.

RÉSUMÉ. Nous prouvons plusieurs nouveaux résultats sur le comportement au bord des solutions non-négatives de l'équation $\mathcal{K}u = 0$, où

$$\mathcal{K} := \sum_{i=1}^{m} \partial_{x_i x_i} + \sum_{i=1}^{m} x_i \partial_{y_i} - \partial_t.$$

Nos résultats sont établis dans un voisinage de la partie non-caractéristique du bord de certains domains locaux Lip_K, qui sont des domains localement Lipschitziens adaptés à la géométrie de $\mathcal{K}$.Nous discutons aussi des généralisations à d'autres opérateurs plus généraux de type Kolmogorov-Fokker-Planck.

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1. Introduction

Let $N = 2m$, where $m \geq 1$ is an integer, and let $\Omega \subset \mathbb{R}^{N+1}$ be a bounded domain, i.e., a bounded, open and connected set. In this paper we establish a number of results concerning the boundary behavior of non-negative solutions to the equation $\mathcal{K}u = 0$ in Ω where

$$\mathcal{K} := \sum_{i=1}^{m} \partial_{x_i x_i} + \sum_{i=1}^{m} x_i \partial_{y_i} - \partial_t, \qquad (x, y, t) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}. \tag{1.1}$$

The operator $\mathcal{K}$, referred to as the Kolmogorov or Kolmogorov-Fokker-Planck operator, was introduced and studied by Kolmogorov in 1934, see [K], as an example of a degenerate parabolic operator having strong regularity properties. Kolmogorov proved that $\mathcal{K}$ has a fundamental solution $\Gamma = \Gamma(x, y, t, \tilde{x}, \tilde{y}, \tilde{t})$ which is smooth in the set $\{(x, y, t) \neq (\tilde{x}, \tilde{y}, \tilde{t})\}$. As a consequence,

$$\mathcal{K}u := f \in C^\infty(\Omega) \quad \Rightarrow \quad u \in C^\infty(\Omega), \tag{1.2}$$

for every distributional solution of $\mathcal{K}u = f$. Property (1.2) can also be stated as

$$\mathcal{K} \text{ is hypoelliptic}, \tag{1.3}$$

see (1.13) below.

The operator $\mathcal{K}$ appears naturally in the context of stochastic processes and in several applications. The fundamental solution $\Gamma(\cdot,\cdot,\cdot,\tilde{x},\tilde{y},\tilde{t})$ is the density of the stochastic process (X_t, Y_t), which solves the Langevin equation

$$\begin{cases} dX_t = \sqrt{2}dW_t, & X_{\tilde{t}} = \tilde{x}, \\ dY_t = X_t dt, & Y_{\tilde{t}} = \tilde{y}, \end{cases} \tag{1.4}$$

where W_t is a m-dimensional Wiener process. The system in (1.4) describes the density of a system with $2m$ degrees of freedom. Given $z = (x, y) \in \mathbb{R}^{2m}$, $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$ are, respectively, the velocity and the position of the system. (1.4) and (1.1) are of fundamental importance in kinetic theory, they form the basis for Langevin type models for particle dispersion and appear in applications in many different areas including finance [BPV], [Pa], and vision [CS1], [CS2].

In [CNP1], [CNP2] and [CNP3], we developed a number of important preliminary estimates concerning the boundary behavior of non-negative solutions to equations of Kolmogorov-Fokker-Planck type in Lipschitz type domains. These papers were the results of our ambition to understand to the extent, and in what sense, scale and translation invariant boundary comparison principles, boundary Harnack inequalities and doubling properties of associated parabolic measures, previously established for uniformly parabolic equations with bounded measurable coefficients in Lipschitz type domains, see [FS], [FSY], [SY], [FGS], [N], [Sa], can be established for non-negative solutions to the equation $\mathcal{K}u = 0$ and for more general equations of Kolmogorov-Fokker-Planck type. In this paper we take this program a large step forward by establishing Theorems 1.1, 1.2 and 1.3 stated below. These results are completely new and represent the starting point for far reaching developments concerning operators of Kolmogorov type. Already in the case of uniformly elliptic and parabolic operators this kind of scale and translation invariant estimates are important in the analysis of free boundary problems, see [C1], [C2] and [ACS] for instance, and in the harmonic analysis approach to partial differential equations in Lipschitz type domains, see [Ke], [HL].

1.1. **Scalings and translations.** The prototype for uniformly parabolic operators in $\mathbb{R}^{m+1}$ is the heat operator

$$\mathcal{H} := \sum_{i=1}^{m} \partial_{x_i x_i} - \partial_t. \tag{1.5}$$

Considering non-smooth domains, here roughly defined as Lipschitz type domains, the ambition to develop estimates for solutions to $\mathcal{H}u = 0$ which respect the standard parabolic scalings, and the standard group of translations on $\mathbb{R}^{m+1}$, naturally leads one to develop estimates for solutions to $\mathcal{H}u = 0$ in the time-dependent setting of Lip(1,1/2)-domains. A notion of (local) Lip(1,1/2)-domains with constants M and r_0 is formulated in the natural way using appropriate local coordinate systems and assuming that in each local chart of size r_0, the boundary can be represented by a Lip(1,1/2)-function f with Lip(1,1/2)-constant M, see [N] for example. Recall that a function $f : \mathbb{R}^{m-1} \times \mathbb{R} \to \mathbb{R}$ is called Lip(1,1/2) with constant M if

$$|f(x', t) - f(\tilde{x}', \tilde{t})| \leq M\big(|x' - \tilde{x}'| + |t - \tilde{t}|^{1/2}\big), \tag{1.6}$$

whenever (x', t), $(\tilde{x}', \tilde{t}') \in \mathbb{R}^{m-1} \times \mathbb{R}$.

Compared to the heat operator, the scalings underlying the operator $\mathcal{K}$ is different, and the change of variables preserving the equation is more involved. As a consequence the appropriate geometric setting for the equation $\mathcal{K}u = 0$ becomes more of an issue. In the case of $\mathcal{K}$, the natural family of dilations $(\delta_r)_{r>0}$ on $\mathbb{R}^{2m+1}$ is defined by

$$\delta_r(x,y,t) = (rx, r^3y, r^2t), \tag{1.7}$$

for every $(x,y,t) \in \mathbb{R}^{2m+1}$ and every positive r. Due to the presence of non constant coefficients in the drift term of $\mathcal{K}$, the usual Euclidean change of variable does not preserve the Kolmogorov equation. Nevertheless, a Galilean change of variable does. Consider a smooth function $u : \Omega \to \mathbb{R}$, choose any point $(\tilde{x}, \tilde{y}, \tilde{t}) \in \mathbb{R}^{2m+1}$ and set $w(x,y,t) = u(\tilde{x}+x, \tilde{y}+y-t\tilde{x}, t+\tilde{t})$. Then

$$\mathcal{K}u(x,y,t) = f(x,y,t) \quad \iff \quad \mathcal{K}w(x,y,t) = f(\tilde{x}+x, \tilde{y}+y-t\tilde{x}, t+\tilde{t}),$$

for every $(x,y,t) \in \Omega$.

The change of variables used above defines a Lie group in $\mathbb{R}^{N+1}$ with group law

$$(\tilde{z},\tilde{t}) \circ (z,t) = (\tilde{x},\tilde{y},\tilde{t}) \circ (x,y,t) = (\tilde{x}+x, \tilde{y}+y-t\tilde{x}, \tilde{t}+t), \tag{1.8}$$

$(z,t),\ (\tilde{z},\tilde{t}) \in \mathbb{R}^{N+1}$. Note that

$$(z,t)^{-1} = (x,y,t)^{-1} = (-x, -y-tx, -t), \tag{1.9}$$

and hence

$$(\tilde{z},\tilde{t})^{-1} \circ (z,t) = (\tilde{x},\tilde{y},\tilde{t})^{-1} \circ (x,y,t) = (x-\tilde{x}, y-\tilde{y}+(t-\tilde{t})\tilde{x}, t-\tilde{t}), \tag{1.10}$$

when $(z,t),\ (\tilde{z},\tilde{t}) \in \mathbb{R}^{N+1}$. Using this notation the operator $\mathcal{K}$ is δ_r-homogeneous of degree two, i.e., $\mathcal{K}\circ\delta_r = r^2(\delta_r\circ\mathcal{K})$, for all $r>0$. The operator $\mathcal{K}$ can be expressed as

$$\mathcal{K} = \sum_{i=1}^m X_i^2 + Y,$$

where

$$X_i = \partial_{x_i}, \quad i=1,\dots,m, \qquad Y = \sum_{i=1}^m x_i\partial_{y_i} - \partial_t, \tag{1.11}$$

and the vector fields $X_1,\dots,X_m$ and Y are left-invariant with respect to the group law (1.8) in the sense that

$$\begin{aligned} X_i\left(u((\tilde{z},\tilde{t})\circ\cdot)\right) &= (X_iu)\left((\tilde{z},\tilde{t})\circ\cdot\right), \quad i=1,\dots,m, \\ Y\left(u((\tilde{z},\tilde{t})\circ\cdot)\right) &= (Yu)\left((\tilde{z},\tilde{t})\circ\cdot\right), \end{aligned} \tag{1.12}$$

for every $(\tilde{z},\tilde{t}) \in \mathbb{R}^{N+1}$. Consequently, $\mathcal{K}\left(u((\tilde{z},\tilde{t})\circ\cdot)\right) = (\mathcal{K}u)\left((\tilde{z},\tilde{t})\circ\cdot\right)$. Taking commutators we see that $[X_i, Y] = \partial_{y_i}$ and that the vector fields $\{X_1,...,X_m,Y\}$ generate the Lie algebra associated to the Lie group $(\circ, \mathbb{R}^{N+1})$. In particular, (1.3) is equivalent to the Hörmander condition,

$$\text{rank Lie}\,(X_1,\dots,X_m,Y)\,(x,y,t) = N+1, \qquad \forall\,(x,y,t) \in \mathbb{R}^{N+1}, \tag{1.13}$$

see [H]. Furthermore, while X_i represents a differential operator of order one, ∂_{y_i} acts as a third order operator. This fact is also reflected in the dilations group $(\delta_r)_{r>0}$ defined above.

Based on the scalings and group of translations discussed above, writing $(x,y,t) = (x_1, x', y_1, y', t)$, $(\tilde{x},\tilde{y},\tilde{t}) = (\tilde{x}_1, \tilde{x}', \tilde{y}_1, \tilde{y}', \tilde{t}) \in \mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R}$, and assuming that x_1 is the dependent variable, it is natural to formulate geometry by using local coordinate charts and expressing the first coordinate x_1 as a function $f : \mathbb{R}^{m-1}\times\mathbb{R}^m\times\mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{aligned} &|f(x', y_1, y', t) - f(\tilde{x}', \tilde{y}_1, \tilde{y}', \tilde{t})| \\ \leq\ & M||(0, x'-\tilde{x}', y_1-\tilde{y}_1+(t-\tilde{t})\tilde{x}_1, y-\tilde{y}'+(t-\tilde{t})\tilde{x}', t-\tilde{t})||_K, \end{aligned} \tag{1.14}$$

for some M, where $\tilde{x}_1 = f(\tilde{x}', \tilde{y}_1, \tilde{y}', \tilde{t})$. Here

$$\|(x,y,t)\|_K = |(x,y)|_K + |t|^{\frac{1}{2}}, \ |(x,y)|_K = |x| + |y|^{1/3}, \tag{1.15}$$

whenever $(x,y,t) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} = \mathbb{R}^{N+1}$, see [CNP2], [CNP3]. Note that $\|\delta_r(x,y,t)\|_K = r\|(x,y,t)\|_K$ for every $r > 0$ and $(x,y,t) \in \mathbb{R}^{N+1}$. Furthermore, as long as f is allowed to depend on the variable y_1, and x_1 is assumed to be the dependent variable, then the term $y_1 - \tilde{y}_1 + (t-\tilde{t})\tilde{x}_1$ has to appear on the right hand side in (1.14) to achieve translation invariance. In line with [CNP2], [CNP3], we call a function f satisfying (1.14) a Lip_K-function, with Lip_K-constant M. From the perspective of scalings and group of translations, Lip_K-functions, and associated (local) domains, are the natural replacement in the context of the operator $\mathcal{K}$ of the Lip(1,1/2)-functions and Lip(1,1/2)-domains considered in the context of $\mathcal{H}$.

1.2. **Geometric aspects: Harnack chains.** While the outline above gives at hand that Lip_K-functions, and associated local Lip_K-domains, may serve as good candidates for geometries in which one may attempt to establish more refined boundary comparison principles for solutions to $\mathcal{K}u = 0$, further considerations are needed. In the corresponding theory for uniformly parabolic operators, the Harnack inequality and a method to connect points and to compare values for non-negative solutions, through Harnack chains in the geometry introduced, are usually very important tools needed to make progress. In this context the progress often builds on the validity of the strong maximum principle, the fact that the spatial variables $(z_1, ..., z_N)$ are decoupled from the time variable t, something which naturally also is reflected in the underlying group of translations, and a flexibility in the very formulation of the Harnack inequality. In contrast, this is where things starts to get complicated for the operator $\mathcal{K}$.

The tool used to build Harnack chains is that of $\mathcal{K}$-admissible paths. A path $\gamma : [0,T] \to \mathbb{R}^{N+1}$ is called $\mathcal{K}$-admissible if it is absolutely continuous and satisfies

$$\frac{d}{d\tau}\gamma(\tau) = \sum_{j=1}^{m} \omega_j(\tau) X_j(\gamma(\tau)) + \lambda(\tau) Y(\gamma(\tau)), \quad \text{for a.e. } \tau \in [0,T], \tag{1.16}$$

where $\omega_j \in L^2([0,T])$, for $j = 1, \dots, m$, and λ are non-negative measurable functions. We say that γ connects $(z,t) = (x,y,t) \in \mathbb{R}^{N+1}$ to $(\tilde{z}, \tilde{t}) = (\tilde{x}, \tilde{y}, \tilde{t}) \in \mathbb{R}^{N+1}$, $\tilde{t} < t$, if $\gamma(0) = (z,t)$ and $\gamma(T) = (\tilde{z}, \tilde{t})$. When considering Kolmogorov operators in the domain $\mathbb{R}^N \times (T_0, T_1)$, it is well known that (1.13) implies the existence of a $\mathcal{K}$-admissible path γ for any points $(z,t), (\tilde{z}, \tilde{t}) \in \mathbb{R}^{N+1}$ with $T_0 < \tilde{t} < t < T_1$.

Given a domain $\Omega \subset \mathbb{R}^{N+1}$, and a point $(z,t) \in \Omega$, we let $A_{(z,t)} = A_{(z,t)}(\Omega)$ denote the set

$$\big\{(\tilde{z}, \tilde{t}) \in \Omega \mid \exists \text{ a } \mathcal{K}\text{-admissible } \gamma : [0,T] \to \Omega \text{ connecting } (z,t) \text{ to } (\tilde{z}, \tilde{t})\big\},$$

and we define $\mathcal{A}_{(z,t)} = \mathcal{A}_{(z,t)}(\Omega) = \overline{A_{(z,t)}(\Omega)}$. Here and in the sequel, $\mathcal{A}_{(z,t)}(\Omega)$ is referred to as the propagation set of the point (z,t) with respect to Ω. The presence of the drift term in $\mathcal{K}$ considerably changes the geometric structure of $A_{(z,t)}(\Omega)$ and $\mathcal{A}_{(z,t)}(\Omega)$ compared to the case of uniformly parabolic equations. Indeed, simply consider $(z,t) = (x,y,t) \in \mathbb{R}^3$ in which case

$$\mathcal{K}u = X^2 u + Yu = 0, \qquad X = \partial_x, \quad \text{and} \quad Y = x\partial_y - \partial_t. \tag{1.17}$$

Consider the domain

$$\Omega = (-R,R) \times (-1,1) \times (-1,1], \tag{1.18}$$

where R is a given positive constant. In this case

$$\mathcal{A}_{(0,0,0)}(\Omega) = \big\{(x,y,t) \in \Omega : |y| \le -tR)\big\}, \tag{1.19}$$

and one can prove, see [CNP1], that there exists a non-negative solution u to $\mathcal{K}u = 0$ in Ω such that $u \equiv 0$ in $\mathcal{A}_{(0,0,0)}(\Omega)$ and such that $u > 0$ in $\Omega \setminus \mathcal{A}_{(0,0,0)}(\Omega)$. In particular, it is impossible to find a positive constant c such that $u(x, y, t) \leq cu(0, 0, 0)$ whenever $(x, y, t) \in \Omega \setminus \mathcal{A}_{(0,0,0)}(\Omega)$. Hence, in this sense the Harnack inequality cannot hold in a set greater than $\mathcal{A}_{(0,0,0)}(\Omega)$ and as a consequence the Harnack inequality we have at our disposal, see Theorem 2.1 stated in the bulk of the paper, is less flexible compared to the corresponding one for uniformly parabolic operators. Naturally this is also related to the Bony maximum principle, see [Bo].

In this context it is fair to mention that the first proof of the scale invariant Harnack inequality which constitutes one of the building blocks of our paper, can be found in [GL]. Furthermore, the introduction of that paper, see p. 776-777 in [GL], also contains a discussion of an example showing why a *uniform* Harnack inequality cannot be expected to hold outside of the propagation set $\mathcal{A}_{(z,t)}$. In [GL] the Harnack inequality is expressed in terms of level sets of the fundamental solution, hence depending implicitly on the underlying Lie group structure. This fact was used in [LP], where the group law (1.8) was used explicitly and the Harnack inequality, in the form we use it, was proved for the first time.

In general, using (1.16) we see that if we want to construct a $\mathcal{K}$-admissible path connecting $(z, t), (\tilde{z}, \tilde{t}) \in \mathbb{R}^{N+1}$, then we have flexibility to define and control the path in the x and t variables by choosing ω_j for $j = 1, \ldots, m$, and λ. However, by choosing $\{\omega_j\}$ and λ, the path in the y variables becomes determined by these choices. In this sense, any such construction renders a certain lack of control of the path in the y variables and it becomes a difficult task (impossible in some cases) to connect arbitrary points $(z, t) = (x, y, t)$ and $(\tilde{z}, \tilde{t}) = (\tilde{x}, \tilde{y}, \tilde{t})$, in a controlled manner, by $\mathcal{K}$-admissible paths and Harnack chains while taking geometric restrictions into account.

An important contribution of this paper is that we are able to overcome this concrete difficulty by imposing one additional restriction on our Lip_K-domains: we consider local Lip_K-domains defined by functions f as in (1.14) with the assumption that f does not depend on the variable y_1. This formulation of the geometry induces an additional degree of freedom which we are able to explore to make progress.

1.3. **Admissible local Lip_K-domains.** Given $(x, y) \in \mathbb{R}^N$, we write

$$(x, y) = (x_1, x', y_1, y') \text{ where } x' = (x_2, ..., x_m),\ y' = (y_2, ..., y_m), \tag{1.20}$$

and we let, with a slight abuse of notation,

$$\|(x', y', t)\|_K = \|((0, x'), (0, y'), t)\|_K = |x'| + |y'|^{1/3} + |t|^{\frac{1}{2}}, \tag{1.21}$$

in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}$. Given positive numbers r_1, r_2, we introduce the open cube

$$\square_{r_1,r_2} = \{(x', y', t) \in \mathbb{R}^{N-2} \times \mathbb{R} \mid |x_i| < r_1,\ |y_i| < r_1^3,\ |t| < r_2^2\}, \tag{1.22}$$

where $i \in \{2, \ldots, m\}$. Given any open set $\square_{r_1,r_2} \subset \mathbb{R}^{N-1} \times \mathbb{R}$, we say that a function f, $f : \square_{r_1,r_2} \to \mathbb{R}$, is a Lip_K-function, with respect to $e_1 = (1, 0, \ldots, 0)$, independent of y_1 and with constant $M \geq 0$, if $x_1 = f(x', y', t)$ and

$$|f(x', y', t) - f(\tilde{x}', \tilde{y}', \tilde{t})| \leq M \left\|(x' - \tilde{x}', y' - \tilde{y}' + (t - \tilde{t})\tilde{x}', t - \tilde{t})\right\|_K, \tag{1.23}$$

whenever $(x', y', t), (\tilde{x}', \tilde{y}', \tilde{t}) \in \square_{r_1,r_2}$. In addition, given positive numbers r_1, r_2, r_3, we let

$$Q_{r_1,r_2,r_3} = \{(x_1, x', y', t) \in \mathbb{R}^N \mid (x', y', t) \in \square_{r_1,r_2},\ |x_1| < r_3\}. \tag{1.24}$$

For positive M and r, we let $Q_{M,r} = Q_{r,\sqrt{2}r,4Mr}$. Finally, given f as above with $f(0, 0, 0) = 0$ and $M, r > 0$, we define

$$\Omega_{f,r} = \{(x_1, x', y_1, y', t) \mid (x_1, x', y', t) \in Q_{M,r},\ x_1 > f(x', y', t),\ |y_1| < r^3\},$$

$$\Delta_{f,r} \;=\; \{(x_1,x',y_1,y',t) \mid (x_1,x',y',t)\in Q_{M,r},\ x_1=f(x',y',t),\ |y_1|<r^3\}.$$

Definition 1. Let f be a Lip_K-function, with respect to $e_1=(1,0,\dots,0)$, independent of y_1 and with constant $M\geq 0$. Let $\Omega_{f,r}$ and $\Delta_{f,r}$ be defined as above. Given M, r_0, we say that $\Omega_{f,2r_0}$ is an *admissible* local Lip_K-domain, with Lip_K-constants M, r_0. Similar we refer to $\Delta_{f,2r_0}$ as an admissible local Lip_K-surface with Lip_K-constants M, r_0.

Remark 1.1. Our results, see Theorems 1.1, 1.2 and 1.3 below, are established near an admissible local Lip_K-surface $\Delta_{f,2r_0}$. The surface $\Delta_{f,2r_0}$ is contained in the non-characteristic part of the boundary of $\Omega_{f,2r_0}$. Recall that a vector $\nu\in\mathbb{R}^{N+1}$ is an outer normal to $\Omega_{f,2r_0}$ at $(z_0,t_0)\in\Delta_{f,2r_0}$ if there exists a positive r such that $B((z_0,t_0)+r\nu,r)\cap\Omega_{f,2r_0}=\emptyset$. Here $B((z_0,t_0)+r\nu,r)$ denotes the (standard) Euclidean ball in $\mathbb{R}^{N+1}$ with center at $(z_0,t_0)+r\nu$ and radius r. Now $\langle X_j(z_0,t_0),\nu\rangle\neq 0$, for some $j=1,\dots,m$, whenever $(z_0,t_0)\in\Delta_{f,2r_0}$. Hence, by definition all points $(z_0,t_0)\in\Delta_{f,2r_0}$ are non-characteristic points for the operator $\mathcal{K}$. For a more thorough discussion of this, regular points for the Dirichlet problem, and Fichera's classification, we refer to subsection 2.4, see (2.15) in particular.

Remark 1.2. We emphasize that an admissible local Lip_K-surface $\Delta_{f,2r_0}$ is defined through a function f which is independent of the y_1 variable. This formulation of the geometry induces an additional degree of freedom which we are able to explore to make progress. In particular, as discussed, due to the lack of flexibility when constructing $\mathcal{K}$-admissible paths and Harnack chains, it is difficult to connect arbitrary points $(z,t)=(x,y,t)$ and $(\tilde z,\tilde t)=(\tilde x,\tilde y,\tilde t)$, in a controlled manner, while taking geometric restrictions into account. However, using that $\Delta_{f,2r_0}$ is independent y_1, and as our equation is invariant under translations in the y_1 variable, we are able to explore this independence in the proof of our main results in a manner similar to how t independence is explored in [FGS]. We refer to subsection 1.5 below for a more thorough discussion, see also Remark 1.5 below.

1.4. **Statement of the main results.** Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain in the sense of Definition 1, with Lip_K-constants M, r_0. The topological boundary is denoted by $\partial\Omega_{f,2r_0}$. As discussed in the bulk of the paper, all points on $\Delta_{f,2r_0}$ are regular for the Dirichlet problem for the operator $\mathcal{K}$ in $\Omega_{f,2r_0}$. For every $(z,t)\in\Omega_{f,2r_0}$, there exists a unique probability measure $\omega_K(z,t,\cdot)$ on $\partial\Omega_{f,2r_0}$ such that the Perron-Wiener-Brelot solution to $\mathcal{K}u=0$ in $\Omega_{f,2r_0}$, with boundary data φ on $\partial\Omega_{f,2r_0}$, equals

$$u(z,t)=\int_{\partial\Omega_{f,2r_0}}\varphi(\tilde z,\tilde t)d\omega_K(z,t,\tilde z,\tilde t). \tag{1.25}$$

We refer to $\omega_K(z,t,\cdot)$ as the Kolmogorov measure relative to (z,t) and $\Omega_{f,2r_0}$. To formulate our results we also have to introduce certain reference points.

Definition 2. Given $\varrho>0$ and $\Lambda>0$ we let

$$\begin{aligned}
A^+_{\varrho,\Lambda} &= \left(\Lambda\varrho,0,-\tfrac{2}{3}\Lambda\varrho^3,0,\varrho^2\right)\in\mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R},\\
A_{\varrho,\Lambda} &= (\Lambda\varrho,0,0,0,0)\in\mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R},\\
A^-_{\varrho,\Lambda} &= \left(\Lambda\varrho,0,\tfrac{2}{3}\Lambda\varrho^3,0,-\varrho^2\right)\in\mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R}.
\end{aligned}\tag{1.26}$$

Furthermore, given $(z_0,t_0)\in\mathbb{R}^{N+1}$ and $\varrho>0$, we let $Q_{M,\varrho}(z_0,t_0)=(z_0,t_0)\circ Q_{M,\varrho}$ and $A_{\varrho,\Lambda}(z_0,t_0)=(z_0,t_0)\circ A_{\varrho,\Lambda}$. In Theorem 1.2 below we use the notation

$$d_K((z,t),(\tilde z,\tilde t)):=\|(\tilde z,\tilde t)^{-1}\circ(z,t)\|_K.$$

Theorem 1.1. *Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Then there exist $\Lambda=\Lambda(N,M)$, $1\leq\Lambda<\infty$, and $c_0=c_0(N,M)$, $1\leq c_0<$*

∞, such that the following is true. Assume that u is a non-negative solution to $\mathcal{K}u = 0$ in $\Omega_{f,2r_0}$ and that u vanishes continuously on $\Delta_{f,2r_0}$. Let $\varrho_0 = r_0/c_0$, introduce

$$m^+ = u(A^+_{\varrho_0,\Lambda}),\ m^- = u(A^-_{\varrho_0,\Lambda}), \tag{1.27}$$

and assume that $m^- > 0$. Then there exist constants $c_1 = c_1(N, M)$, $1 \leq c_1 < \infty$, $c_2 = c_2(N, M, m^+/m^-)$, $1 \leq c_2 < \infty$, such that if we let $\varrho_1 = \varrho_0/c_1$, then

$$u(z, t) \leq c_2 u(A_{\varrho,\Lambda}(z_0, t_0)),$$

whenever $(z, t) \in \Omega_{f,2r_0} \cap Q_{M,\varrho/c_1}(z_0, t_0)$, for some $0 < \varrho < \varrho_1$ and $(z_0, t_0) \in \Delta_{f,\varrho_1}$.

Theorem 1.2. *Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Then there exist $\Lambda = \Lambda(N, M)$, $1 \leq \Lambda < \infty$, and $c_0 = c_0(N, M)$, $1 \leq c_0 < \infty$, such that the following is true. Assume that u and v are non-negative solutions to $\mathcal{K}u = 0$ in $\Omega_{f,2r_0}$ and that v and u vanish continuously on $\Delta_{f,2r_0}$. Let $\varrho_0 = r_0/c_0$, introduce*

$$\begin{aligned} m_1^+ &= v(A^+_{\varrho_0,\Lambda}),\ m_1^- = v(A^-_{\varrho_0,\Lambda}),\\ m_2^+ &= u(A^+_{\varrho_0,\Lambda}),\ m_2^- = u(A^-_{\varrho_0,\Lambda}), \end{aligned} \tag{1.28}$$

and assume that $m_1^- > 0$, $m_2^- > 0$. Then there exist constants $c_1 = c_1(N, M)$, $c_2 = c_2(N, M, m_1^+/m_1^-, m_2^+/m_2^-)$, $\sigma = \sigma(N, M, m_1^+/m_1^-, m_2^+/m_2^-)$, $1 \leq c_1, c_2 < \infty$, $\sigma \in (0, 1)$, such that if we let $\varrho_1 = \varrho_0/c_1$, then

$$\left|\frac{v(z,t)}{u(z,t)} - \frac{v(\tilde z,\tilde t)}{u(\tilde z,\tilde t)}\right| \leq c_2 \left(\frac{d_K((z,t),(\tilde z,\tilde t))}{\varrho}\right)^{\sigma} \frac{v(A_{\varrho,\Lambda}(z_0,t_0))}{u(A_{\varrho,\Lambda}(z_0,t_0))},$$

whenever $(z, t), (\tilde z, \tilde t) \in \Omega_{f,2r_0} \cap Q_{M,\varrho/c_1}(z_0, t_0)$, for some $0 < \varrho < \varrho_1$ and $(z_0, t_0) \in \Delta_{f,\varrho_1}$.

Theorem 1.3. *Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Then there exist $\Lambda = \Lambda(N, M)$, $1 \leq \Lambda < \infty$, $c_1 = c_1(N, M)$, $1 \leq c_1 < \infty$, such that the following is true. Let $\varrho_1 = r_0/c_1$, and consider ϱ, $0 < \varrho < \varrho_1$. Let $(z_0, t_0) \in \Delta_{f,\varrho_1}$ and let $\omega_K(A^+_{\varrho,\Lambda}(z_0, t_0), \cdot)$ be the Kolmogorov measure relative to $A^+_{\varrho,\Lambda}(z_0, t_0)$ and $\Omega_{f,2r_0}$. Then there exist $c_2 = c_2(N, M)$, $1 \leq c_2 < \infty$, and $c_3 = c_3(N, M)$, $1 \leq c_3 < \infty$, such that*

$$\begin{aligned} &\omega_K(A^+_{\varrho,\Lambda}(z_0, t_0), \Delta_{f,2r_0} \cap Q_{M,2\tilde\varrho}(\bar z_0, \bar t_0))\\ &\leq c_2 \omega_K(A^+_{\varrho,\Lambda}(z_0, t_0), \Delta_{f,2r_0} \cap Q_{M,\tilde\varrho}(\bar z_0, \bar t_0)), \end{aligned}$$

whenever $(\bar z_0, \bar t_0) \in \Delta_{f,2r_0}$ and $Q_{M,\tilde\varrho}(\bar z_0, \bar t_0) \subset Q_{M,\varrho/c_3}(z_0, t_0)$.

Remark 1.3. Theorem 1.1 and Theorem 1.2 give scale and translation invariant quantitative estimates concerning the behavior, at the boundary, for non-negative solutions vanishing on $\Delta_{f,2r_0}$. The constants in the estimates depend only on N, M and certain reference quotients for (of) the solution(s) at well-defined interior points of reference. Theorem 1.3 gives a scale and translation invariant doubling property of the Kolmogorov measure.

Remark 1.4. Theorems 1.1, 1.2 and 1.3 are completely new and we believe that these theorems represent the starting point for far reaching developments concerning operators of Kolmogorov type. Using this notion of local Lip_K-domains we in [CNP2], [CNP3], in greater generality, developed a number of important preliminary results concerning the boundary behavior of non-negative solutions like, for example, the Carleson estimate. This paper can be seen as a rather far reaching continuation of these papers.

Remark 1.5. In Theorem 1.1, Theorem 1.2 and Theorem 1.3, as well as in the generalizations stated in Theorem 7.1 and Theorem 7.2 below, the underlying function f defining the local domain is assumed to be independent of a set of properly chosen variables. It is fair to pose the question if this is really necessary for the validity of this type of results. Though our argument relies heavily on independence, we believe that the answer to this question likely is no. We believe that the results established in this paper can serve as a starting point for the development of the corresponding results under weaker assumptions. We here leave this problem for future research.

1.5. **Brief discussion of the proof and organization of the paper.** Section 2 is of preliminary nature and we here state facts about the fundamental solution associated to $\mathcal{K}$, we state the Harnack inequality, we discuss the Dirichlet problem and we introduce the Kolmogorov measure and the Green function. In Section 3 we elaborate on the Harnack inequality, $\mathcal{K}$-admissible paths and Harnack chains under geometric restrictions. Some of the material in this section builds on results established in [CNP2], [CNP3]. In Section 4 we establish an important relation between the Kolmogorov measure and the Green function. In Section 5 we first prove Lemma 5.1 which gives a weak comparison principle at the boundary. Using Lemma 5.1 we in Section 5 then prove an important lemma: Lemma 5.3. In fact, it is Lemma 5.3 which enables us to, in the end, complete the proofs of Theorems 1.1, 1.2 and 1.3. In the context of admissible local Lip_K-domains, Lemma 5.3 states that there exist constants $c_i = c_i(N, M)$, $1 \leq c_i < \infty$, $i \in \{0, 1, 2, 3\}$, such that if u is a non-negative solution to $\mathcal{K}u = 0$ in $\Omega_{f,2r_0}$, vanishing continuously on $\Delta_{f,2r_0}$, $\varrho_0 = r_0/c_0$, $\varrho_1 = \varrho_0/c_1$, then

$$c_2^{-1} \frac{u(A^+_{\varrho_0,\Lambda})}{u(A^-_{\varrho_0,\Lambda})} \leq \frac{u(x_1, x', 0, y', t)}{u(x_1, x', y_1, y', t)} \leq c_2 \frac{u(A^+_{\varrho_0,\Lambda})}{u(A^-_{\varrho_0,\Lambda})}, \tag{1.29}$$

whenever $(x_1, x', y_1, y', t) \in \Omega_{f,\varrho_1/c_3}$. I.e., for (x_1, x', y', t) fixed and up to the boundary, all values of the function

$$y_1 \mapsto u(x_1, x', y_1, y', t)$$

are comparable to $u(x_1, x', 0, y', t)$, uniformly in (x_1, x', y', t), but with constants depending on the (acceptable) quotient $u(A^+_{\varrho_0,\Lambda})/u(A^-_{\varrho_0,\Lambda})$. Using this result we have a crucial additional degree of freedom at our disposal when building Harnack chains to connect points: we can freely connect points in the x_1 variable, taking geometric restriction into account, accepting that the path in the y_1 variable will most probably not end up in 'the right spot'. In the proof of Lemma 5.3 we use the fact that by the very definition of an admissible local Lip_K-domain, the surface $\Delta_{f,2r_0}$ is independent of y_1, hence we are able to translate with respect to this variable. Section 6 is devoted to the proof of Theorem 1.1, Theorem 1.2 and Theorem 1.3. Section 7 is devoted to a discussion of to what extent Theorems 1.1, 1.2 and 1.3 can be extended to more general operators of Kolmogorov type.

2. Preliminaries

In general we will establish our estimates in an admissible local Lip_K-domain $\Omega_{f,2r_0} \subset \mathbb{R}^{N+1}$, with Lip_K-constants M, r_0. Therefore, throughout the paper c will in general denote a positive constant $c \geq 1$, not necessarily the same at each occurrence, depending at most on N and M. Naturally $c = c(a_1, \dots, a_l)$ denotes a positive constant $c \geq 1$ which may depend only on $a_1, \dots, a_l$ and which is not necessarily the same at each occurrence. Two quantities A and B are said to be comparable, or $A \approx B$, if $c^{-1} \leq A/B \leq c$ for some $c = c(N, M)$, $c \geq 1$.

2.1. **Notation.** Recall the definition of $|(x,y)|_K$, $(x,y)\in\mathbb{R}^N$, in (1.15) and that $\|\delta_r(x,y,t)\|_K = r\|(x,y,t)\|_K$ for every $r>0$ and $(x,y,t)\in\mathbb{R}^{N+1}$. We recall the following pseudo-triangular inequality: there exists a positive constant $\mathbf{c}$ such that

$$\begin{aligned}\|(x,y,t)^{-1}\|_K &\leq \mathbf{c}\|(x,y,t)\|_K,\\ \|(x,y,t)\circ(\tilde x,\tilde y,\tilde t)\|_K &\leq \mathbf{c}(\|(x,y,t)\|_K + \|(\tilde x,\tilde y,\tilde t)\|_K),\end{aligned}\tag{2.1}$$

whenever $(x,y,t),(\tilde x,\tilde y,\tilde t)\in\mathbb{R}^{N+1}$. We define the quasi-distance d_K by setting

$$d_K((z,t),(\tilde z,\tilde t)) := \|(\tilde z,\tilde t)^{-1}\circ(z,t)\|_K,\tag{2.2}$$

and we introduce the ball

$$\mathcal{B}_K((x,y,t),r) := \{(\tilde x,\tilde y,\tilde t)\in\mathbb{R}^{N+1} \mid d_K((\tilde x,\tilde y,\tilde t),(x,y,t))<r\}.\tag{2.3}$$

Note that from (2.1) it follows directly that

$$d_K((x,y,t),(\tilde x,\tilde y,\tilde t)) \leq \mathbf{c}(d_K((x,y,t),(\hat x,\hat y,\hat t)) + d_K((\hat x,\hat y,\hat t),(\tilde x,\tilde y,\tilde t))),\tag{2.4}$$

whenever $(x,y,t),(\hat x,\hat y,\hat t),(\tilde x,\tilde y,\tilde t)\in\mathbb{R}^{N+1}$. For any $(x,y,t)\in\mathbb{R}^{N+1}$ and $H\subset\mathbb{R}^{N+1}$, we define

$$d_K((x,y,t),H) := \inf\{d_K((x,y,t),(\tilde x,\tilde y,\tilde t)) \mid (\tilde x,\tilde y,\tilde t)\in H\}.\tag{2.5}$$

Using this notation we say that a function $f:\mathcal{O}\to\mathbb{R}$ is Hölder continuous of order $\alpha\in(0,1]$, in short $f\in C_K^{0,\alpha}(\mathcal{O})$, if there exists a positive constant c such that

$$|f(x,y,t)-f(\tilde x,\tilde y,\tilde t)| \leq c\, d_K((x,y,t),(\tilde x,\tilde y,\tilde t))^\alpha,\tag{2.6}$$

for every $(x,y,t),(\tilde x,\tilde y,\tilde t)\in\mathcal{O}$. We let

$$\|u\|_{C_K^{0,\alpha}(\mathcal{O})} = \sup_{\mathcal{O}}|u| + \sup_{\substack{(x,y,t),(\tilde x,\tilde y,\tilde t)\in\mathcal{O}\\ (x,y,t)\neq(\tilde x,\tilde y,\tilde t)}} \frac{|u(x,y,t)-u(\tilde x,\tilde y,\tilde t)|}{\|(\tilde x,\tilde y,\tilde t)^{-1}\circ(x,y,t)\|_K^\alpha}.\tag{2.7}$$

Note that, if $\mathcal{O}$ is any bounded subset of $\mathbb{R}^{N+1}$, then every $u\in C_K^{0,\alpha}(\mathcal{O})$ is Hölder continuous in the usual sense as

$$\|(\tilde x,\tilde y,\tilde t)^{-1}\circ(x,y,t)\|_K \leq c_{\mathcal{O}}|(x,y,t)-(\tilde x,\tilde y,\tilde t)|^{\frac13}.$$

2.2. **Fundamental solution.** Following [K] and [LP] it is well known that an explicit fundamental solution, Γ, associated to $\mathcal{K}$ can be written down. Let

$$B := \begin{pmatrix} 0 & I_m\\ 0 & 0\end{pmatrix},\ E(s)=\exp(-sB^*),$$

for $s\in\mathbb{R}$, where I_m, 0, represent the identity matrix and the zero matrix in $\mathbb{R}^m$, respectively. * denotes the transpose. Furthermore, let

$$\mathcal{C}(t) \ := \ \int_0^t E(s)\begin{pmatrix} I_m & 0\\ 0 & 0\end{pmatrix}E^*(s)ds = \begin{pmatrix} tI_m & -\frac{t^2}{2}I_m\\ -\frac{t^2}{2}I_m & \frac{t^3}{3}I_m\end{pmatrix},$$

whenever $t\in\mathbb{R}$. Note that $\det\mathcal{C}(t)=t^{4m}/12$ and that

$$(\mathcal{C}(t))^{-1} = 12\begin{pmatrix} \frac{t^{-1}}{3}I_m & \frac{t^{-2}}{2}I_m\\ \frac{t^{-2}}{2}I_m & t^{-3}I_m\end{pmatrix}.$$

Using this notation we have that

$$\Gamma(z,t,\tilde z,\tilde t) = \Gamma(z-E(t-\tilde t)\tilde z, t-\tilde t,0,0)\tag{2.8}$$

where $\Gamma(z,t,0,0)=0$ if $t\leq 0$, $z\neq 0$, and

$$\Gamma(z,t,0,0) = \frac{(4\pi)^{-N/2}}{\sqrt{\det\mathcal{C}(t)}}\exp\left(-\frac14\langle\mathcal{C}(t)^{-1}z,z\rangle\right) \qquad \text{if } t>0.\tag{2.9}$$

Here $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^N$. We also note that

$$\Gamma(z,t,\tilde z,\tilde t)\le \frac{c}{\|(\tilde z,\tilde t)^{-1}\circ(z,t)\|_K^{\mathbf{q}}} \text{ for all } (z,t),\ (\tilde z,\tilde t)\in\mathbb{R}^N\times(0,T),\ t>\tilde t, \tag{2.10}$$

where $\mathbf{q}=4m$ and $c=c(N)$. Often $\mathbf{q}+2$ is referred to as the homogeneous dimension of $\mathbb{R}^{N+1}$ with respect to the dilations group $(\delta_r)_{r>0}$.

2.3. **The Harnack inequality.** To formulate the Harnack inequality we first need to introduce some additional notation. We let, for $r>0$ and $(z_0,t_0)\in\mathbb{R}^{N+1}$,

$$\begin{aligned} Q^- &= \Big(B(\tfrac12 e_1,1)\cap B(-\tfrac12 e_1,1)\Big)\times[-1,0],\\ Q_r^-(z_0,t_0) &= (z_0,t_0)\circ\delta_r\left(Q^-\right), \end{aligned} \tag{2.11}$$

where e_1 is the unit vector pointing in the direction of x_1 and $B(\frac12 e_1,1)$ and $B(-\frac12 e_1,1)$ are standard Euclidean balls of radius 1, centered at $\frac12 e_1$ and $-\frac12 e_1$, respectively. Similarly, we let

$$\begin{aligned} Q &= \Big(B(\tfrac12 e_1,1)\cap B(-\tfrac12 e_1,1)\Big)\times[-1,1],\\ Q_r(z_0,t_0) &= (z_0,t_0)\circ\delta_r\left(Q\right). \end{aligned} \tag{2.12}$$

Given $\alpha,\beta,\gamma,\theta\in\mathbb{R}$ such that $0<\alpha<\beta<\gamma<\theta^2$, we set

$$\widetilde Q_r^+(z_0,t_0)=\big\{(x,t)\in Q_{\theta r}^-(z_0,t_0)\mid t_0-\alpha r^2\le t\le t_0\big\},$$
$$\widetilde Q_r^-(z_0,t_0)=\big\{(x,t)\in Q_{\theta r}^-(z_0,t_0)\mid t_0-\gamma r^2\le t\le t_0-\beta r^2\big\}.$$

In the following we formulate two versions of the Harnack inequality. Recall, given a domain $\Omega\subset\mathbb{R}^{N+1}$ and a point $(z,t)\in\Omega$, the sets $A_{(z,t)}(\Omega)$ and $\mathcal{A}_{(z,t)}(\Omega)=\overline{A_{(z,t)}(\Omega)}$ defined in the introduction.

Theorem 2.1. *There exist constants $c>1$ and $\alpha,\beta,\gamma,\theta\in(0,1)$, with $0<\alpha<\beta<\gamma<\theta^2$, such that the following is true. Assume u is a non-negative solution to $\mathcal{K}u=0$ in $Q_r^-(z_0,t_0)$ for some $r>0$, $(z_0,t_0)\in\mathbb{R}^{N+1}$. Then,*

$$\sup_{\widetilde Q_r^-(z_0,t_0)} u\le c\inf_{\widetilde Q_r^+(z_0,t_0)} u.$$

Theorem 2.2. *Let $\Omega\subset\mathbb{R}^{N+1}$ be domain and let $(z_0,t_0)\in\Omega$. Let K be a compact set contained in the interior of $A_{(z_0,t_0)}(\Omega)$. Then there exists a positive constant c_K, depending only on Ω and K, such that*

$$\sup_K u\le c_K\, u(z_0,t_0),$$

green for every non-negative solution u of $\mathcal{K}u=0$ in Ω.

Remark 2.1. We emphasize, and this is different compared to the case of uniform parabolic equations, that the constants $\alpha,\beta,\gamma,\theta$ in Theorem 2.1 cannot be arbitrarily chosen. In particular, according to Theorem 2.2, the cylinder $\widetilde Q_r^-(z_0,t_0)$ has to be contained in the interior of the propagation set $\mathcal{A}_{(z_0,t_0)}(Q_r^-(z_0,t_0))$.

2.4. **The Dirichlet problem.** Let $\Omega\subset\mathbb{R}^{N+1}$ be a bounded domain with topological boundary $\partial\Omega$. Given $\varphi\in C(\partial\Omega)$ we consider here the well posedness of the boundary value problem

$$\begin{cases}\mathcal{K}u=0 & \text{in } \Omega,\\ u=\varphi & \text{on } \partial\Omega.\end{cases} \tag{2.13}$$

The existence of a solution to this problem can be established by using the Perron-Wiener-Brelot method and, in the sequel, u_φ will denote this solution to (2.13). In the following we first introduce what we refer to as the Kolmogorov boundary of Ω, denoted $\partial_K\Omega$. The notion of the Kolmogorov boundary replaces the notion of the parabolic boundary used in the context of uniformly parabolic equations.

Definition 3. The Kolmogorov boundary of Ω, denoted $\partial_K\Omega$, is defined as

$$\partial_K\Omega = \bigcup_{(z,t)\in\Omega} (\mathcal{A}_{(z,t)}(\Omega) \cap \partial\Omega).$$

By Definition 3, $\partial_K\Omega \subset \partial\Omega$ is the set of all points on the topological boundary of Ω which is contained in the closure of the propagation of at least one interior point in Ω. The importance of the Kolmogorov boundary of Ω is highlighted by the following lemma.

Lemma 2.1. *Consider the Dirichlet problem in* (2.13) *with boundary data* $\varphi \in C(\partial\Omega)$ *and let* $u = u_\varphi$ *be the corresponding Perron-Wiener-Brelot solution. Then*

$$\sup_{\Omega} |u| \leq \sup_{\partial_K\Omega} |\varphi|.$$

In particular, if $\varphi \equiv 0$ *on* $\partial_K\Omega$*, then* $u \equiv 0$ *in* Ω.

Proof. The lemma is a consequence of the Bony maximum principle, see [Bo]. □

$\partial_K\Omega$ is the largest subset of the topological boundary of Ω on which we can attempt to impose boundary data if we want to construct non trivial solutions. Hence, also the notion of regular points for the Dirichlet problem only makes sense for points on the Kolmogorov boundary and we let $\partial_R\Omega$ be the set of all $(z_0, t_0) \in \partial_K\Omega$ such that

$$\lim_{(z,t)\to(z_0,t_0)} u_\varphi(z,t) = \varphi(z_0,t_0) \text{ for any } \varphi \in C(\partial_K\Omega). \tag{2.14}$$

We refer to $\partial_R\Omega$ as the regular boundary of Ω with respect to the operator $\mathcal{K}$. By definition $\partial_R\Omega \subseteq \partial_K\Omega$.

Given a bounded domain $\Omega \subset \mathbb{R}^{N+1}$, in [M, Proposition 6.1] Manfredini gives sufficient conditions for regularity of boundary points. Recall that a vector $\nu \in \mathbb{R}^{N+1}$ is an outer normal to Ω at $(z_0, t_0) \in \partial\Omega$ if there exists a positive r such that $B((z_0, t_0) + r\nu, r) \cap \Omega = \emptyset$. Here $B((z_0, t_0) + r\nu, r)$ denotes the (standard) Euclidean ball in $\mathbb{R}^{N+1}$ with center at $(z_0, t_0) + r\nu$ and radius r. In consistency with Fichera's classification, sufficient conditions for the regularity can be expressed in geometric terms as follows. If $(z_0, t_0) \in \partial\Omega$ and $\nu = (\nu_1, ..., \nu_{N+1})$ is an outer normal to Ω at (z_0, t_0), then the following holds:

$$\begin{aligned} &(a) \text{ if } (\nu_1, \dots, \nu_m) \neq 0, \text{ then } (z_0,t_0) \in \partial_R\Omega, \\ &(b) \text{ if } (\nu_1, \dots, \nu_m) = 0 \text{ and } \langle Y(z_0,t_0), \nu\rangle > 0, \text{ then } (z_0,t_0) \in \partial_R\Omega, \\ &(c) \text{ if } (\nu_1, \dots, \nu_m) = 0 \text{ and } \langle Y(z_0,t_0), \nu\rangle < 0, \text{ then } (z_0,t_0) \notin \partial_R\Omega, \end{aligned} \tag{2.15}$$

where Y is the vector field defined in (1.11). Condition (a) can be equivalently expressed in terms of the vector fields X_j's as follows: $\langle X_j(z_0, t_0), \nu\rangle \neq 0$ for some $j = 1, \dots, m$. If this condition holds, then in the literature (z_0, t_0) is often referred to as a non-characteristic point for the operator $\mathcal{K}$.

A more refined sufficient condition for the regularity of the boundary points of $\partial\Omega$ is given in [M, Theorem 6.3] in terms of an exterior cone condition.

Lemma 2.2. *Let* $\Omega_{f,2r_0}$ *be an admissible local* Lip_K*-domain, with* Lip_K*-constants* M, r_0*. Then*

$$\partial_R\Omega_{f,2r_0} = \partial_K\Omega_{f,2r_0},$$

i.e., all points on the Kolmogorov boundary are regular for the operator $\mathcal{K}$.

Proof. First, using Lemma 3.6 below and the sufficient condition for the regularity of the boundary points in terms of the existence of exterior cones referred to above, see [M, Theorem 6.3], we have that

$$\Delta_{f,2r_0} \subset \partial_R \Omega_{f,2r_0}.$$

Furthermore, that

$$\partial_K \Omega_{f,2r_0} \setminus \Delta_{f,2r_0} \subset \partial_R \Omega_{f,2r_0}$$

follows, as discussed above, also by using the results in [M]. □

Remark 2.2. The operator adjoint to $\mathcal{K}$ is

$$\mathcal{K}^* = \sum_{i=1}^{m} \partial_{x_i x_i} - \sum_{i=1}^{m} x_i \partial_{y_i} + \partial_t. \tag{2.16}$$

In the case of the adjoint operator $\mathcal{K}^*$ we denote the associated Kolmogorov boundary of $\Omega_{f,2r_0}$ by $\partial_K^* \Omega_{f,2r_0}$. The above discussion and lemmas then apply to $\mathcal{K}^*$ subject to natural modifications.

Lemma 2.3. *Let $\Omega = \Omega_{f,2r_0}$. Then there exists, for any $\varphi \in C(\partial_K \Omega)$, $\varphi^* \in C(\partial_K^* \Omega)$, unique solutions $u = u_\varphi$, $u \in C^\infty(\Omega)$, $u^* = u_{\varphi^*}$, $u^* \in C^\infty(\Omega)$, to the Dirichlet problem in* (2.13) *and to the corresponding Dirichlet problem for $\mathcal{K}^*$, respectively. Furthermore, u is continuous up to the boundary at all boundary points contained in $\partial_K \Omega$ and u^* is continuous up to the boundary at all boundary points contained in $\partial_K^* \Omega$, Moreover, there exist, for every $(z,t) \in \Omega$, unique probability measures $\omega_K(x,t,\cdot)$ and $\omega_K^*(z,t,\cdot)$ on $\partial_K \Omega$ and $\partial_K^* \Omega$, respectively, such that*

$$\begin{aligned} u(z,t) &= \int_{\partial_K \Omega} \varphi(\tilde{z},\tilde{t}) d\omega_K(z,t,\tilde{z},\tilde{t}), \\ u^*(z,t) &= \int_{\partial_K^* \Omega} \varphi^*(\tilde{z},\tilde{t}) d\omega_K^*(x,t,\tilde{z},\tilde{t}). \end{aligned} \tag{2.17}$$

Proof. The lemma is an immediate consequence of Lemma 2.1 and Lemma 2.2. □

Definition 4. Let $(z,t) \in \Omega = \Omega_{f,2r_0}$. Then $\omega_K(z,t,\cdot)$ is referred to as the Kolmogorov measure relative to (z,t) and $\Omega = \Omega_{f,2r_0}$, and $\omega_K^*(z,t,\cdot)$ is referred to as the adjoint Kolmogorov measure relative to (z,t) and $\Omega = \Omega_{f,2r_0}$.

We define the Green function for $\Omega_{f,2r_0}$, with pole at $(\hat{z},\hat{t}) \in \Omega_{f,2r_0}$, as

$$\begin{aligned} G(z,t,\hat{z},\hat{t}) &= \Gamma(z,t,\hat{z},\hat{t}) \\ &\quad - \int_{\partial_K \Omega_{f,2r_0}} \Gamma(\tilde{z},\tilde{t},\hat{z},\hat{t}) d\omega_K(z,t,\tilde{z},\tilde{t}), \end{aligned} \tag{2.18}$$

where Γ is the fundamental solution to the operator $\mathcal{K}$ introduced in (2.8). If we instead consider $(z,t) \in \Omega_{f,2r_0}$ as fixed, then, for $(\hat{z},\hat{t}) \in \Omega_{f,2r_0}$,

$$\begin{aligned} G(z,t,\hat{z},\hat{t}) &= \Gamma(z,t,\hat{z},\hat{t}) \\ &\quad - \int_{\partial_K^* \Omega_{f,2r_0}} \Gamma(z,t,\tilde{z},\tilde{t}) d\omega_K^*(\hat{z},\hat{t},\tilde{z},\tilde{t}), \end{aligned} \tag{2.19}$$

where now $\partial_K^* \Omega_{f,2r_0}$ is the Kolmogorov boundary for the equation adjoint to $\mathcal{K}$ and $\omega_K^*(\hat{z},\hat{t},\cdot)$ is the associated adjoint Kolmogorov measure relative to $(\hat{z},\hat{t})$ and $\Omega_{f,2r_0}$. Given $\theta \in C_0^\infty(\mathbb{R}^{N+1})$, we have the representation formulas

$$\begin{aligned} \theta(z,t) &= \int_{\partial_K \Omega_{f,2r_0}} \theta(\tilde{z},\tilde{t}) d\omega_K(z,t,\tilde{z},\tilde{t}) + \int G(z,t,\hat{z},\hat{t}) \mathcal{K}\theta(\hat{z},\hat{t}) d\hat{z} d\hat{t}, \\ \theta(\hat{z},\hat{t}) &= \int_{\partial_K^* \Omega_{f,2r_0}} \theta(\tilde{z},\tilde{t}) d\omega_K^*(\hat{z},\hat{t},\tilde{z},\tilde{t}) + \int G(z,t,\hat{z},\hat{t}) \mathcal{K}^*\theta(z,t) dz dt, \end{aligned} \tag{2.20}$$

whenever $(z,t),\ (\hat z,\hat t)\in\Omega_{f,2r_0}$. In particular,

$$
\begin{aligned}
\int G(z,t,\hat z,\hat t)\mathcal{K}\theta(\hat z,\hat t)d\hat z d\hat t &= -\int \theta(\tilde z,\tilde t)\,d\omega_K(z,t,\tilde z,\tilde t),\\
\int G(z,t,\hat z,\hat t)\mathcal{K}^*\theta(z,t)dzdt &= -\int \theta(\tilde z,\tilde t)\,d\omega_K^*(\hat x,\hat t,\tilde z,\tilde t),
\end{aligned}
\tag{2.21}
$$

whenever $\theta\in C_0^\infty(\mathbb{R}^{N+1}\setminus\{(z,t)\})$ and $\theta\in C_0^\infty(\mathbb{R}^{N+1}\setminus\{(\hat z,\hat t)\})$, respectively.

3. Harnack chains under geometric restrictions

In this section we discuss the construction of Harnack chains in domains $\Omega\subset\mathbb{R}^{N+1}$ and we derive some important lemmas. The following lemma gives the general connection between appropriate $\mathcal{K}$-admissible paths and the possibility to compare values of non-negative solutions to $\mathcal{K}u=0$ in Ω.

Lemma 3.1. *Let $\Omega\subset\mathbb{R}^{N+1}$ be domain and let*

$$
\Omega_\epsilon := \{(z,t)\in\Omega \mid d_K((z,t),\partial\Omega)>\epsilon\}. \tag{3.1}
$$

for some $\epsilon\in(0,1)$ small enough to ensure that $\Omega_\epsilon\neq\emptyset$. Consider $(z,t),(\tilde z,\tilde t)\in\Omega_\epsilon$, $\tilde t<t$. Then the following is true for every non-negative solution u of $\mathcal{K}u=0$ in Ω. Consider a $\mathcal{K}$-admissible path $(\gamma(\tau),T-\tau):[0,T]\to\mathbb{R}^{N+1}$, defined by non-negative measurable functions $\omega_j\in L^2([0,T])$, for $j=1,\dots,m$, and λ. Assume that $(\gamma(\tau),T-\tau)\in\Omega_\epsilon$ for all $\tau\in[0,T]$, $\inf_{\tau\in[0,T]}\lambda(\tau)>0$, and that $(z,t)=\gamma(0)$, $(\tilde z,\tilde t)=\gamma(T)$. Then there exists a positive constant c, depending only on N, such that if we define $c(\gamma,\epsilon)$ through

$$
\ln(c(\gamma,\epsilon)) = c\left(1+\frac{t-\tilde t}{\epsilon^2}+\int_0^T \frac{\omega_1^2(s)+\cdots+\omega_m^2(s)}{\lambda(s)}\,ds\right),
$$

then

$$
u(\tilde z,\tilde t)\le c(\gamma,\epsilon)u(z,t).
$$

Remark 3.1. The problem when attempting to apply Lemma 3.1 is that, in general, we have no method at our disposal based on which we, in concrete situations, can construct a $\mathcal{K}$-admissible path $(\gamma(\tau),T-\tau):[0,T]\to\mathbb{R}^{N+1}$, connecting $(z,t),(\tilde z,\tilde t)\in\Omega_\epsilon$, while at the same time ensuring that $(\gamma(\tau),T-\tau)\in\Omega_\epsilon$ for all $\tau\in[0,T]$.

Definition 5. Let $\Omega\subset\mathbb{R}^{N+1}$ be domain. Let $(z,t),\ (\tilde z,\tilde t)\in\Omega,\ \tilde t<t$, be given. Let $\{r_j\}_{j=1}^k$ be a finite sequence of real numbers such that $0<r_j\le r_0$, for any $j=1,\dots,k$, and let $\{(z_j,t_j)\}_{j=1}^k$ be a sequence of points such that $(z_1,t_1)=(z,t)$. Then $\{\{(z_j,t_j)\}_{j=1}^k,\{r_j\}_{j=1}^k\}$ is said to be a Harnack chain in Ω connecting (z,t) to $(\tilde z,\tilde t)$ if

$$
\begin{aligned}
&(i) && Q_{r_j}^-(z_j,t_j)\subset\Omega, \text{ for every } j=1,\dots,k,\\
&(ii) && (z_{j+1},t_{j+1})\in\widetilde{Q}_{r_j}^-(z_j,t_j), \text{ for every } j=1,\dots,k-1,\\
&(iii) && (\tilde z,\tilde t)\in\widetilde{Q}_{r_k}^-(z_k,t_k).
\end{aligned}
\tag{3.2}
$$

Let $\Omega\subset\mathbb{R}^{N+1}$ be domain. Let $(z,t),\ (\tilde z,\tilde t)\in\Omega,\ \tilde t<t$, be given. Let u be a non-negative solution to $\mathcal{K}u=0$ in Ω. Assume that $\{\{(z_j,t_j)\}_{j=1}^k,\{r_j\}_{j=1}^k\}$ is a Harnack chain in Ω connecting $(\tilde z,\tilde t)$ to (z,t) and let c be the constant appearing in Theorem 2.1. Then, using Theorem 2.1, we see that

$$
u(z_{j+1},t_{j+1})\le cu(z_j,t_j), \text{ for every } j=1,\dots,k-1, \tag{3.3}
$$

and hence,

$$
u(\tilde z,\tilde t)\le cu(z_k,t_k)\le c^k u(z,t). \tag{3.4}
$$

Next we recall the following lemmas, Lemma 3.2 and Lemma 3.3. Lemma 3.2 is Lemma 2.2 in [BP].

Lemma 3.2. *Let* $(\gamma(\tau), T-\tau) : [0,T] \to \mathbb{R}^{N+1}$ *be a* $\mathcal{K}$*-admissible path and let* a, b *be constants such that* $0 \le a < b \le T$*. Then there exist positive constants* h *and* β*, depending only on* N*, such that*

$$\int_a^b |\omega(s)|^2 ds \le h \quad \Rightarrow \quad \gamma(b) \in Q_r^-(\gamma(a), T-a) \text{ where } r = \sqrt{\frac{b-a}{\beta}}. \tag{3.5}$$

Lemma 3.3. *Let* $\Omega \subset \mathbb{R}^{N+1}$ *be domain. Let* (z,t)*,* $(\tilde{z},\tilde{t}) \in \Omega$*,* $\tilde{t} < t$*, be given. Consider the path* $(\gamma(\tau), t-\tau) : [0, t-\tilde{t}] \to \mathbb{R}^{N+1}$ *where*

$$\gamma(\tau) = E(-\tau)\left(z + \mathcal{C}(\tau)\mathcal{C}^{-1}(t-\tilde{t})(E(t-\tilde{t})\tilde{z} - z)\right). \tag{3.6}$$

Then $\gamma(0) = z$*,* $\gamma(t-\tilde{t}) = \tilde{z}$ *and* $(\gamma(\tau), t-\tau)$ *is a* $\mathcal{K}$*-admissible path. Moreover, the path satisfies* (1.16) *with*

$$\omega(\tau) = (\omega_1(\tau), .., \omega_m(\tau)) = E(\tau)^*\mathcal{C}^{-1}(t-\tilde{t})(E(t-\tilde{t})\tilde{z} - z). \tag{3.7}$$

Let h *and* β *be as in Lemma 3.2 and define* $\{\tau_j\}$ *as follows. Let* $\tau_0 = 0$*, and define* τ_j*, for* $j \ge 1$*, recursively as follows:*

$$(i) \qquad \text{if } \int_{\tau_j}^{t-\tilde{t}} \frac{|\omega(\tau)|^2}{h} d\tau > 1 \text{ then } \tau_{j+1} = \inf\left\{\sigma \in (\tau_j, t-\tilde{t}] : \int_{\tau_j}^{\sigma} \frac{|\omega(\tau)|^2}{h} d\tau > 1\right\},$$

$$(ii) \qquad \text{if } \int_{\tau_j}^{t-\tilde{t}} \frac{|\omega(\tau)|^2}{h} d\tau \le 1 \text{ then } \tau_{j+1} := t-\tilde{t}.$$

Let k *be smallest index such that* $\tau_{k+1} = t-\tilde{t}$*. Define, based on* $\{\tau_j\}_{j=0}^{k+1}$*,*

$$r_j = \sqrt{\frac{\tau_{j+1} - \tau_j}{\beta}}, \qquad j = 1, \ldots, k, \tag{3.8}$$

and let $(z_j, t_j) = (\gamma(\tau_j), t-\tau_j)$ *for* $j = 1, \ldots, k$*. Assume that*

$$(\gamma(\tau), t-\tau) : [0, t-\tilde{t}] \to \Omega, \text{ and } Q_{r_j}^-(z_j, t_j) \subset \Omega, \tag{3.9}$$

for every $j = 1, \ldots, k$*. Then there exists a constant* $c = c(N)$*,* $1 \le c < \infty$*, such that if* u *is a non-negative solution to* $\mathcal{K}u = 0$ *in* Ω*, then*

$$u(\tilde{z},\tilde{t}) \le c^{\left(1 + \frac{1}{h}\langle \mathcal{C}^{-1}(t-\tilde{t})(z - E(t-\tilde{t})\tilde{z}), z - E(t-\tilde{t})\tilde{z}\rangle\right)} u(z,t). \tag{3.10}$$

Proof. This lemma is essentially proved in [BP]. In particular, that $(\gamma(\tau), t-\tau) : [0, t-\tilde{t}] \to \mathbb{R}^{N+1}$ is a $\mathcal{K}$-admissible path, and that (3.7) holds, follow by a direct computation. Similarly,

$$\int_0^{t-\tilde{t}} |\omega(\tau)|^2 d\tau = \langle \mathcal{C}^{-1}(t-\tilde{t})(z - E(t-\tilde{t})\tilde{z}), z - E(t-\tilde{t})\tilde{z}\rangle. \tag{3.11}$$

We now apply Lemma 3.2 to the path in (3.6). Let $\{\{(z_j, t_j)\}_{j=1}^k, \{r_j\}_{j=1}^k\}$ be constructed as in the statement of Lemma 3.3. Then, using Lemma 3.2, and the assumption in (3.9), it follows that

$$\{\{(z_j, t_j)\}_{j=1}^k, \{r_j\}_{j=1}^k\}$$

is a Harnack chain in $\mathbb{R}^{N+1}$ connecting $(\tilde{z},\tilde{t})$ to (z,t). Furthermore, the length of the chain, k, can be estimated and

$$k \le 1 + \frac{1}{h}\langle \mathcal{C}^{-1}(t-\tilde{t})(z - E(t-\tilde{t})\tilde{z}), z - E(t-\tilde{t})\tilde{z}\rangle. \tag{3.12}$$

This completes the proof of the lemma. □

Remark 3.2. The crucial assumption to be verified when applying Lemma 3.3 is (3.9), i.e., we have to ensure that $(\gamma(\tau), t-\tau) : [0, t-\tilde{t}] \to \Omega$ and that $Q^-_{r_j}(z_j, t_j) \subset \Omega$, for every $j = 1, \dots, k$. This condition is trivially satisfied when $\Omega = \mathbb{R}^N \times (T_0, T_1)$ for some $T_0 < \tau - r^2 < t < T_1$. In this case, the path constructed in Lemma 3.3 is the solution of an optimal control problem giving the $\mathcal{K}$-admissible path connecting (z, t), $(\tilde{z}, \tilde{t})$, $\tilde{t} < t$, which minimizes the energy

$$\int_0^{t-\tilde{t}} |\omega(\tau)|^2 d\tau. \tag{3.13}$$

This path is constructed without reference to any geometric restrictions and it is not a straight line. Clearly, this introduces new difficulties when we impose some geometric restrictions on the domain Ω as it is, in Lemma 3.3, the path which imposes restrictions on Ω. In reality we want the opposite: we want to construct a path subject to the geometric restrictions imposed by Ω. Finally, following [BP] we can also conclude that Lemma 3.3 holds for much more general operators of Kolmogorov type.

Remark 3.3. Consider Lemma 3.3 and let $\delta = t - \tilde{t}$. Then

$$\gamma(\tau) = E(-\tau)\left(z + \mathcal{C}(\tau)\mathcal{C}^{-1}(\delta)(E(\delta)\tilde{z} - z)\right). \tag{3.14}$$

By a straightforward computation we see that

$$\begin{aligned}
\mathcal{C}(\tau)\mathcal{C}^{-1}(\delta) &= 12\begin{pmatrix} \tau I_m & -\frac{\tau^2}{2} I_m \\ -\frac{\tau^2}{2} I_m & \frac{\tau^3}{3} I_m \end{pmatrix}\begin{pmatrix} \frac{\delta^{-1}}{3} I_m & \frac{\delta^{-2}}{2} I_m \\ \frac{\delta^{-2}}{2} I_m & \delta^{-3} I_m \end{pmatrix} \\
&= 12\begin{pmatrix} (\frac{1}{3}\tau\delta^{-1} - \frac{1}{4}(\tau\delta^{-1})^2) I_m & (\frac{1}{2}\tau\delta^{-2} - \frac{1}{2}(\tau^2\delta^{-3})) I_m \\ (-\frac{1}{6}\tau^2\delta^{-1} + \frac{1}{6}(\tau^3\delta^{-2})) I_m & (-\frac{1}{4}\tau^2\delta^{-2} + \frac{1}{3}(\tau\delta^{-1})^3) I_m \end{pmatrix} \\
&= \begin{pmatrix} A_{11}(\tau/\delta) I_m & \delta^{-1} A_{12}(\tau/\delta) I_m \\ \tau A_{21}(\tau/\delta) I_m & A_{22}(\tau/\delta) I_m \end{pmatrix}, \qquad (3.15)
\end{aligned}$$

where A_{ij} are bounded functions defined on the interval $[0, 1]$ and $A_{ij}(0) = 0$. Note also that

$$\begin{pmatrix} A_{11}(1) & A_{12}(1) \\ A_{21}(1) & A_{22}(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.16}$$

Furthermore, simply using the short notation $z = (x, y)$, $\tilde{z} = (\tilde{x}, \tilde{y})$, $A_{ij} = A_{ij}(\tau/\delta)$, we get, after some computations, that

$$\begin{aligned}
\gamma(\tau) &= E(-\tau)\left(z + \mathcal{C}(\tau)\mathcal{C}^{-1}(\delta)(E(\delta)\tilde{z} - z)\right) \\
&= \begin{pmatrix} I_m & 0 \\ \tau I_m & I_m \end{pmatrix}\begin{pmatrix} x + A_{11}(\tilde{x} - x) + \delta^{-1} A_{12}(\tilde{y} - y - \delta\tilde{x}) \\ y + \tau A_{21}(\tilde{x} - x) + A_{22}(\tilde{y} - y - \delta\tilde{x}) \end{pmatrix} = \begin{pmatrix} \gamma_x(\tau) \\ \gamma_y(\tau) \end{pmatrix}, \qquad (3.17)
\end{aligned}$$

where

$$\begin{aligned}
\gamma_x(\tau) &= x - A_{12}(\tau/\delta)\tilde{x} + A_{11}(\tau/\delta)(\tilde{x} - x) + \delta^{-1} A_{12}(\tau/\delta)(\tilde{y} - y), \\
\gamma_y(\tau) &= \tau(x + A_{11}(\tau/\delta)(\tilde{x} - x) + A_{21}(\tau/\delta)(\tilde{x} - x)) \\
&\quad + y + \tilde{A}_{12}(\tau/\delta)(\tilde{y} - y - \delta\tilde{x}) + A_{22}(\tau/\delta)(\tilde{y} - y - \delta\tilde{x}), \qquad (3.18)
\end{aligned}$$

for some new function $\tilde{A}_{12}$ with the same properties as A_{12}.

Remark 3.4. Consider Lemma 3.3 and let $\delta = t - \tilde{t}$. Consider the path $(\gamma(\tau), t-\tau) : [0, t-\tilde{t}] \to \mathbb{R}^{N+1}$. Using Remark 3.3 we see that

$$\begin{aligned}
(z, t)^{-1} \circ (\gamma(\tau), t - \tau) &= (-x, -y - tx, -t) \circ (\gamma(\tau), t - \tau) \\
&= (\delta_x(\tau), \delta_y(\tau), -\tau), \\
(\tilde{z}, \tilde{t})^{-1} \circ (\gamma(\tau), t - \tau) &= (-\tilde{x}, -\tilde{y} - \tilde{t}\tilde{x}, -\tilde{t}) \circ (\gamma(\tau), t - \tau) \\
&= (\tilde{\delta}_x(\tau), \tilde{\delta}_y(\tau), \delta - \tau),
\end{aligned}$$

where

$$\begin{aligned}\delta_x(\tau) &= -A_{12}(\tau/\delta)\tilde{x} + A_{11}(\tau/\delta)(\tilde{x}-x) + \delta^{-1}A_{12}(\tau/\delta)(\tilde{y}-y),\\ \delta_y(\tau) &= \tau(A_{11}(\tau/\delta)(\tilde{x}-x) + A_{21}(\tau/\delta)(\tilde{x}-x))\\ &\quad +\tilde{A}_{12}(\tau/\delta)(\tilde{y}-y-\delta\tilde{x}) + A_{22}(\tau/\delta)(\tilde{y}-y-\delta\tilde{x}),\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\tilde{\delta}_x(\tau) &= x-\tilde{x} - A_{12}(\tau/\delta)\tilde{x} + A_{11}(\tau/\delta)(\tilde{x}-x) + \delta^{-1}A_{12}(\tau/\delta)(\tilde{y}-y),\\ \tilde{\delta}_y(\tau) &= \tau(x-\tilde{x} + A_{11}(\tau/\delta)(\tilde{x}-x) + A_{21}(\tau/\delta)(\tilde{x}-x))\\ &\quad +(y-\tilde{y}) + \delta\tilde{x} + \tilde{A}_{12}(\tau/\delta)(\tilde{y}-y-\delta\tilde{x}) + A_{22}(\tau/\delta)(\tilde{y}-y-\delta\tilde{x}).\end{aligned} \tag{3.20}$$

Remark 3.5. Consider Lemma 3.3 and let $\delta = t - \tilde{t}$. Then, by similarly considerations as in Remark 3.3 we see that

$$\begin{aligned}\langle \mathcal{C}^{-1}(\delta)(z - E(\delta)\tilde{z}), z - E(\delta)\tilde{z}\rangle &= 4\delta^{-1}|x-\tilde{x}|^2 + 12\delta^{-3}|y-\tilde{y}+\delta\tilde{x}|^2\\ &\quad +12\delta^{-2}\langle y-\tilde{y}+\delta\tilde{x}, x-\tilde{x}\rangle\\ &\le 100(\delta^{-1}|x-\tilde{x}|^2 + \delta^{-3}|y-\tilde{y}+\delta\tilde{x}|^2).\end{aligned} \tag{3.21}$$

Remark 3.6. Inequality (3.10) in Lemma 3.3 gives the sharp bound for a non-negative solution in $\mathbb{R}^N$. The exponent appearing in (3.10) is found by solving an optimal control problem as briefly discussed in Remark 3.2. However, in the context of the equation $\mathcal{K}u = 0$ it is also possible to give a more intuitive construction of Harnack chains, a construction that gives a non sharp, but equivalent, exponent. In the following we show how to construct such a $\mathcal{K}$-admissible path connecting $(x, y, t) \in \mathbb{R}^N \times \mathbb{R}^+$ to $(0, 0, 0)$. Consider $\gamma : [0, t] \to \mathbb{R}^{N+1}$ such that

$$\frac{d}{d\tau}\gamma(\tau) = \sum_{j=1}^{m}\omega_j(\tau)X_j + Y(\gamma(\tau)),$$

for some piecewise constant vector $\omega = (\omega_i, \dots, \omega_m) \in \mathbb{R}^m$. Writing $\gamma(\tau) = (x(\tau), y(\tau), t-\tau)$ we have that

$$\frac{d}{d\tau}x(\tau) = \omega(\tau), \qquad \frac{d}{d\tau}y(\tau) = x(\tau).$$

We now let, for suitable vectors $\overline{\omega}, \widetilde{\omega} \in \mathbb{R}^m$ to be chosen, $\omega(\tau) = \overline{\omega}$ for $\tau \in [0, \frac{t}{2})$, $\omega(\tau) = \widetilde{\omega}$ for $\tau \in [\frac{t}{2}, \frac{3}{4}t)$, $\omega(\tau) = -\widetilde{\omega}$ for $\tau \in \left[\frac{3}{4}t, t\right]$. Specifically, we choose $\overline{\omega}$ so that $x\left(\frac{t}{2}\right) = 0$. A direct computation shows that

$$x\left(\tfrac{t}{2}\right) = x + \tfrac{t}{2}\overline{\omega}, \qquad y\left(\tfrac{t}{2}\right) = y + \tfrac{t}{2}x + \tfrac{t^2}{8}\overline{\omega},$$

and if we choose $\overline{\omega} = -\frac{2}{t}x$, then $x\left(\frac{t}{2}\right) = 0$ and $y\left(\frac{t}{2}\right) = y + \frac{t}{4}x$. In particular,

$$x(\tau) = \left(\tfrac{t}{4} - \left|\tau - \tfrac{3}{4}t\right|\right)\widetilde{\omega}, \qquad y(t) = y + \tfrac{t}{4}x + \tfrac{t^2}{16}\widetilde{\omega},$$

for $\tau \in \left[\frac{t}{2}, t\right]$ and $(x(t), (y(t)) = (0, 0)$ if we choose $\widetilde{\omega} = -\frac{16}{t^2}\left(y + \frac{t}{4}x\right)$. Based on this construction we now use Lemma 3.2 to give an estimate for the constant k in (3.4). Indeed, let k_0 be the positive integer which satisfies

$$k_0 h < \tfrac{2}{t}|x|^2 = \tfrac{t}{2}|\overline{\omega}|^2 = \int_0^{t/2}|\omega(s)|^2 ds \le (k_0+1)h.$$

By Lemma 3.2, the points $z_j = \gamma\left(\frac{t}{2\beta j}\right)$, $1 \le j \le k_0$, form a Harnack chain of length k_0. Analogously, we let k_1 be the positive integer which satisfies

$$k_1 h < \tfrac{8}{t^3}|y + \tfrac{t}{4}x|^2 = \tfrac{t}{4}|\widetilde{\omega}|^2 = \int_{t/2}^{3t/4}|\omega(s)|^2 ds \le (k_1+1)h,$$

and we form a Harnack chain of length k_1. The construction made in the interval $[\frac{t}{2}, \frac{3}{4}t)$ gives a Harnack chain also for $\left[\frac{3}{4}t, t\right]$. We eventually obtain a Harnack chain of length $k = k_0 + 2k_1 + 3$. Put together, the above two inequalities imply that $u(0,0,0) \leq c^k u(x,y,t)$ with k satisfying

$$k \leq c\left(\frac{|x|^2}{t} + \frac{|y + \frac{t}{4}x|^2}{t^3}\right), \tag{3.22}$$

for some positive constant c depending only on N. This argument was introduced in [P].

Lemma 3.4. *Let* Λ *be a positive constant. Define*

$$z_\Lambda = \left(\Lambda, 0, -\tfrac{2}{3}\Lambda, 0\right) \in \mathbb{R} \times \mathbb{R}^{m-1} \times \mathbb{R} \times \mathbb{R}^{m-1}. \tag{3.23}$$

Then, the path $[0,1] \ni \tau \to \gamma(\tau) = \delta_{1-\tau}(z_\Lambda, 1)$ *is* $\mathcal{K}$*-admissible.*

Proof. Note that by definition

$$\gamma(\tau) = \left((1-\tau)\Lambda, 0, -\tfrac{2}{3}(1-\tau)^3\Lambda, 0, (1-\tau)^2\right), \tau \in [0,1].$$

Hence, by a direct computation

$$\frac{d}{d\tau}\gamma(\tau) = (-\Lambda, 0, 2(1-\tau)^2\Lambda, -2(1-\tau)),\ \tau \in [0,1].$$

In particular,

$$\frac{d}{d\tau}\gamma(\tau) = \sum_{j=1}^m \omega_j(\tau)X_j(\gamma(\tau)) + \lambda(\tau)Y(\gamma(\tau)),\ \tau \in [0,1], \tag{3.24}$$

where $\omega_1 = -\Lambda$, $\omega_j \equiv 0$ for $j \in \{2,..,m\}$ and $\lambda(\tau) = 2(1-\tau)$. □

3.1. **The Harnack inequality in cones in local** Lip_K**-domains.** Given $(z_0,t_0) \in \mathbb{R}^{N+1}$, $\bar{z} \in \mathbb{R}^N$, $\bar{t} \in \mathbb{R}_+$, consider an open neighborhood $U \subset \mathbb{R}^N$ of $\bar{z}$, and let

$$\begin{aligned} Z^+_{\bar{z},\bar{t},U}(z_0,t_0) &= \left\{(z_0,t_0) \circ \delta_s(x,\bar{t}) \mid x \in U,\ 0 < s \leq 1\right\}, \\ Z^-_{\bar{z},\bar{t},U}(z_0,t_0) &= \left\{(z_0,t_0) \circ \delta_s(x,-\bar{t}) \mid x \in U,\ 0 < s \leq 1\right\}. \end{aligned} \tag{3.25}$$

Then $Z^+_{\bar{z},\bar{t},U}(z_0,t_0)$ and $Z^-_{\bar{z},\bar{t},U}(z_0,t_0)$ are cones with vertex at (z_0,t_0). Note that this notation was introduced in [CNP3]. Given $\varrho > 0$ and $\Lambda > 0$, recall the points $A^+_{\varrho,\Lambda}$, $A_{\varrho,\Lambda}$, $A^-_{\varrho,\Lambda}$, introduced in (1.26). In addition we here introduce

$$\begin{aligned} \tilde{A}^+_{\varrho,\Lambda} &= (-\Lambda\varrho, 0, \frac{2}{3}\Lambda\varrho^3, 0, \varrho^2), \\ \tilde{A}^-_{\varrho,\Lambda} &= (-\Lambda\varrho, 0, -\frac{2}{3}\Lambda\varrho^3, 0, -\varrho^2). \end{aligned} \tag{3.26}$$

Furthermore, given $(z_0,t_0) \in \mathbb{R}^{N+1}$ we let $A^\pm_{\varrho,\Lambda}(z_0,t_0) = (z_0,t_0) \circ A^\pm_{\varrho,\Lambda}$, $A_{\varrho,\Lambda}(z_0,t_0) = (z_0,t_0) \circ A_{\varrho,\Lambda}$, $\tilde{A}^\pm_{\varrho,\Lambda}(z_0,t_0) = (z_0,t_0) \circ \tilde{A}^\pm_{\varrho,\Lambda}$. Consider the cones $Z^\pm_{\cdot,\cdot,\cdot}(z_0,t_0)$ defined in (3.25). Given η, $0 < \eta \ll 1$, Λ, and $\rho > 0$, we let

$$\begin{aligned} C^+_{\varrho,\eta,\Lambda}(z_0,t_0) &= Z^+_{A^+_{\varrho,\Lambda},\mathcal{B}_K((z^+_{\varrho,\Lambda},0),\eta\varrho)\cap\{(z,t)\in\mathbb{R}^{N+1}:t=0\}}(z_0,t_0), \\ C^-_{\varrho,\eta,\Lambda}(z_0,t_0) &= Z^-_{A^-_{\varrho,\Lambda},\mathcal{B}_K((z^-_{\varrho,\Lambda},0),\eta\varrho)\cap\{(z,t)\in\mathbb{R}^{N+1}:t=0\}}(z_0,t_0), \\ \tilde{C}^+_{\varrho,\eta,\Lambda}(z_0,t_0) &= Z^+_{\tilde{A}^+_{\varrho,\Lambda},\mathcal{B}_K((\tilde{z}^+_{\varrho,\Lambda},0),\eta\varrho)\cap\{(z,t)\in\mathbb{R}^{N+1}:t=0\}}(z_0,t_0), \\ \tilde{C}^-_{\varrho,\eta,\Lambda}(z_0,t_0) &= Z^-_{\tilde{A}^-_{\varrho,\Lambda},\mathcal{B}_K((\tilde{z}^+_{\varrho,\Lambda},0)\eta\varrho)\cap\{(z,t)\in\mathbb{R}^{N+1}:t=0\}}(z_0,t_0), \end{aligned} \tag{3.27}$$

where the points $z^+_{\varrho,\Lambda}$, $z^-_{\varrho,\Lambda}$, $\tilde z^+_{\varrho,\Lambda}$, $\tilde z^-_{\varrho,\Lambda}$ are defined through the relations $A^+_{\varrho,\Lambda} = (z^+_{\varrho,\Lambda},\varrho^2)$, $A^-_{\varrho,\Lambda} = (z^-_{\varrho,\Lambda},-\varrho^2)$, $\tilde A^+_{\varrho,\Lambda} = (\tilde z^+_{\varrho,\Lambda},\varrho^2)$, $\tilde A^-_{\varrho,\Lambda} = (\tilde z^-_{\varrho,\Lambda},-\varrho^2)$. The balls $\mathcal{B}_K((z^\pm_{\varrho,\Lambda},0),\eta\varrho)$, $\mathcal{B}_K((\tilde z^\pm_{\varrho,\Lambda},0),\eta\varrho)$, are defined as in (2.3). Note that

$$C^\pm_{\varrho,\eta,\Lambda}(z_0,t_0),\ \tilde C^\pm_{\varrho,\eta,\Lambda}(z_0,t_0), \tag{3.28}$$

represent, for η small, cones 'centered' around appropriate ($\mathcal{K}$-admissible) paths passing through (z_0,t_0) as well as the reference points $A^\pm_{\varrho,\Lambda}(z_0,t_0)$, $\tilde A^\pm_{\varrho,\Lambda}(z_0,t_0)$.

Lemma 3.5. *Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Then there exist $\Lambda=\Lambda(N,M)$, $1\le\Lambda<\infty$, and $c_0=c_0(N,M)$, $1\le c_0<\infty$, such that the following is true. Let $\varrho_0=r_0/c_0$, consider $(z_0,t_0)\in\Delta_{f,\varrho_0}$, $0<\varrho<\varrho_0$, and let $A^\pm_{\varrho,\Lambda}(z_0,t_0)$, $\tilde A^\pm_{\varrho,\Lambda}(z_0,t_0)$, be defined as above. Then*

$$A^\pm_{\varrho,\Lambda}(z_0,t_0),\ \tilde A^\pm_{\varrho,\Lambda}(z_0,t_0)\in\Omega_{f,r_0}, \tag{3.29}$$

and there exists a constant $c=c(N,M)$, $1\le c<\infty$, such that

$$\begin{aligned} &(i) \qquad c^{-1}\varrho<d_K(P_{\varrho,\Lambda}(z_0,t_0),(z_0,t_0))<c\varrho,\\ &(ii) \qquad c^{-1}\varrho<d_K(P_{\varrho,\Lambda}(z_0,t_0),\Delta_{f,2r_0}),\end{aligned} \tag{3.30}$$

whenever $P_{\varrho,\Lambda}(z_0,t_0)\in\{A^\pm_{\varrho,\Lambda}(z_0,t_0),\tilde A^\pm_{\varrho,\Lambda}(z_0,t_0)\}$. Furthermore, the paths

$$\gamma^+(\tau)=A^+_{(1-\tau)\varrho,\Lambda}(z_0,t_0),\ \gamma^-(\tau)=A^-_{(1-\tau)\varrho,\Lambda}(z_0,t_0),\tau\in[0,1], \tag{3.31}$$

are $\mathcal{K}$-admissible paths.

Proof. (3.29) and (3.30) are consequences of Lemma 4.4 in [CNP3]. That the paths in (3.31) are $\mathcal{K}$-admissible follows from Lemma 3.4. □

Lemma 3.6. *Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Then there exist $\Lambda=\Lambda(N,M)$, $1\le\Lambda<\infty$, and $c_0=c_0(N,M)$, $1\le c_0<\infty$, such that the following is true. Let $\varrho_0=r_0/c_0$, consider $(z_0,t_0)\in\Delta_{f,\varrho_0}$ and $0<\varrho<\varrho_0$. Then there exists $\eta=\eta(N,M)$, $0<\eta\ll1$, such that if we introduce $C^\pm_{\varrho,2\eta,\Lambda}(z_0,t_0)$, $\tilde C^\pm_{\varrho,2\eta,\Lambda}(z_0,t_0)$, as in (3.27), then*

$$\begin{aligned} &(i) \quad C^\pm_{\varrho,2\eta,\Lambda}(z_0,t_0)\subset\Omega_{f,r_0},\\ &(ii) \quad \tilde C^\pm_{\varrho,2\eta,\Lambda}(z_0,t_0)\subset\mathbb{R}^{N+1}\setminus\Omega_{f,r_0}.\end{aligned} \tag{3.32}$$

Proof. This is a consequence of Lemma 4.4 in [CNP3]. □

Lemma 3.7. *Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Let $\Lambda=\Lambda(N,M)$, $1\le\Lambda<\infty$, be as in Lemma 3.6. Then there exists $c_0=c_0(N,M)$, $1\le c_0<\infty$, such that the following holds. Let $\varrho_0=r_0/c_0$, $\varrho_1=\varrho_0/c_0$, assume $(z,t)\in\Omega_{f,\varrho}$, $0<\varrho<\varrho_1$, and let $d=d_K((z,t),\Delta_{f,2r_0})$. Then there exist $(z_0^\pm,t_0^\pm)\in\Delta_{f,c_0\varrho}$ and $\varrho^\pm$ such that*

$$(z,t)=A^\pm_{\varrho^\pm,\Lambda}(z_0^\pm,t_0^\pm) \text{ and } c^{-1}d\le\varrho^\pm\le cd,$$

for some $c=c(N,M)$, $1\le c<\infty$.

Proof. This result is Lemma 4.6 in [CNP3], but we here give a simplified proof. Let in the following c_0 be a degree of freedom as stated in the lemma, let $\varrho_0=r_0/c_0$, $\varrho_1=\varrho_0/c_0$, and consider $(z,t)=(x_1,x',y_1,y',t)\in\Omega_{f,\varrho}$ for some $0<\varrho<\varrho_1$. Let $d=d_K((z,t),\Delta_{f,2r_0})$. In the following we prove that $(z,t)=A^+_{\varrho^+,\Lambda}(z_0^+,t_0^+)$ for some (z_0^+,t_0^+), ϱ^+, as stated in the lemma. Consider the path

$$\begin{aligned}\gamma(\tau) &= (x_1,x',y_1,y',t)\circ\delta_{d\tau}(A^+_{1,\Lambda})^{-1}\\ &= (x_1,x',y_1,y',t)\circ\delta_{d\tau}(-\Lambda,0,-\frac13\Lambda,0,-1)\end{aligned}$$

$$= (x_1 - \Lambda d\tau, x', y_1 - \frac{1}{3}\Lambda(d\tau)^3 - (d\tau)^2 x_1, y' - (d\tau)^2 x', t - (d\tau)^2) \tag{3.33}$$

for $\tau \geq 0$. Then $\gamma(0) = (x_1, x', y_1, y', t)$. Let $\tau_0 \geq 0$ be the first value of τ for which $\gamma(\tau) \in \Delta_{f,2r_0}$. Now, using that $\Omega_{f,2r_0}$ is an admissible local Lip_K-domain, with Lip_K-constant M, we first note that $d \approx |x_1 - f(x', y', t)|$, with constants of comparison depending only on N and M, and then that there exists $c = c(N, M)$, $1 \leq c < \infty$, such that $c^{-1} \leq \tau_0 \leq c$. Let $(z_0^+, t_0^+) = \gamma(\tau_0)$, then $(z, t) = A^+_{d\tau_0,\Lambda}(z_0^+, t_0^+)$ and the conclusions of the lemma follows immediately. □

Remark 3.7. Given an admissible local Lip_K-domain $\Omega_{f,2r_0}$, with Lip_K-constants M, r_0, we let, from now on, $\Lambda = \Lambda(N, M)$, $1 \leq \Lambda < \infty$, $c_0 = c_0(N, M)$, $1 \leq c_0 < \infty$, and $\eta = \eta(N, M)$, $0 < \eta \ll 1$, be such that Lemma 3.5 and Lemma 3.6 hold whenever $(z_0, t_0) \in \Delta_{f,\varrho_0}$ and $0 < \varrho < \varrho_0$, and such that Lemma 3.7 holds whenever $(z, t) \in \Omega_{f,\varrho}$, $0 < \varrho < \varrho_1$.

Lemma 3.8. *Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Let Λ, c_0, η, ϱ_0, ϱ_1, be in accordance with Remark 3.7. Let δ, $0 < \delta < 1$, be a degree of freedom. Then there exists $c = c(N, M, \delta)$, $1 \leq c < \infty$, such that following holds. Assume that u is a non-negative solution to $\mathcal{K}u = 0$ in $\Omega_{f,2\varrho_0}$, that $(z_0, t_0) \in \Delta_{f,\varrho_1}$, and consider ϱ such that $0 < \varrho < \varrho_1$. Then*

$$\begin{aligned}
&(i) \quad \sup_{\mathcal{B}_K(A^+_{\delta\varrho,\Lambda}(z_0,t_0),\varrho/c)} u \leq c \inf_{\mathcal{B}_K(A^+_{\varrho,\Lambda}(z_0,t_0),\varrho/c)} u,\\
&(ii) \quad \inf_{\mathcal{B}_K(A^-_{\delta\varrho,\Lambda}(z_0,t_0),\varrho/c)} u \geq c^{-1} \sup_{\mathcal{B}_K(A^-_{\varrho,\Lambda}(z_0,t_0),\varrho/c)} u,
\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}
&(i') \quad A^+_{\delta\varrho,\Lambda}(\hat{z}_0, \hat{t}_0) \in \mathcal{B}_K(A^+_{\delta\varrho,\Lambda}(z_0, t_0), \varrho/c),\\
&(ii') \quad A^-_{\delta\varrho,\Lambda}(\hat{z}_0, \hat{t}_0) \in \mathcal{B}_K(A^-_{\delta\varrho,\Lambda}(z_0, t_0), \varrho/c),
\end{aligned} \tag{3.35}$$

whenever $(\hat{z}_0, \hat{t}_0) \in \Delta_{f,\varrho/c}(z_0, t_0)$.

Proof. We first note that there exists, given δ, $0 < \delta < 1$, $\bar{c}$ depending only on N, M and δ, such that

$$\mathcal{B}_K(A^{\pm}_{\delta\varrho,\Lambda}(z_0, t_0), \varrho/\bar{c}) \subset C^{\pm}_{\varrho,2\eta,\Lambda}(z_0, t_0) \subset \Omega_{f,r_0} \tag{3.36}$$

where the second inclusion follows from Lemma 3.6 (i). Furthermore, to prove the lemma we note that we can, without loss of generality, assume that $\varrho = 1$ and that $(z_0, t_0) = (0, 0)$. We then want to prove, given δ, $0 < \delta < 1$, that there exist c_1, c_2, c_3, depending only on N, M and δ, such that

$$\begin{aligned}
&(i) \quad \sup_{\mathcal{B}_K(A^+_{\delta,\Lambda}(0,0),1/c_1)} u \leq c_2 u(A^+_{1,\Lambda}(0, 0)),\\
&(ii) \quad \inf_{\mathcal{B}_K(A^-_{\delta,\Lambda}(0,0),1/c_1)} u \geq c_2^{-1} u(A^-_{1,\Lambda}(0, 0)),
\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}
&(i') \quad A^+_{\delta,\Lambda}(\hat{z}_0, \hat{t}_0) \in \mathcal{B}_K(A^+_{\delta,\Lambda}(0, 0), 1/c_1),\\
&(ii') \quad A^-_{\delta,\Lambda}(\hat{z}_0, \hat{t}_0) \in \mathcal{B}_K(A^-_{\delta,\Lambda}(0, 0), 1/c_1),
\end{aligned} \tag{3.38}$$

whenever $(\hat{z}_0, \hat{t}_0) \in \Delta_{f,1/c_3}(0, 0)$. Note that the statements in (3.37) depend only on the geometry of $\Omega_{f,2r_0}$ through Λ. To prove (3.37) we now first note, using (3.36), the construction, Lemma 3.9 and its proof, that

$$A^+_{\delta,\Lambda}(0, 0) \in A_{A^+_{1,\Lambda}(0,0)}(C^+_{2,\eta,\Lambda}(0, 0)).$$

In particular, $A^+_{\delta,\Lambda}(0,0)$ is an interior point of the propagation set of $A^+_{1,\Lambda}(0,0)$ in $C^+_{2,\eta,\Lambda}(0,0)$ $(A_{A^+_{1,\Lambda}(0,0)}(C^+_{2,\eta,\Lambda}(0,0))$. Using this we immediately see that there exists $\tilde c=\tilde c(N,M,\delta)$, $1\leq \tilde c<\infty$, such that

$$\mathcal{B}_K(A^+_{\delta,\Lambda}(0,0),1/\tilde c)\subset A_{A^+_{1,\Lambda}(0,0)}(C^+_{2,\eta,\Lambda}(0,0)). \tag{3.39}$$

By essentially the same argument we have that

$$A^-_{1,\Lambda}(0,0)\in A_{(z,t)}(C^-_{2,\eta,\Lambda}(0,0)), \tag{3.40}$$

whenever $(z,t)\in\mathcal{B}_K(A^-_{\delta,\Lambda}(0,0),1/\tilde c)$. Letting $c_1=\max\{\bar c,\tilde c\}$ and appealing to Theorem 2.2 we see that (3.37) follows. To prove (i') and (ii') we first note that $A^\pm_{\delta,\Lambda}(0,0)\in\mathcal{B}_K(A^\pm_{\delta,\Lambda}(0,0),1/c_1)$. Hence, the statements in (3.38) simply follow by continuity of the maps

$$(\hat z_0,\hat t_0)\to(\hat z_0,\hat t_0)\circ A^\pm_{\delta,\Lambda}(0,0)=A^\pm_{\delta,\Lambda}(\hat z_0,\hat t_0).$$

This completes the proof of the lemma. □

Lemma 3.9. *Let $\Omega_{f,2r_0}$ be an admissible local* Lip_K*-domain, with* Lip_K*-constants M, r_0. Let Λ, c_0, η, ϱ_0, ϱ_1, be in accordance with Remark 3.7. Then there exist $c=c(N,M)$, $1\leq c<\infty$, and $\gamma=\gamma(N,M)$, $0<\gamma<\infty$, such that following holds. Assume that u is a non-negative solution to $\mathcal{K}u=0$ in $\Omega_{f,2\varrho_0}$, that $(z_0,t_0)\in\Delta_{f,\varrho_1}$, and consider ϱ, $\tilde\varrho$, $0<\tilde\varrho\leq\varrho<\varrho_1$. Then*

$$\begin{aligned} u(A^+_{\tilde\varrho,\Lambda}(z_0,t_0)) &\leq c(\varrho/\tilde\varrho)^\gamma u(A^+_{\varrho,\Lambda}(z_0,t_0)),\\ u(A^-_{\tilde\varrho,\Lambda}(z_0,t_0)) &\geq c^{-1}(\tilde\varrho/\varrho)^\gamma u(A^-_{\varrho,\Lambda}(z_0,t_0)). \end{aligned} \tag{3.41}$$

Proof. The lemma follows from the construction of Harnack chain along the paths in (3.31) and Lemma 3.8. For the details we refer to Lemma 4.3 in [CNP3]. □

3.2. **Additional estimates based on the Harnack inequality.** Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Recall that given f with $f(0,0,0)=0$ and $M,r>0$, we defined

$$\begin{aligned} \Omega_{f,r} &= \{(x_1,x',y_1,y',t)\mid (x_1,x',y',t)\in Q_{M,r},\ x_1>f(x',y',t),\ |y_1|<r^3\},\\ \Delta_{f,r} &= \{(x_1,x',y_1,y',t)\mid (x_1,x',y',t)\in Q_{M,r},\ x_1=f(x',y',t),\ |y_1|<r^3\}, \end{aligned}$$

where $Q_{M,r}=Q_{r,\sqrt{2}r,4Mr}$ was introduced below (1.24). Let Λ, c_0, η, ϱ_0, ϱ_1, be in accordance with Remark 3.7 and consider $(z_0,t_0)\in\Delta_{f,\varrho_1}$, $0<\varrho<\varrho_1$. Let $Q_{M,r}(z_0,t_0)=(z_0,t_0)\circ Q_{M,r}$ and consider the sets $\Omega_{f,2r_0}\cap Q_{M,r_0/2}(z_0,t_0)$ and $\Omega_{f,2r_0}\cap Q_{M,\varrho}(z_0,t_0)$. Then, by a change of variables,

$$\begin{aligned} \Omega_{f,2r_0}\cap Q_{M,r_0/2}(z_0,t_0)=\Omega_{\tilde f,r_0/4},\ \Omega_{f,2r_0}\cap Q_{M,\varrho}(z_0,t_0)=\Omega_{\tilde f,\varrho},\\ \Delta_{f,2r_0}\cap Q_{M,r_0/2}(z_0,t_0)=\Delta_{\tilde f,r_0/4},\ \Delta_{f,2r_0}\cap Q_{M,\varrho}(z_0,t_0)=\Delta_{\tilde f,\varrho}, \end{aligned} \tag{3.42}$$

for a new function $\tilde f$, $\tilde f(0,0,0)=0$, having the same properties as f. Keeping this in mind we will in the following, with a slight abuse of notation, simply use the following notation:

$$\begin{aligned} \Omega_{f,2r_0}(z_0,t_0) &:= \Omega_{f,2r_0}\cap Q_{M,\varrho}(z_0,t_0),\\ \Delta_{f,2r_0}(z_0,t_0) &:= \Delta_{f,2r_0}\cap Q_{M,\varrho}(z_0,t_0). \end{aligned} \tag{3.43}$$

Lemma 3.10. *Let $\Omega_{f,2r_0}$ be an admissible local* Lip_K*-domain, with* Lip_K*-constants M, r_0. Let Λ, c_0, η, ϱ_0, ϱ_1, be in accordance with Remark 3.7. Then there exist $c=c(N,M)$, $1\leq c<\infty$, and $\gamma=\gamma(N,M)$, $0<\gamma<\infty$, such that the following holds. Assume that u is a non-negative solution to $\mathcal{K}u=0$ in $\Omega_{f,2\varrho_0}$ and that $(z_0,t_0)\in\Delta_{f,\varrho_1}$. Then*

$$u(z,t)\leq c(\varrho/d)^\gamma u(A^+_{\varrho,\Lambda}(z_0,t_0)),$$

$$u(z,t) \geq c^{-1}(d/\varrho)^{\gamma} u(A^{-}_{\varrho,\Lambda}(z_0,t_0)), \tag{3.44}$$

whenever $(z,t) \in \Omega_{f,2\varrho/c}(z_0,t_0)$, $0<\varrho<\varrho_1$, *and where* $d = d_K((z,t),\Delta_{f,2r_0})$.

Proof. We just give the proof in case $(z_0,t_0) = (0,0)$ as our estimates will only depend on N and the Lip_K-constant of f, and as we may, by construction and as by discussed above, after a redefinition $f \to \tilde{f}$, also reduce the general case $(z_0,t_0) \in \Delta_{f,\varrho_1}$ to this situation $(z_0,t_0) = (0,0)$. By Lemma 3.7 we see that there exist, given $(z,t) \in \Omega_{f,\varrho}$ and $0<\varrho<\varrho_1$, points $(z_0^{\pm},t_0^{\pm}) \in \Delta_{f,c_0\varrho}$ and $\varrho^{\pm}$ such that

$$(z,t) = A^{\pm}_{\varrho^{\pm},\Lambda}(z_0^{\pm},t_0^{\pm}) \text{ and } c^{-1}d \leq \varrho^{\pm} \leq cd,$$

for some $c = c(N,M)$, $1 \leq c < \infty$. Hence, it suffices to prove the lemma with (z,t) replaced with $A^{\pm}_{\varrho^{\pm},\Lambda}(z_0^{\pm},t_0^{\pm})$ as above. In the following we let δ, $0<\delta \ll 1$, $\tilde{\delta}$, $0<\tilde{\delta} \ll 1$, $\tilde{\delta} \leq \delta$, be fixed degrees of freedom to be chosen. Based on δ, $\tilde{\delta}$ we impose the restriction that $(z,t) \in \Omega_{f,\tilde{\delta}\varrho}$ and we let $\bar{\varrho} = \delta\varrho$. Then, using Lemma 3.9 we see that

$$\begin{aligned} u(z,t) &= u(A^{+}_{\varrho^{+},\Lambda}(z_0^{+},t_0^{+})) \leq c(\bar{\varrho}/\varrho^{+})^{\gamma} u(A^{+}_{\bar{\varrho},\Lambda}(z_0^{+},t_0^{+})),\\ u(z,t) &= u(A^{-}_{\varrho^{-},\Lambda}(z_0^{-},t_0^{-})) \geq c^{-1}(\varrho^{-}/\bar{\varrho})^{\gamma} u(A^{-}_{\bar{\varrho},\Lambda}(z_0^{-},t_0^{-})). \end{aligned} \tag{3.45}$$

Keeping δ fixed we choose $\tilde{\delta} = \tilde{\delta}(N,M,\delta)$ such that, in the above construction, we have

$$(z_0^{\pm},t_0^{\pm}) \in \Delta_{f,\varrho/c}(0,0) \tag{3.46}$$

where c is the constant appearing in Lemma 3.8. Then, using Lemma 3.8 we can conclude that

$$\begin{aligned} u(A^{+}_{\bar{\varrho},\Lambda}(z_0^{+},t_0^{+})) &\leq c u(A^{+}_{\varrho,\Lambda}(0,0)),\\ u(A^{-}_{\bar{\varrho},\Lambda}(z_0^{-},t_0^{-})) &\geq c^{-1} u(A^{-}_{\varrho,\Lambda}(0,0)), \end{aligned} \tag{3.47}$$

for some constant $c = c(N,M,\delta)$, $1 \leq c < \infty$. Combining (3.45), (3.47), and the above, the lemma follows. □

Lemma 3.11. *Let* $\Omega_{f,2r_0}$ *be an admissible local* Lip_K*-domain, with* Lip_K*-constants* M, r_0. *Let* Λ, c_0, η, ϱ_0, ϱ_1, *be in accordance with Remark 3.7. Let* $\varepsilon \in (0,1)$ *be given. Then there exists* $c = c(N,M,\varepsilon)$, $1<c<\infty$, *such that following holds. Assume* $(z_0,t_0) \in \Delta_{f,\varrho_1}$, $0<\varrho<\varrho_1$, *and that* u *is a non-negative solution to* $\mathcal{K}u = 0$ *in* $\Omega_{f,2\varrho}(z_0,t_0)$, *vanishing continuously on* $\Delta_{f,2\varrho}(z_0,t_0)$. *Then*

$$\sup_{\Omega_{f,\varrho/c}(z_0,t_0)} u \leq \varepsilon \sup_{\Omega_{f,\varrho}(z_0,t_0)} u. \tag{3.48}$$

Proof. This lemma can be proved by a straightforward barrier argument. We refer to Lemma 3.1 in [CNP2] and Lemma 4.5 in [CNP3] for the details. □

Lemma 3.12. *Let* $\Omega_{f,2r_0}$ *be an admissible local* Lip_K*-domain, with* Lip_K*-constants* M, r_0. *Let* Λ, c_0, η, ϱ_0, ϱ_1, *be in accordance with Remark 3.7. Then there exists* $c = c(N,M)$, $1 \leq c < \infty$, *such that following holds. Assume that* u *is a non-negative solution to* $\mathcal{K}u = 0$ *in* $\Omega_{f,2\varrho_0}$, *vanishing continuously in* Δ_{f,r_0}, *and that* $(z_0,t_0) \in \Delta_{f,\varrho_1}$. *Then*

$$u(z,t) \leq c u(A^{+}_{\varrho,\Lambda}(z_0,t_0))$$

whenever $(z,t) \in \Omega_{f,\varrho/c}(z_0,t_0)$, $0<\varrho<\varrho_1$.

Proof. This is essentially Theorem 1.1 in [CNP3]. □

Remark 3.8. Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Based on the above lemmas, from now on we will let Λ, c_0, η, ϱ_0, ϱ_1, be in accordance with Remark 3.7 and we recall that $\varrho_1 \ll \varrho_0$. In this work we then prove estimates related to a scale ϱ satisfying $0 < \varrho < \varrho_1$.

4. Kolmogorov measure and the Green function: relations

Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Let $(z,t) \in \Omega_{f,2r_0}$ and recall the notion of the Kolmogorov measure relative to (z,t) and $\Omega_{f,2r_0}$, $\omega_K(z,t,\cdot)$, introduced in Definition 4 and Lemma 2.3. The purpose of this section is to prove the following lemma.

Lemma 4.1. *Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Let Λ, c_0, η, ϱ_0, ϱ_1, be in accordance with Remark 3.7. Let $\omega_K(z,t,\cdot)$ be the Kolmogorov measure relative to $(z,t) \in \Omega_{f,2r_0}$ and $\Omega_{f,2r_0}$ and let $G(z,t,\cdot)$ be the adjoint Green function for $\Omega_{f,2r_0}$ with pole at (z,t). Then there exists $c = c(N,M)$, $1 \leq c < \infty$, such that*

$$\begin{aligned} &(i) \qquad c^{-1}\varrho^{\mathbf{q}} G(z,t,A^+_{\varrho,\Lambda}) \leq \omega_K(z,t,\Delta_{f,\varrho}),\\ &(ii) \qquad \omega_K(z,t,\Delta_{f,\varrho/c}) \leq c\varrho^{\mathbf{q}} G(z,t,A^-_{\varrho,\Lambda}), \end{aligned}$$

whenever $(z,t) \in \Omega_{f,2\varrho_0}$, $t \geq 8\varrho^2$, $0 < \varrho < \varrho_1$.

Proof. Let in the following $(z,t) \in \Omega_{f,2\varrho_0}$. We first prove statement (i). By definition 2.18 we have

$$G(z,t,A^+_{\varrho,\Lambda}) = \Gamma(z,t,A^+_{\varrho,\Lambda}) - \int_{\partial_K \Omega_{f,2r_0}} \Gamma(\tilde z,\tilde t,A^+_{\varrho,\Lambda}) d\omega_K(z,t,\tilde z,\tilde t). \tag{4.1}$$

Obviously, we have that

$$G(z,t,A^+_{\varrho,\Lambda}) \leq \Gamma(z,t,A^+_{\varrho,\Lambda}), \tag{4.2}$$

whenever $(z,t) \in \Omega_{f,2r_0}$. Let δ, $0 < \delta \ll 1$, be a degree of freedom such that $Q_{\delta\varrho}(A^+_{\varrho,\Lambda}) \subset \Omega_{f,2r_0}$ where $Q_{\delta\varrho}(A^+_{\varrho,\Lambda})$ is defined in (2.12). Recalling that the t-coordinate of the point $A^+_{\varrho,\Lambda}$ is ρ^2 we introduce the sets

$$\begin{aligned} S_1 &= \{(z,t) \in \Omega_{f,2r_0} : \ t = \varrho^2\} \setminus Q_{\delta\varrho/2}(A^+_{\varrho,\Lambda}),\\ S_2 &= \{(z,t) \in \Omega_{f,2r_0} : \ t > \varrho^2\} \cap \partial(Q_{\delta\varrho/2}(A^+_{\varrho,\Lambda})). \end{aligned} \tag{4.3}$$

Using (2.10) and (4.2) we see that

$$G(z,t,A^+_{\varrho,\Lambda}) \leq c(N,\delta)\varrho^{-\mathbf{q}} \text{ whenever } (z,t) \in S_2. \tag{4.4}$$

Next, using a simple argument based on Lemma 3.11 we see that there exists $c = c(N,M)$, $1 \leq c < \infty$, such that

$$\omega_K(A^+_{\varrho/c,\Lambda}, \Delta_{f,\varrho}) \geq c^{-1}. \tag{4.5}$$

Indeed, let $v(z,t) = \omega_K(z,t,\Delta_{f,\varrho})$ for $(z,t) \in \Omega_{f,2r_0}$. Then $\mathcal{K}v = 0$ in $\Omega_{f,2r_0}$, $0 \leq v(z,t) \leq 1$ in $\Omega_{f,2r_0}$ and $v(z,t) = 1$ in $\Delta_{f,\varrho}$. Hence the function $u(z,t) = 1 - v(z,t)$ satisfies the assumptions of Lemma 3.11 and (4.5) follows. Next, we note that if we choose δ sufficiently small, then $S_2 \subset \mathcal{B}_K(A^+_{\varrho,\Lambda}, \varrho/c)$ where the constant c is the one appearing in (3.34) of Lemma 3.8. In particular, we can conclude that we can choose $\delta = \delta(N,M)$, $0 < \delta \ll 1$, use (4.5) and apply inequality *(i)* of (3.34) to the function $v(z,t) = \omega_K(z,t,\Delta_{f,\varrho})$, to conclude that

$$\omega_K(z,t,\Delta_{f,\varrho}) \geq \tilde c^{-1} \text{ whenever } (z,t) \in S_2, \tag{4.6}$$

for some $\tilde c=\tilde c(N,M)$, $1\le\tilde c<\infty$. Note that $G(z,t,A^+_{\varrho,\Lambda})=0$ if $(z,t)\in S_1$. Hence, from (4.4), (4.6), and from the maximum principle, it follows that

$$r^{\mathbf{q}}G(z,t,A^+_{\varrho,\Lambda})\le c\omega_K(z,t,\Delta_{f,\varrho}),\tag{4.7}$$

whenever $(z,t)\in\Omega_{f,2\varrho_0}\cap\{(z,t):\ t\ge 8\varrho^2\}$. This completes the proof of (i).

We next prove statement (ii). Let $(z,t)\in\Omega_{f,2r_0}\cap\{(z,t):\ t\ge 8\varrho^2\}$ and let δ, $0<\delta\ll 1$, be a degree of freedom to be chosen. Recall that

$$\begin{aligned}\Omega_{f,\varrho}&=\{(x_1,x',y_1,y',t)\mid (x_1,x',y',t)\in Q_{M,\varrho},\ x_1>f(x',y',t),\ |y_1|<\varrho^3\},\\ \Delta_{f,\varrho}&=\{(x_1,x',y_1,y',t)\mid (x_1,x',y',t)\in Q_{M,\varrho},\ x_1=f(x',y',t),\ |y_1|<\varrho^3\}.\end{aligned}$$

Based on this we in the following let

$$\tilde Q_\varrho=\{(x_1,x',y_1,y',t)\mid (x_1,x',y',t)\in Q_{M,\varrho},\ |y_1|<\varrho^3\}.\tag{4.8}$$

Using this notation, and given δ, we let $\theta\in C^\infty(\mathbb{R}^{N+1})$ be such that $\theta\equiv 1$ on the set $\tilde Q_{\delta\varrho/2}$ and $\theta\equiv 0$ on the complement of $\tilde Q_{3\delta\varrho/4}$. Such a function θ can be constructed so that $|\mathcal{K}\theta(z,t)|\le c(\delta\varrho)^{-2}$, whenever $(z,t)\in\mathbb{R}^{N+1}$. Using θ we immediately see that

$$\omega_K(z,t,\Delta_{f,\delta\varrho/2})\le\int_{\partial_K\Omega_{f,2r_0}}\theta(\tilde z,\tilde t)d\omega_K(z,t,\tilde z,\tilde t).\tag{4.9}$$

By the representation formula in (2.20) we have that

$$\begin{aligned}\theta(z,t)&=\int_{\partial_K\Omega_{f,2r_0}}\theta(\tilde z,\tilde t)d\omega_K(z,t,\tilde z,\tilde t)\\ &\quad+\int_{\Omega_{f,2r_0}}G(z,t,\hat z,\hat t)\mathcal{K}\theta(\hat z,\hat t)d\hat zd\hat t.\end{aligned}\tag{4.10}$$

By construction $\theta(z,t)=0$ whenever $(z,t)\in\Omega_{f,2r_0}\cap\{(z,t):\ t\ge 8\varrho^2\}$ and hence we deduce that

$$\omega_K(z,t,\Delta_{f,\delta\varrho/2})\le c\,(\delta\varrho)^{-2}\int_{\tilde Q_{\delta\varrho}}G(z,t,\tilde z,\tilde t)d\tilde zd\tilde t.\tag{4.11}$$

Next, using the adjoint version of Lemma 3.12 and (4.11) we see that we can choose $\delta=\delta(N,M)$, $0<\delta\ll 1$, so that

$$\omega_K(z,t,\Delta_{f,\delta\varrho/2})\le c\varrho^{\mathbf{q}}G(z,t,A^-_{\varrho,\Lambda}),\tag{4.12}$$

for some constant $c=c(N,M)$, $1\le c<\infty$. This completes the proof of (ii). □

Lemma 4.2. *Let $\Omega_{f,2r_0}$ be an admissible local* Lip_K*-domain, with* Lip_K*-constants M, r_0. Let Λ, c_0, η, ϱ_0, ϱ_1, be in accordance with Remark 3.7. Let $\omega_K(A^+_{\varrho_0,\Lambda},\cdot)$ be the Kolmogorov measure relative to $A^+_{\varrho_0,\Lambda}\in\Omega_{f,2r_0}$ and $\Omega_{f,2r_0}$ and let $G(A^+_{\varrho_0,\Lambda},\cdot)$ be the adjoint Green function for $\Omega_{f,2r_0}$ with pole at $A^+_{\varrho_0,\Lambda}$. Then there exists $c=c(N,M)$, $1\le c<\infty$, such that*

$$\begin{aligned}&(i)\qquad c^{-1}\varrho^{\mathbf{q}}G(A^+_{\varrho_0,\Lambda},A^+_{\varrho,\Lambda})\le\omega_K(A^+_{\varrho_0,\Lambda},\Delta_{f,\varrho}),\\ &(ii)\qquad \omega_K(A^+_{\varrho_0,\Lambda},\Delta_{f,\varrho/c})\le c\varrho^{\mathbf{q}}G(A^+_{\varrho_0,\Lambda},A^-_{\varrho,\Lambda}),\end{aligned}$$

whenever $0<\varrho<\varrho_1$.

Proof. The lemma is an immediate consequence of Lemma 4.1. □

Remark 4.1. Following the arguments used in the proof of Lemma 4.1 we can prove the

$$\omega_K(A^+_{\varrho,\Lambda},\Delta_{f,2r_0}\cap Q_{M,2\tilde\varrho}(\bar z_0,\bar t_0))\le c\tilde\varrho^{\mathbf{q}}G(A^+_{\varrho,\Lambda},A^-_{2c\tilde\varrho,\Lambda}(\bar z_0,\bar t_0))\tag{4.13}$$

provided $(\bar z_0,\bar t_0)\in\Delta_{f,2r_0}$ and $Q_{M,\tilde\varrho}(\bar z_0,\bar t_0)\subset Q_{M,\varrho/c_3}$. This inequality will be useful in the sequel.

Remark 4.2. Adjoint versions of Lemma 4.1 and Lemma 4.2 also hold. Indeed, an adjoint version of Lemma 4.2 can be stated as follows. Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Let Λ, c_0, η, ϱ_0, ϱ_1, be in accordance with Remark 3.7. Let $\omega_K^*(A^-_{\varrho_0,\Lambda},\cdot)$ be the adjoint Kolmogorov measure relative to $A^-_{\varrho_0,\Lambda}\in\Omega_{f,2r_0}$ and $\Omega_{f,2r_0}$ and let $G(\cdot,A^-_{\varrho_0,\Lambda})$ be the Green function for $\Omega_{f,2r_0}$ with pole at $A^-_{\varrho_0,\Lambda}$. Then there exists $c=c(N,M)$, $1\le c<\infty$, such that

$$
\begin{aligned}
&(i) \qquad c^{-1}\varrho^{\mathbf{q}}G(A^-_{\varrho,\Lambda},A^-_{\varrho_0,\Lambda})\le\omega_K^*(A^-_{\varrho_0,\Lambda},\Delta_{f,\varrho}),\\
&(ii) \qquad \omega_K^*(A^-_{\varrho_0,\Lambda},\Delta_{f,\varrho/c})\le c\varrho^{\mathbf{q}}G(A^+_{\varrho,\Lambda},A^-_{\varrho_0,\Lambda}),
\end{aligned}
$$

whenever $0<\varrho<\varrho_1$.

5. A weak comparison principle and its consequences

The main purpose of this section is to prove Lemma 5.1 and Lemma 5.3 stated below.

Lemma 5.1. *Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Let Λ, c_0, η, ϱ_0, ϱ_1, be in accordance with Remark 3.7. Then there exists $c=c(N,M)$, $1\le c<\infty$, such that the following is true. Assume that u, v, are non-negative solutions to $\mathcal{K}u=0$ in $\Omega_{f,2r_0}$ and that u and v vanish continuously on $\Delta_{f,2r_0}$. Then*

$$
c^{-1}\frac{v(A^-_{\varrho,\Lambda})}{u(A^+_{\varrho,\Lambda})}\le\frac{v(z,t)}{u(z,t)}\le c\frac{v(A^+_{\varrho,\Lambda})}{u(A^-_{\varrho,\Lambda})}, \tag{5.1}
$$

whenever $(z,t)\in\Omega_{f,\varrho/c}$ and $0<\varrho\le\varrho_1$.

Proof. Let in the following $\varepsilon=\varepsilon(N,M)$, $0<\varepsilon\ll1$, be a degree of freedom to be chosen. Consider the set $\Delta_{f,6\varepsilon\varrho}\setminus\Delta_{f,4\varepsilon\varrho}$. We claim that there exist $\delta=\delta(N,M)$, $0<\delta\ll1$, and a set of points $\{(z_i,t_i)\}_{i=1}^L$ such that $(z_i,t_i)\in\Delta_{f,6\varepsilon\varrho}\setminus\Delta_{f,4\varepsilon\varrho}$,

$$
\{\Delta_{f,\delta\varepsilon\varrho}(z_i,t_i)\}_{i=1}^L \text{ is a covering of } \Delta_{f,6\varepsilon\varrho}\setminus\Delta_{f,4\varepsilon\varrho}, \tag{5.2}
$$

and such that

$$
\Delta_{f,\delta\varepsilon\varrho/\mathbf{k}}(z_i,t_i)\cap\Delta_{f,\delta\varepsilon\varrho/\mathbf{k}}(z_j,t_j)=\emptyset \text{ whenever } i\ne j, \tag{5.3}
$$

for some $\mathbf{k}$ only depending on the diameter of the cylinder $Q_{M,1}$ and on the constant $\mathbf{c}$ appearing in the triangular inequality (2.4). Furthermore, the construction can be made so that

$$
\sum_{i=1}^L\omega_K\big(z,t,\Delta_{f,\delta\varepsilon\varrho}(z_i,t_i)\big)\ge c^{-1}, \tag{5.4}
$$

for some $c=c(N,M,\delta(N,M))=c(N,M)$, $1\le c<\infty$, whenever

$$
(z,t)\in\partial_K\Omega_{f,5\varepsilon\varrho}\cap\{(z,t)\in\Omega_{f,2r_0}\mid d_K(z,t,\Delta_{f,2r_0})\le\delta^2\varepsilon\varrho\}. \tag{5.5}
$$

The claim is a direct consequence of a Vitali covering argument and the method used in the proof of (4.5). Using the claim we introduce the auxiliary function

$$
\Psi(z,t) \;=\; \sum_{i=1}^L\omega_K\big(z,t,\Delta_{f,\delta\varepsilon\varrho}(z_i,t_i)\big)+(\varepsilon\varrho)^{\mathbf{q}}G\big(z,t,A^-_{k\varepsilon\varrho}\big), \tag{5.6}
$$

where $k\gg1$ is a large degree of freedom to be chosen below, and we let

$$
\begin{aligned}
&\Gamma_1:=\partial_K\Omega_{f,5\varepsilon\varrho}\cap\{(z,t)\in\Omega_{f,2r_0}\mid d_K(z,t,\Delta_{f,2r_0})\le\delta^2\varepsilon\varrho\},\\
&\Gamma_2:=\partial_K\Omega_{f,5\varepsilon\varrho}\cap\{(z,t)\in\Omega_{f,2r_0}\mid d_K(z,t,\Delta_{f,2r_0})>\delta^2\varepsilon\varrho\}. \qquad (5.7)
\end{aligned}
$$

Using Lemma 3.12 we see that there exist $k = k(N, M)$ and $c = c(N, M)$, $1 \leq c < \infty$, such that

$$v(z,t) \leq cv(A^+_{k\varepsilon\varrho,\Lambda}), \tag{5.8}$$

whenever $(z,t) \in \Omega_{f,6\varepsilon\varrho}$. By construction, see (5.4),

$$\Psi(z,t) \geq c^{-1} \text{ whenever } (z,t) \in \Gamma_1, \tag{5.9}$$

and for some $c = c(N, M)$, $1 \leq c < \infty$. Considering $(z,t) \in \Gamma_2$ we see, using Lemma 3.10, that there exist $k = k(N, M)$ and $c = c(N, M)$, $1 \leq c < \infty$, such that

$$(\varepsilon\varrho)^{\mathbf{q}} G\big(z,t, A^-_{k\varepsilon\varrho,\Lambda}\big) \geq c^{-1} (\varepsilon\varrho)^{\mathbf{q}} G\big(A^-_{k\varepsilon\varrho/\mathbf{k},\Lambda}, A^-_{k\varepsilon\varrho,\Lambda}\big), \tag{5.10}$$

whenever $(z,t) \in \Gamma_2$. Furthermore, we claim that, if k is big enough, then

$$(\varepsilon\varrho)^{\mathbf{q}} G\big(A^-_{k\varepsilon\varrho/\mathbf{k},\Lambda}, A^-_{k\varepsilon\varrho,\Lambda}\big) \geq c^{-1}, \tag{5.11}$$

by elementary estimates and the Harnack inequality. To give a more detailed proof of this claim, recall the notation introduced in (3.27) and (4.8). Let $\widetilde{\Omega} = A^-_{k\varepsilon\varrho,\Lambda} \circ \widetilde{Q}_{4\varepsilon\varrho}$ and let $\widetilde{G}$ denote the Green function for the set $\widetilde{\Omega}$. Using the dilation invariance of the fundamental solution Γ, and of the cone $C^-_{\rho,\eta,\Lambda}(0,0)$, we see that we can use (2.18) to prove that

$$(\varepsilon\varrho)^{\mathbf{q}} \tilde{G}\big(A^-_{(k-\eta)\varepsilon\varrho,\Lambda}, A^-_{k\varepsilon\varrho,\Lambda}\big) \geq c^{-1}, \tag{5.12}$$

for some $\eta = \eta(N, M)$, $0 < \eta \ll 1$. Using this, we see that

$$(\varepsilon\varrho)^{\mathbf{q}} G\big(A^-_{(k-\eta)\varepsilon\varrho,\Lambda}, A^-_{k\varepsilon\varrho,\Lambda}\big) \geq c^{-1}, \tag{5.13}$$

by the comparison principle. (5.11) now follows from (5.13) and as, by the Harnack inequality,

$$G\big(A^-_{k\varepsilon\varrho/\mathbf{k},\Lambda}, A^-_{k\varepsilon\varrho,\Lambda}\big) \geq c^{-1} G\big(A^-_{(k-\eta)\varepsilon\varrho,\Lambda}, A^-_{k\varepsilon\varrho,\Lambda}\big) \geq c^{-1}. \tag{5.14}$$

To proceed with the proof of Lemma 5.1 we next note, combining (5.8)-(5.11), and using the maximum principle, we can conclude that there exist $k = k(N, M)$ and $c = c(N, M)$, $1 \leq c < \infty$, such that

$$v(z,t) \leq cv(A^+_{k\varepsilon\varrho,\Lambda})\Psi(z,t), \tag{5.15}$$

whenever $(z,t) \in \Omega_{f,5\varepsilon\varrho}$. To continue, having estimated v from above we next want to estimate u from below. To start the estimate we introduce the sets

$$\begin{aligned} \widetilde{S}_1 &= \{(z,t) \in \Omega_{f,2r_0} :\ t = -(k\varepsilon\varrho)^2\} \setminus Q_{\delta\varrho/2}(A^-_{k\varepsilon\varrho,\Lambda}), \\ \widetilde{S}_2 &= \{(z,t) \in \Omega_{f,2r_0} :\ t > -(k\varepsilon\varrho)^2\} \cap \partial(Q_{\delta\varrho/2}(A^-_{k\varepsilon\varrho,\Lambda})). \end{aligned} \tag{5.16}$$

and, by arguing as in Lemma 4.1, we see that

$$(\varepsilon\varrho)^{\mathbf{q}} G\big(z,t, A^-_{k\varepsilon\varrho,\Lambda}\big) \leq c, \tag{5.17}$$

holds whenever $(z,t) \in \Omega_{f,5\varepsilon\varrho}$. Then, by using the continuity of u, choosing δ sufficiently small and also using the maximum principle, we find that there exist $k = k(N, M)$ and $c = c(N, M)$, $1 \leq c < \infty$, such that

$$u(z,t) \geq c^{-1} (\varepsilon\varrho)^{\mathbf{q}} G\big(z,t, A^-_{k\varepsilon\varrho,\Lambda}\big) u(A^-_{k\varepsilon\varrho,\Lambda}), \tag{5.18}$$

whenever $(z,t) \in \Omega_{f,5\varepsilon\varrho}$. We now claim that there exists $c = c(N, M)$, $1 \leq c < \infty$ such that

$$c(\varepsilon\varrho)^{\mathbf{q}} G\big(z,t, A^-_{k\varepsilon\varrho,\Lambda}\big) \geq \Psi(z,t), \tag{5.19}$$

whenever $(z,t) \in \partial_K \Omega_{f,\varepsilon\varrho}$. Assuming (5.19) it follows from (5.18), (5.19), and the maximum principle, that that exist $k = k(N,M)$ and $c = c(N,M)$, $1 \leq c < \infty$, such that

$$u(z,t) \geq c^{-1} u(A^-_{k\varepsilon\varrho,\Lambda}) \Psi(z,t), \tag{5.20}$$

whenever $(z,t) \in \Omega_{f,\varepsilon\varrho}$ and hence the proof of the lemma is complete once we define ε through the relation $k\varepsilon = 1$. Finally, to prove (5.19) it follows, by construction, that we only have to prove that

$$\omega_K\big(z,t,\Delta_{f,\delta\varepsilon\varrho}(z_i,t_i)\big) \leq (\varepsilon\varrho)^{\mathbf{q}} G\big(z,t,A^-_{K\varepsilon\varrho,\Lambda}\big), \tag{5.21}$$

whenever $(z,t) \in \partial_K \Omega_{f,\varepsilon\varrho}$ and $i = 1,\dots,L$. However, arguing as in the proof of statement *(ii)* in Lemma 4.1 we see that (5.21) holds. This completes the proof of Lemma 5.1. □

Lemma 5.2. *Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Let Λ, c_0, η, ϱ_0, ϱ_1, be in accordance with Remark 3.7. Then there exists $c = c(N,M)$, $1 \leq c < \infty$, such that following holds. Let $(z_0,t_0) \in \Delta_{f,\varrho_1}$, consider $0 < \varrho \leq \varrho_1$, assume that u, v are non-negative solutions to $\mathcal{K}u = 0$ in $\Omega_{f,2\varrho}(z_0,t_0)$ and that u and v vanish continuously on $\Delta_{f,2\varrho}(z_0,t_0)$. Then*

$$c^{-1}\frac{v(A^-_{\varrho,\Lambda})}{u(A^+_{\varrho,\Lambda})} \leq \frac{v(z,t)}{u(z,t)} \leq c\frac{v(A^+_{\varrho,\Lambda})}{u(A^-_{\varrho,\Lambda})}, \tag{5.22}$$

whenever $(z,t) \in \Omega_{f,\varrho/c}(z_0,t_0)$.

Proof. Note that Lemma 5.2 is a localized version of Lemma 5.1. In fact, analyzing the proof of Lemma 5.1, using appropriate localized versions of Lemma 3.8, Lemma 3.9, Lemma 3.10 and Lemma 3.12, localized in the sense that u does not have to be a solution in all of $\Omega_{f,2r_0}$ or $\Omega_{f,2\varrho_0}$, we see that the conclusion of Lemma 5.2 is true. We omit further details. □

5.1. Implications of the weak comparison principle.

Lemma 5.3. *Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Let Λ, c_0, η, ϱ_0, ϱ_1, be in accordance with Remark 3.7. Then there exists $c = c(N,M)$, $1 \leq c < \infty$, such that the following is true. Assume that u is a non-negative solution to $\mathcal{K}u = 0$ in $\Omega_{f,2r_0}$ and that u vanishes continuously on $\Delta_{f,2r_0}$. Then*

$$c^{-1}\frac{u(A^-_{\varrho_0,\Lambda})}{u(A^+_{\varrho_0,\Lambda})} \leq \frac{u(x_1,x',0,y',t)}{u(x_1,x',y_1,y',t)} \leq c\frac{u(A^+_{\varrho_0,\Lambda})}{u(A^-_{\varrho_0,\Lambda})}, \tag{5.23}$$

whenever $(x_1,x',y_1,y',t) \in \Omega_{f,\varrho_1/c}$.

Proof. Consider $u = u(x,y,t) = u(x_1,x',y_1,y',t)$ as in the statement of the lemma and let $v = v(x,y,t) = v(x_1,x',y_1,y',t) = u(x_1,x',y_1 \pm \delta,y',t)$ for some $\delta > 0$ small. Let $\tilde r_0 = (r_0 - \delta)/4$. Then $\mathcal{K}v = 0$ in $\Omega_{f,2\tilde r_0}$ and v vanishes continuously on $\Delta_{f,2\tilde r_0}$ since we are assuming that the function defining $\Delta_{f,2r_0}$ is independent of the y_1-coordinate. We can now apply Lemma 5.1 to the functions v and u, with $r_0, \varrho_0, \varrho_1$ replaced by $\tilde r_0, \tilde\varrho_0, \tilde\varrho_1$, and conclude that

$$c_{-1}\frac{v(A^-_{\tilde\varrho_0,\Lambda})}{u(A^+_{\tilde\varrho_0,\Lambda})} \leq \frac{v(x,y,t)}{u(x,y,t)} \leq c\frac{v(A^+_{\tilde\varrho_0,\Lambda})}{u(A^-_{\tilde\varrho_0,\Lambda})}, \tag{5.24}$$

whenever $(x,y,t) \in \Omega_{f,\varrho/c}$ and $0 < \tilde\varrho_0 \leq \tilde\varrho_1$. We now fix $\tilde\varrho_0, \tilde\varrho_1$ as above, and we claim that there exists $c = c(N,M)$, $1 \leq c < \infty$, such that

$$u(A^+_{\tilde\varrho_1,\Lambda}) \leq c u(A^+_{\tilde\varrho_0,\Lambda}), \qquad u(A^-_{\tilde\varrho_1,\Lambda}) \geq c^{-1} u(A^-_{\tilde\varrho_0,\Lambda}). \tag{5.25}$$

and

$$v(A^+_{\tilde\varrho_1,\Lambda}) \le \bar c u(A^+_{\tilde\varrho_0,\Lambda}), \qquad v(A^-_{\tilde\varrho_1,\Lambda}) \ge \bar c^{-1} u(A^-_{\tilde\varrho_0,\Lambda}), \tag{5.26}$$

whenever $(x_1, x', y_1, y', t) \in \Omega_{f,\tilde\varrho_1/\bar c}$. To prove this we first make the trivial observations that, for any degree of freedom $\varepsilon = \varepsilon(N, M)$, $0 < \varepsilon \ll 1$,

$$\begin{aligned} A^+_{\tilde\varrho_1,\Lambda} + (0,0,0,\pm\delta,0) &\in \mathcal{B}_K(A^+_{\tilde\varrho_1,\Lambda}, \varepsilon\tilde\varrho_1),\\ A^-_{\tilde\varrho_1,\Lambda} + (0,0,0,\pm\delta,0) &\in \mathcal{B}_K(A^-_{\tilde\varrho_1,\Lambda}, \varepsilon\tilde\varrho_1), \end{aligned} \tag{5.27}$$

provided $\delta \le (\varepsilon\tilde\varrho_1)^3$. Hence,

$$v(A^+_{\tilde\varrho_1,\Lambda}) \le \sup_{\mathcal{B}_K(A^+_{\tilde\varrho_1,\Lambda},\varepsilon\tilde\varrho_1)} u, \ v(A^-_{\tilde\varrho_1,\Lambda}) \ge \inf_{\mathcal{B}_K(A^+_{\tilde\varrho_1,\Lambda},\varepsilon\tilde\varrho_1)} u. \tag{5.28}$$

Next, based on the quotient $\tilde\varrho_1/\tilde\varrho_0 = 1/c_0$ we choose $\varepsilon = \varepsilon(1/c_0, N, M)$ so that we can apply Lemma 3.8. In particular, based on (5.28) the inequalities in (5.25) and (5.26) now follow from Lemma 3.8. This completes the proof of the lemma. □

Lemma 5.4. *Let $\Omega_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Let Λ, c_0, η, ϱ_0, ϱ_1, be in accordance with Remark 3.7. Assume that u is a non-negative solution to $\mathcal{K}u = 0$ in $\Omega_{f,2r_0}$ and that u vanishes continuously on $\Delta_{f,2r_0}$. Let*

$$m^+ = u(A^+_{\varrho_0,\Lambda}), \qquad m^- = u(A^-_{\varrho_0,\Lambda}),$$

and assume that $m^- > 0$. Then there exist constants $c_1 = c_1(N, M)$, $1 \le c_1 < \infty$, $c_2 = c_2(N, M, m^+/m^-)$, $1 \le c_2 < \infty$, such that

$$u(A^-_{\varrho,\Lambda}(z_0,t_0)) \le c_2 u(A_{\varrho,\Lambda}(z_0,t_0)) \le c_2^2 u(A^+_{\varrho,\Lambda}(z_0,t_0)),$$

whenever $0 < \varrho < \varrho_1/c_1$ and $(z_0, t_0) \in \Delta_{f,\varrho_1}$.

Proof. Assuming that $m^- > 0$ we see that Lemma 3.10 implies that $m^+ > 0$. By Lemma 5.3 we have

$$c^{-1}\frac{m^-}{m^+} \le \frac{u(x_1, x', 0, y', t)}{u(x_1, x', y_1, y', t)} \le c\frac{m^+}{m^-}, \tag{5.29}$$

whenever $(x_1, x', y_1, y', t) \in \Omega_{f,\varrho_1/c}$. Let $(z_0, t_0) \in \Delta_{f,\varrho_1}$, and recall that

$$\begin{aligned} A_{\varrho,\Lambda}(z_0,t_0) &= (z_0,t_0) \circ (\Lambda\varrho, 0, 0, 0, 0),\\ A^-_{\varrho,\Lambda}(z_0,t_0) &= (z_0,t_0) \circ (\Lambda\varrho, 0, \tfrac{2}{3}\Lambda\varrho^3, 0, -\varrho^2). \end{aligned}$$

We now consider the path

$$\gamma(\tau) = (z_0,t_0) \circ (\Lambda\varrho, 0, \tau\Lambda\varrho, 0, -\tau), \qquad \tau \in [0, \varrho^2],$$

which is a $\mathcal{K}$-admissible such that

$$\gamma(0) = A_{\varrho,\Lambda}(z_0,t_0), \ \gamma(\varrho^2) = (z_0,t_0) \circ (\Lambda\varrho, 0, \Lambda\varrho^3, 0, -\varrho^2).$$

By construction, the definition of the points $A^-_{\varrho,\Lambda}(z_0,t_0)$, $A_{\varrho,\Lambda}(z_0,t_0)$, and the fact that the function defining $\Delta_{f,2r_0}$ is independent of the y_1-coordinate, the path γ is contained in $\Omega_{f,2r_0}$. Thus we can construct a Harnack chain connecting $A_{\varrho,\Lambda}(z_0,t_0)$ and $\gamma(\varrho^2)$, based on which we can conclude that

$$u(\gamma(\varrho^2)) \le c u(A_{\varrho,\Lambda}(z_0,t_0)), \tag{5.30}$$

for some $c = c(N, M)$, $1 \le c < \infty$. Note that the coordinates $A^-_{\varrho,\Lambda}(z_0,t_0)$ and $\gamma(\varrho^2)$ only differ in the y_1-coordinate. In particular, using (5.29) we have

$$c^{-2}\left(\frac{m^-}{m^+}\right) \le \frac{u(\gamma(\varrho^2))}{u(A^-_{\varrho,\Lambda}(z_0,t_0))} \le c^2\left(\frac{m^+}{m^-}\right), \tag{5.31}$$

whenever $0 < \varrho < \varrho_1/c$. Combining (5.30) and (5.31) we see that

$$u(A^-_{\varrho,\Lambda}) \le c^2 \left(\frac{m^+}{m^-}\right) u(\gamma(\varrho^2)) \le c^3 \left(\frac{m^+}{m^-}\right) u(A_{\varrho,\Lambda}),$$

whenever $0 < \varrho < \varrho_1/c$. The other inequality is proved analogously. □

6. Proof of the main results

In this section we prove Theorem 1.1, Theorem 1.2 and Theorem 1.3. The proofs rely heavily on Lemma 5.3. We prove the theorems based on the set up concluded in Remark 3.8. Using a by now familiar argument it suffices to prove Theorem 1.1, Theorem 1.2 and Theorem 1.3 in the case $(z_0, t_0) = (0, 0)$ only. Thus, throughout this section we will assume $(z_0, t_0) = (0, 0)$. Furthermore, we again note that Lemma 3.10 implies, assuming $m^- > 0$ in Theorem 1.1 and $m_1^- > 0$, $m_2^- > 0$ in Theorem 1.2, that $m^+ > 0$ and $m_1^+ > 0$, $m_2^+ > 0$.

6.1. **Proof of Theorem 1.1.** Assume that u is a non-negative solution to $\mathcal{K}u = 0$ in $\Omega_{f,2r_0}$ and that u vanishes continuously on $\Delta_{f,2r_0}$. In the sequel, the constants $\Lambda, c_0, \eta, \varrho_0, \varrho_1$ will be chosen in accordance with Remark 3.8. Hence, to prove Theorem 1.1 we have to show that there exist constants $c_1 = c_1(N, M)$, $1 \le c_1 < \infty$, $c_2 = c_2(N, M, m^+/m^-)$, $1 \le c_2 < \infty$, such that

$$u(z, t) \le c_2 u(A_{\varrho,\Lambda}),$$

whenever $(z, t) \in \Omega_{f,\varrho/c_1}$ and $0 < \varrho < \varrho_1$. Based on this we from now on consider ϱ_0 and ϱ, $0 < \varrho < \varrho_1$, as fixed. To start the proof we introduce

$$h(\hat{\varrho}) = \hat{\varrho}^{-\gamma} u(A^+_{\hat{\varrho},\Lambda}), \qquad 0 < \hat{\varrho} \le \varrho_0, \tag{6.1}$$

where γ is the constant appearing in Lemma 3.9. Furthermore, we let

$$\tilde{\varrho} = \max\{\hat{\varrho} : \ \varrho \le \hat{\varrho} \le \varrho_0,\ h(\hat{\varrho}) \ge h(\varrho)\}. \tag{6.2}$$

By the definition of $\tilde{\varrho}$ in (6.2) we see that

$$u(A^+_{\varrho,\Lambda}) \le (\varrho/\tilde{\varrho})^\gamma u(A^+_{\tilde{\varrho},\Lambda}). \tag{6.3}$$

Furthermore, using Lemma 3.9 we see that

$$u(A^-_{\tilde{\varrho},\Lambda}) \le c(\tilde{\varrho}/\varrho)^\gamma u(A^-_{\varrho,\Lambda}). \tag{6.4}$$

In the following we prove that there exists a constant $\bar{c} = \bar{c}(N, M, m^+/m^-)$, $1 \le \bar{c} < \infty$, such that

$$u(A^+_{\tilde{\varrho},\Lambda}) \le \bar{c}\, u(A^-_{\tilde{\varrho},\Lambda}), \tag{6.5}$$

for this particular choice of $\tilde{\varrho}$. In fact, assuming (6.5) we first see, combining Lemma 3.12, (6.3), (6.4) and (6.5), that

$$\begin{aligned} \sup_{\Omega_{f,\varrho/c}} u(x,t) &\le cu(A^+_{\varrho,\Lambda}) \le c(\varrho/\tilde{\varrho})^\gamma u(A^+_{\tilde{\varrho},\Lambda}) \\ &\le c\bar{c}(\varrho/\tilde{\varrho})^\gamma u(A^-_{\tilde{\varrho},\Lambda}) \le c^2\bar{c}u(A^-_{\varrho,\Lambda}), \end{aligned} \tag{6.6}$$

where c, $1 \le c < \infty$, depends only on N, M. An application of Lemma 5.4 then completes the proof of Theorem 1.1.

To prove (6.5) we let $K \gg 1$ be an other degree of freedom based on which we divide the proof into two cases.

The case $\varrho_0/(8K) < \tilde{\varrho}$. In this case we immediately obtain from Lemma 3.9 that

$$\frac{u(A^+_{\tilde{\varrho},\Lambda})}{u(A^-_{\tilde{\varrho},\Lambda})} \le c^2(\varrho_0/\tilde{\varrho})^{2\gamma} \frac{u(A^+_{\varrho_0,\Lambda})}{u(A^-_{\varrho_0,\Lambda})} = c^2(\varrho_0/\tilde{\varrho})^{2\gamma} \frac{m^+}{m^-},$$

and the conclusion follows immediately.

The case $\tilde\varrho \le \varrho_0/(8K)$. In this case we first note, by the definition of $\tilde\varrho$, that $\varrho < \tilde\varrho < \varrho_0$ and that $h(2K\tilde\varrho) < h(\tilde\varrho)$, i.e.,

$$u(A^+_{\tilde\varrho,\Lambda}) > (2K)^{-\gamma} u(A^+_{2K\tilde\varrho,\Lambda}).$$

Using Lemma 3.12 we see that the above inequality implies that

$$u(A^+_{\tilde\varrho,\Lambda}) \ge c^{-1}(2K)^{-\gamma} \sup_{\Omega_{f,2K\tilde\varrho/c}} u, \tag{6.7}$$

for some $c = c(N, M)$, $1 \le c < \infty$. In the following we can, without loss of generality, assume that $\tilde\varrho = 1$. Based on this we let $\tilde K = K/c$ and we introduce

$$TC_{f,2\tilde K} := \Omega_{f,2\tilde K} \cap \{(z,t) \in \mathbb{R}^{N+1} : -4 < t < 1\}, \tag{6.8}$$

where TC stands for Thin Cylinder. Using this notation, (6.7) implies that

$$u(A^+_{1,\Lambda}) \ge \tilde c^{-1}(2\tilde K)^{-\gamma} \sup_{TC_{f,2\tilde K}} u, \tag{6.9}$$

again for some $\tilde c = \tilde c(N, M)$, $1 \le \tilde c < \infty$. We emphasize that $\tilde K$ is a degree of freedom which remains to be chosen. Furthermore, we can, by a redefinition of u, and without loss of generality, assume that

$$\sup_{TC_{f,2\tilde K}} u = 1. \tag{6.10}$$

Hence (6.9) becomes

$$u(A^+_{1,\Lambda}) \ge \tilde c^{-1}(2\tilde K)^{-\gamma}. \tag{6.11}$$

We now let

$$\begin{aligned} \Gamma_{\tilde K,B} &:= \partial_K(TC_{f,2\tilde K}) \cap \{(z,t) \in \mathbb{R}^{N+1} : t = -4\}, \\ \Gamma_{\tilde K,IL} &:= (\partial_K(TC_{f,2\tilde K}) \setminus \Gamma_{\tilde K,B}) \setminus \Delta_{f,2\tilde K}. \end{aligned} \tag{6.12}$$

Then $\Gamma_{\tilde K,B}$ represents the Bottom (in time) of the domain $TC_{f,2\tilde K}$ and $\Gamma_{\tilde K,IL}$ is the lateral part of $\partial_K(TC_{f,2\tilde K})$ which is contained in $\Omega_{f,2r_0}$: the Interior Lateral part of the boundary of $TC_{f,2\tilde K}$. Since $u = 0$ on $\Delta_{f,2\tilde K}$ we note that $u(A^+_{1,\Lambda})$ is determined by the values of u on $\Gamma_{\tilde K,B}$ and $\Gamma_{\tilde K,IL}$. Specifically, if we let $\omega_K(A^+_{1,\Lambda}, \cdot)$ be the Kolmogorov measure relative to $A^+_{1,\Lambda}$ and $TC_{f,2\tilde K}$, then

$$\begin{aligned} u(A^+_{1,\Lambda}) &= \int_{\Gamma_{\tilde K,B}} u(z,t) d\omega_K(A^+_{1,\Lambda}, z, t) + \int_{\Gamma_{\tilde K,IL}} u(z,t) d\omega_K(A^+_{1,\Lambda}, z, t) \\ &=: I_1 + I_2. \end{aligned} \tag{6.13}$$

We now use the following lemma, the proof of which we postpone to the next subsection.

Lemma 6.1. *Let* $\tilde c$ *and* γ *be as in* (6.11). *Then there exists* $\tilde K = \tilde K(N, M)$, $\tilde K \gg 1$, *such that*

$$I_2 \le \frac{1}{2}\tilde c^{-1}(2\tilde K)^{-\gamma}.$$

Using Lemma 6.1 and (6.11) we see that

$$u(A^+_{1,\Lambda}) \le I_1 + \frac{1}{2}\tilde c^{-1}(2\tilde K)^{-\gamma} \le I_1 + \frac{1}{2}u(A^+_{1,\Lambda}). \tag{6.14}$$

We can therefore conclude that

$$u(A^+_{1,\Lambda}) \le 2 \sup_{(z,t)\in\Gamma_{\tilde K,B}} u(z,t). \tag{6.15}$$

In particular, using Lemma 3.12 we see that there exists ε, $0<\varepsilon\ll 1$, depending on N and M, such that

$$\sup_{(z,t)\in\Gamma_{\tilde K,B}\cap\Omega_{f,\varepsilon}(z_1,t_1)} u(z,t)\leq cu(A^+_{c\varepsilon,\Lambda}(z_1,t_1))$$

for every $(z_1,t_1)\in\Gamma_{\tilde K,B}\cap\Delta_{f,2r_0}$. In the above inequality c is the constant appearing in Lemma 3.12. Then, using also Lemma 3.5, we can conclude that

$$u(A^+_{1,\Lambda})\leq 2c\, u(\tilde z,\tilde t), \tag{6.16}$$

for some $(\tilde z,\tilde t)\in\tilde\Gamma^\varepsilon_{\tilde K,B}$, where

$$\begin{aligned}\tilde\Gamma^\varepsilon_{\tilde K,B} &= TC_{f,2\tilde K}\cap\big\{(z,t)\in\mathbb{R}^{N+1}:-4\leq t\leq -4+(c\varepsilon)^2\big\}\\ &\quad\cap\big\{(z,t)\in\mathbb{R}^{N+1}:d_K((z,t),\Delta_{f,2r_0})\geq\varepsilon/c\big\}.\end{aligned}\tag{6.17}$$

To complete the proof we now use the following lemma, the proof of which we also postpone to the next subsection.

Lemma 6.2. *Let $(\tilde z,\tilde t)$ be any point of $\tilde\Gamma^\varepsilon_{\tilde K,B}$. Then there exists a constant $\bar c$, depending at most on N, M, ε, and m^+/m^-, such that*

$$u(\tilde z,\tilde t)\leq\bar c u(A^-_{1,\Lambda}).$$

Using Lemma 6.2 and (6.16) we can conclude that (6.5) also holds in this case. This completes the proof of Theorem 1.1 modulo the proofs of Lemma 6.1 and Lemma 6.2 given below. □

6.2. **Proof of Lemma 6.1 and Lemma 6.2.** We here prove Lemma 6.1 and Lemma 6.2. We we note that Lemma 6.2, together with Lemma 5.3, represent the the main (novel) technical components of the paper.

Proof of Lemma 6.1. Using the normalization in (6.10) we see that

$$I_2\leq\omega_K(A^+_{1,\Lambda},\Gamma_{\tilde K,IL}). \tag{6.18}$$

Recall the sets $\tilde Q_\cdot$ introduced in (4.8), and let λ, $1\leq\lambda\ll\tilde K$, be an additional degree of freedom. Let θ be a smooth function defined on $\{(z,t)\in\mathbb{R}^{N+1}\mid t=-4\}$, satisfying $0\leq\theta(z,t)\leq 1$, and

$$\begin{aligned}\theta(z,t)&=1 &&\text{on } (\tilde Q_{\tilde K+\lambda}\setminus\tilde Q_{\tilde K-\lambda})\cap\{(z,t)\in\mathbb{R}^{N+1}\mid t=-4\},\\ \theta(z,t)&=0 &&\text{on } \tilde Q_{\tilde K-\lambda-1}\cap\{(z,t)\in\mathbb{R}^{N+1}\mid t=-4\},\\ \theta(z,t)&=0 &&\text{on } (\mathbb{R}^{N+1}\setminus\tilde Q_{\tilde K+\lambda+1})\cap\{(z,t)\in\mathbb{R}^{N+1}\mid t=-4\}.\end{aligned}\tag{6.19}$$

Then θ is a (smooth) approximation of the characteristic function for the set $(\tilde Q_{\tilde K+\lambda}\setminus\tilde Q_{\tilde K-\lambda})\cap\{(z,t)\in\mathbb{R}^{N+1}\mid t=-4\}$. Given θ we let w satisfy $\mathcal{K}w=0$ in $\{(z,t)\in\mathbb{R}^{N+1}\mid t>-4\}$ with Cauchy data on $\{(z,t)\in\mathbb{R}^{N+1}\mid t=-4\}$ defined by the function θ. Given $\tilde K\gg 1$ we claim that there exist $\lambda\geq 1$ and a constant c, both just depending on N, and hence independent of $\tilde K$, such that

$$w(z,t)\geq c^{-1}\text{ whenever }(z,t)\in\Gamma_{\tilde K,IL}. \tag{6.20}$$

Indeed,

$$w(z,t)=\int_{\mathbb{R}^N}\Gamma(z,t,\tilde z,-4)\theta(\tilde z)d\tilde z, \tag{6.21}$$

where the fundamental solution associated to $\mathcal{K}$, Γ, is defined in (2.8). Using (6.21) we see that the bound from below in (6.20) follows from elementary estimates. Next, using (6.18), (6.20), and the maximum principle, we see that

$$I_2\leq cw(A^+_{1,\Lambda}). \tag{6.22}$$

Note that

$$w(A^+_{1,\Lambda}) = \int_{\mathbb{R}^N} \Gamma(A^+_{1,\Lambda}, \tilde z, -4)\theta(\tilde z)d\tilde z, \tag{6.23}$$

and that, by (2.8) and (2.9), we have

$$\begin{aligned}\Gamma(A^+_{1,\Lambda}, \tilde z, -4) &= \Gamma(A^+_{1,\Lambda}, \tilde x, \tilde y, -4)\\ &\leq c\exp(-(|\tilde x|^2 + |\tilde y|^2)) \leq c^2 \exp(-c\tilde K^2),\end{aligned} \tag{6.24}$$

whenever $\theta(\tilde z) \neq 0$ and for some harmless constant c, $1 \leq c < \infty$. In particular, combining the above we see that

$$I_2 \leq c\exp(-c\tilde K^2)\tilde K^{\mathbf{q}+2} \tag{6.25}$$

and hence Lemma 6.1 follows for $\tilde K$ large enough. □

Proof of Lemma 6.2. Consider an arbitrary point $(\tilde z, \tilde t) \in \tilde\Gamma^\varepsilon_{\tilde K,B}$ where $\tilde\Gamma^\varepsilon_{\tilde K,B}$ is the set defined in (6.17). We want to prove that there exists a constant $\tilde c$, depending at most on N, M, and ε, such that

$$u(\tilde z, \tilde t) \leq \tilde c u(A^-_{1,\Lambda}). \tag{6.26}$$

To do this we will construct a $\mathcal{K}$-admissible path $(\gamma(\tau), -1-\tau) : [0, -1-\tilde t] \to \mathbb{R}^{N+1}$,

$$\gamma(\tau) = (\gamma_{1,x}(\tau), \gamma'_x(\tau), \gamma_{1,y}(\tau), \gamma'_y(\tau)),$$

such that $(\gamma(0), -1) = A^-_{1,\Lambda} = (\Lambda, 0, \frac{2}{3}\Lambda, 0, -1) = (x_1, x', y_1, y', -1) =: (z, t)$, and an associated Harnack chain, targeting $(\tilde z, \tilde t) = (\tilde x_1, \tilde x', \tilde y_1, \tilde y', \tilde t)$. Note that $3 \geq -1 - \tilde t \geq 3 - \varepsilon$ and hence (z, t) and $(\tilde z, \tilde t)$ are well separated in time. In the following we let $\delta := -1-\tilde t$. As the first step in the construction we construct a path $\gamma'(\tau) := (\gamma'_x(\tau), \gamma'_y(\tau))$ in $\mathbb{R}^{N-2}$ connecting $z' := (0, 0)$ to $\tilde z' := (\tilde x', \tilde y')$. Indeed we simply let $\gamma'(\tau)$ be the path in (3.6), i.e., we consider $(\gamma'(\tau), -1-\tau) : [0, \delta] \to \mathbb{R}^{N-2}$ where

$$\gamma'(\tau) = E(-\tau)\left(z' + \mathcal{C}(\tau)\mathcal{C}^{-1}(\delta)(E(\delta)\tilde z' - z')\right). \tag{6.27}$$

We now first note, using Remark 3.4 and the fact that $(\tilde z, \tilde t) \in \tilde\Gamma^\varepsilon_{\tilde K,B}$, that

$$\begin{aligned}d'_K((\gamma'(\tau), -1-\tau), (0, 0, -1)) &\leq c(F(\tau/\delta)K^{3/2} + \tau^{1/2}),\\ d'_K((\gamma'(\tau), -1-\tilde t-\tau), (\tilde x', \tilde y', \tilde t)) &\leq c(F(\tau/\delta)K^{3/2} + \tau^{1/2}),\end{aligned} \tag{6.28}$$

whenever $\tau \in [0, \delta]$ and where d'_K denotes the natural and corresponding quasi-distance function in $\mathbb{R}^{N-2} \times \mathbb{R}$. Furthermore, F is a non-negative function such that $F(0) = 0$ and $F(\tau/\delta) \leq ct/\tau$ for some $c = c(N)$, $1 < c < \infty$. In particular, given $0 < \varepsilon'$ small we see that we can find $\delta' = \delta'(N, K, \varepsilon') = \delta'(N, M, \varepsilon')$, such that

$$\begin{aligned}d'_K((\gamma'(\tau), -1-\tau), (0, 0, -1)) &\leq \varepsilon',\\ d'_K((\gamma'(\tau), -1-\tilde t-\tau), (\tilde x', \tilde y', \tilde t)) &\leq \varepsilon',\end{aligned} \tag{6.29}$$

whenever $\tau \in [0, \delta']$. To proceed, we let

$$d = \Lambda - f(0, 0, t) \quad \text{and} \quad \tilde d = \tilde x_1 - f(\tilde x', \tilde y', \tilde t),$$

and we note that there exists, by construction of the set $\tilde\Gamma^\varepsilon_{\tilde K,B}$, $\bar c = \bar c(N, M)$, $1 \leq \bar c < \infty$, such that

$$\min\{d, \tilde d\} \geq \bar c^{-1} \min\{\varepsilon, 1/100\}. \tag{6.30}$$

Furthermore, using (6.29), and the Lip_K-character of f, we can conclude that there exists $\delta' = \delta'(N, K, \varepsilon) = \delta'(N, M, \varepsilon)$, $0 < \delta' \ll \delta$, such that

$$x_1 - f(\gamma'(\tau), -1-\tau) \geq d/2,\ \tilde x_1 - f(\gamma'(\tau), -1-\tilde t-\tau) \geq \tilde d/2, \tag{6.31}$$

whenever $\tau \in [0, \delta']$. Next, using the analysis in Remark 3.3, see (3.14), and the construction, we also see that there exists $c = c(N, \Lambda, K) = c(N, M, K)$, $1 \leq c < \infty$, such that

$$||(\gamma'(\tau), -1-\tau)||_K \leq c \text{ whenever } \tau \in [0, \delta]. \tag{6.32}$$

In particular, using that $(0, 0, 0) \in \Delta_{f,2r_0}$, (6.32), and the Lip_K-character of f, we can conclude that there exists a constant $\tilde{c} = \tilde{c}(N, M, K)$ such that

$$|f(\gamma'(\tau), -1-\tau)| \leq \tilde{c} \text{ whenever } \tau \in [0, \delta]. \tag{6.33}$$

We will now use (6.31), (6.33), to construct a path $\gamma_{1,x}(\tau)$ connecting x_1 to $\tilde{x}_1$. Indeed, for δ', $\tilde{c}$, as above we let

$$\begin{aligned}
&(i) && \gamma_{1,x}(0) = x_1, \\
&(ii) && \frac{d}{d\tau}\gamma_{1,x}(\tau) = 2(4\tilde{c} - d)/\delta' \text{ for } \tau \in [0, \delta'/2], \\
&(iii) && \frac{d}{d\tau}\gamma_{1,x} = 0 \qquad\qquad \text{for } \tau \in (\delta'/2, \delta - \delta'/2), \\
&(iv) && \frac{d}{d\tau}\gamma_{1,x}(\tau) = 2(\tilde{d} - 4\tilde{c})/\delta' \text{ for } \tau \in [\delta - \delta'/2, \delta].
\end{aligned} \tag{6.34}$$

Note that to construct $\gamma_{1,x}$ we start at x_1 and we then travel very fast into the domain. We then stay in the interior for a substantial amount of time before travel back towards the boundary ending up at $\gamma_{1,x}(\delta) = \tilde{x}_1$. Given the path $\gamma_{1,x}(\tau)$, the path in y_1-variable becomes

$$\begin{aligned}
&(i) && \gamma_{1,y}(0) = y_1, \\
&(ii) && \frac{d}{d\tau}\gamma_{1,y}(\tau) = \gamma_{1,x}(\tau) \text{ for } \tau \in [0, \delta].
\end{aligned} \tag{6.35}$$

In particular, further control of the path in y_1-variable is impossible but we note that

$$|\gamma_{1,y}(\tau)| \leq c = c(N, M, K) = c(N, M) \text{ whenever } \tau \in [0, \delta], \tag{6.36}$$

and for some (potentially large) constant c. Put together, (6.27), (6.34), and (6.35) complete the construction of a $\mathcal{K}$-admissible path

$$(\gamma(\tau), -1-\tau) = (\gamma_{1,x}(\tau), \gamma_x'(\tau), \gamma_{1,y}(\tau), \gamma_y'(\tau), -1-\tau),$$

such that $(\gamma(0), -1) = A^-_{1,\Lambda} = (\Lambda, 0, \frac{2}{3}\Lambda, 0, -1) = (x_1, x', y_1, y', -1)$ and such that

$$(\gamma_{1,x}(\delta), \gamma_x'(\delta), \gamma_y'(\delta), -1-\delta) = (\tilde{x}_1, \tilde{x}', \tilde{y}', -1-\delta).$$

Note that we can not ensure that $\gamma_{1,y}(\delta) = \tilde{y}_1$. However, using (6.30), (6.31), (6.33), and the construction in (6.34), we can conclude that

$$d_K((\gamma(\tau), -1-\tau), \Delta_{f,2r_0}) \geq \bar{c}^{-1} \min\{\varepsilon, 1/100\}, \tag{6.37}$$

whenever $\tau \in [0, \delta]$ and for some $\bar{c} = \bar{c}(N, M)$, $1 \leq \bar{c} < \infty$, where we of course also have used that the function f defining $\Delta_{f,2r_0}$ is independent of y_1.

Using the $\mathcal{K}$-admissible path $(\gamma(\tau), -1-\tau) : [0, T] \to \mathbb{R}^{N+1}$, and in particular (6.37), we now build a Harnack chain connecting $(\gamma(0), -1) = (z, t)$ to $(\gamma(\delta), -1-\delta)$ using Lemma 3.5 and Lemma 3.6. Indeed, we see that

$$\frac{d}{d\tau}\gamma(\tau) = \sum_{j=1}^{m} \omega_j(\tau) X_j(\gamma(\tau)) + Y(\gamma(\tau)), \qquad \text{for } \tau \in [0, \delta], \tag{6.38}$$

where we have explicit expressions for $\omega = (\omega_1, \omega') = (\omega_1, ..., \omega_m)$ through (6.34) (ω_1) and Lemma 3.6 (ω'). Using Lemma 3.5 we know that

$$\int_a^b |\omega(\tau)|^2 d\tau \leq h \quad \Rightarrow \quad \gamma(b) \in Q_r^-(\gamma(a), -1-a) \text{ where } r = \sqrt{\frac{b-a}{\beta}}, \tag{6.39}$$

whenever $0 \leq a \leq b \leq \delta$. Using (6.39) we will construct a finite sequence of real numbers $\{r_j\}_{j=1}^k$, and a sequence of points $\{(z_j, t_j)\}_{j=1}^k$, such that $(z_1, t_1) = (z, t)$ and such that

$$\begin{aligned}
&(i) \qquad Q_{r_j}^-(z_j, t_j) \subset \Omega_{f,2r_0}, \text{ for every } j = 1, \ldots, k,\\
&(ii) \qquad (z_{j+1}, t_{j+1}) \in \widetilde{Q}_{r_j}^-(z_j, t_j), \text{ for every } j = 1, \ldots, k-1,\\
&(iii) \qquad \gamma(\delta) \in \widetilde{Q}_{r_k}^-(z_k, t_k).
\end{aligned} \tag{6.40}$$

To start the construction we note, see (6.37), that we can in the following use that there exists $\bar\varepsilon = \bar\varepsilon(N, M, \varepsilon)$, $0 < \bar\varepsilon \ll 1$, such that

$$Q_{2\bar\varepsilon}(\gamma(\tau), -1-\tau) \subset \Omega_{f,2r_0} \text{ whenever } \tau \in [0, \delta], \tag{6.41}$$

and we will build a Harnack chain with $r_j = \bar\varepsilon$ for all j. We construct $\{(z_j, t_j)\}_{j=1}^k$ inductively as follows. Let $(z_1, t_1) = (z, t)$ and assume that $(z_j, t_j) = (\gamma(\tau_j), -1-\tau_j)$ has been constructed for some $j \geq 1$. If $\tau_j = \delta$, then the construction is stopped and we let $k = j$. If $\tau_j < \delta$ then we construct $(z_{j+1}, t_{j+1}) = (\gamma(\tau_{j+1}), -1-\tau_{j+1})$ by arguing as follows. There are two options, either

$$(i)\ \tau_j + \bar\varepsilon^2\beta < \delta \text{ or } (ii)\ \tau_j + \bar\varepsilon^2\beta \geq \delta, \tag{6.42}$$

where β is the constant appearing in Lemma 2.1 and hence in the definition of the sets $\{\widetilde{Q}_{r_k}^-(z_k, t_k)\}$. We consider (i) first and we note that there are now two additional options: either

$$\begin{aligned}
&(i') \qquad \int_{\tau_j}^{\tau_j+\bar\varepsilon^2\beta} \frac{|\omega(\tau)|^2}{h}\, d\tau \leq 1 \text{ or}\\
&(ii') \qquad \int_{\tau_j}^{\tau_j+\bar\varepsilon^2\beta} \frac{|\omega(\tau)|^2}{h}\, d\tau > 1.
\end{aligned} \tag{6.43}$$

If (i') is true, then we set $\tau_{j+1} = \tau_j + \bar\varepsilon^2\beta$, $z_{j+1} = \gamma(\tau_{j+1})$. If (ii') is true, then we set

$$\tau_{j+1} = \sup\left\{\sigma \in (\tau_j, \tau_j + \bar\varepsilon^2\beta) \mid \int_{\tau_j}^{\sigma} \frac{|\omega(\tau)|^2}{h}\, d\tau \leq 1\right\}, \tag{6.44}$$

$z_{j+1} = \gamma(\tau_{j+1})$. In either case we can conclude, using (2.1), (6.39), and (6.41), that there exists $c = c(N, M)$, $1 \leq c < \infty$, such that

$$\begin{aligned}
u(z_{j+1}, t_{j+1}) &= u(\gamma(\tau_{j+1}), -1-\tau_{j+1})\\
&\leq cu(\gamma(\tau_j), -1-\tau_j) = cu(z_j, t_j).
\end{aligned} \tag{6.45}$$

We next consider (ii). In this case $\tau_j \geq \delta - \bar\varepsilon^2\beta$. Assume first that, in addition,

$$(i') \qquad \int_{\tau_j}^{\delta} \frac{|\omega(\tau)|^2}{h}\, d\tau \leq 1. \tag{6.46}$$

In this case we set $\tau_{j+1} = \delta$, $z_{j+1} = \gamma(\tau_{j+1})$, and we can again conclude that (6.45) holds. If, on the contrary, (6.46) does not hold, then we set

$$\tau_{j+1} = \sup\left\{\sigma \in (\tau_j, \delta) \mid \int_{\tau_j}^{\sigma} \frac{|\omega(\tau)|^2}{h}\, d\tau \leq 1\right\}, \tag{6.47}$$

$z_{j+1} = \gamma(\tau_{j+1})$, and we again see that (6.45) holds. We note that by this construction there will be a first j such that $\tau_j = \delta$ and we then set $k = j$. The next step is to estimate k and we note that $0<\tau_{j+1}-\tau_j\leq\bar\varepsilon^2\beta$ for all j. Let $\mathcal{I}_1$ denote the set of all index j for which either $(i)+(ii')$ or (ii), and the scenario leading up to (6.47), occur. Let $\mathcal{I}_2$ denote the set of all index j for which either $(i)+(i')$ or $(ii)+(i')$, occur. Note the union of the sets $\mathcal{I}_1$ and $\mathcal{I}_2$ is the set of all indices occurring in the construction. Now, by continuity of $\omega(\tau)=(\omega_1(\tau),\omega'(\tau))=(\omega_1(\tau),...,\omega_m(\tau))$ we first see that

$$\int_{\tau_j}^{\tau_{j+1}}\frac{|\omega(\tau)|^2}{h}\,d\tau=1,\ \text{for all } j\in\mathcal{I}_1. \tag{6.48}$$

In particular,

$$|\mathcal{I}_1|\leq\int_0^\delta\frac{|\omega(\tau)|^2}{h}\,d\tau. \tag{6.49}$$

Furthermore, we easily see that

$$|\mathcal{I}_2|\leq\frac{\delta}{\bar\varepsilon^2\beta}. \tag{6.50}$$

In particular,

$$k\leq|\mathcal{I}_1|+|\mathcal{I}_2|\leq\frac{\delta}{\bar\varepsilon^2\beta}+\int_0^\delta\frac{|\omega(\tau)|^2}{h}\,d\tau. \tag{6.51}$$

Hence, using (6.51), Lemma 3.3, Remark 3.5, the fact that $3\geq\delta\geq3-\varepsilon$, and the explicit construction in (6.34), we can conclude that there exists $c=c(N,M)$, $1\leq c<\infty$, such that

$$u(\gamma(\delta),-1-\delta)\leq cu(A^-_{1,\Lambda}). \tag{6.52}$$

By construction $(\gamma(\delta),-1-\delta)=(\tilde x_1,\tilde x',\gamma_{1,y}(\delta),\tilde y',\tilde t)$ and $(\gamma(\delta),-1-\delta)$ only differ from $(\tilde z,\tilde t)=(\tilde x_1,\tilde x',\tilde y_1,\tilde y',\tilde t)$ in the y_1-coordinate. However, using (6.36) and Lemma 5.3 we see that there exists $c=c(N,M,m^+/m^-)$, $1\leq c<\infty$, such that

$$u(\tilde z,\tilde t)\leq cu(\gamma(\delta),-1-\delta). \tag{6.53}$$

In particular, combining (6.52) and (6.53) we see that the proof of Lemma 6.2 is complete. □

6.3. **Proof of Theorem 1.2.** Assume that u and v are non-negative solutions to $\mathcal{K}u=0$ in $\Omega_{f,2r_0}$ and that v and u vanish continuously on $\Delta_{f,2r_0}$. Relying on the set up concluded in Remark 3.8 we introduce $m_1^\pm$, $m_2^\pm$, as in (1.28). As previously noted, the assumption $\min\{m_1^-,m_2^-\}>0$ implies $\min\{m_1^+,m_2^+\}>0$. We intend to prove that there exist constants $c_1=c_1(N,M)$, $1\leq c_1<\infty$, $c_2=c_2(N,M,m_1^+/m_1^-,m_2^+/m_2^-)$, $1\leq c_2<\infty$, $\sigma=\sigma(N,M,m_1^+/m_1^-,m_2^+/m_2^-)$, $\sigma\in(0,1)$, such that

$$\left|\frac{v(z,t)}{u(z,t)}-\frac{v(\tilde z,\tilde t)}{u(\tilde z,\tilde t)}\right|\leq c_2\left(\frac{d_K((z,t),(\tilde z,\tilde t))}{\varrho}\right)^\sigma\frac{v(A_{\varrho,\Lambda})}{u(A_{\varrho,\Lambda})},$$

whenever (z,t), $(\tilde z,\tilde t)\in\Omega_{f,\varrho/c_1}$ and $0<\varrho<\varrho_1$. The proof is based on interior Hölder continuity estimates, Lemma 5.1, Lemma 5.2, Theorem 1.1 and its proof, see (6.5) in particular. To start the proof, let

$$\mathcal{O}_{v,u}(z,t,\tilde\varrho)=\sup_{\Omega_{f,2r_0}\cap Q_{M,\tilde\varrho}(z,t)}\frac{v}{u}-\inf_{\Omega_{f,2r_0}\cap Q_{M,\tilde\varrho}(z,t)}\frac{v}{u}, \tag{6.54}$$

whenever (z,t) and $\tilde\varrho$ are such that $Q_{M,\tilde\varrho}(z,t)$ is contained in the closure of the set $\Omega_{f,\varrho_1/(100c_1)}$, where c_1 are as in the statement of Theorem 1.1. Using Lemma 5.1, and the assumptions on $m_1^\pm$, $m_2^\pm$, we first see that $\mathcal{O}_{v,u}(0,0,\varrho_1/c_1)<\infty$. Let now ϱ be fixed and let $\bar\varrho=\delta\varrho$ for some degree of freedom $\delta=\delta(N,M)$, $0<\delta\ll1$, to be chosen. Consider $0 < \tilde{\varrho} \leq \bar{\varrho}$, pick $(z,t) \in \Omega_{f,\bar{\varrho}}$ and let $d = d_K(z,t,\Delta_{f,2r_0})$. Given $\tilde{\varrho}$, (z,t), d, we consider two cases: $\tilde{\varrho} \leq d$ (interior case) and $\tilde{\varrho} > d$ (boundary case).

We first consider the case $\tilde{\varrho} \leq d$. Let

$$\hat{v}(\tilde{z},\tilde{t}) := \big(\mathcal{O}_{v,u}(z,t,\tilde{\varrho})\big)^{-1}\left(v(\tilde{z},\tilde{t}) - \left(\inf_{\Omega_{f,2r_0}\cap Q_{M,\tilde{\varrho}}(z,t)} v/u\right) u(\tilde{z},\tilde{t})\right),$$

and note that

$$(i) \qquad 0 \leq \frac{\hat{v}(\tilde{z},\tilde{t})}{u(\tilde{z},\tilde{t})} \leq 1, \text{ whenever } (\tilde{z},\tilde{t}) \in \Omega_{f,2r_0} \cap Q_{M,\tilde{\varrho}}(z,t),$$

$$(ii) \qquad \mathcal{O}_{\hat{v},u}(z,t,\tilde{\varrho}) = 1. \tag{6.55}$$

Let γ, $0 < \gamma \ll 1$ be a degree of freedom and assume first, in addition, that

$$\frac{\hat{v}((z,t)\circ(0,-\gamma\tilde{\varrho}^2))}{u((z,t)\circ(0,-\gamma\tilde{\varrho}^2))} \geq \frac{1}{2}. \tag{6.56}$$

Note that $\mathcal{K}\hat{v} = 0$ in $\Omega_{f,2r_0}$ and that $\hat{v}$ is non-negative in $\Omega_{f,2r_0} \cap Q_{M,\tilde{\varrho}}(z,t)$. Therefore, using the Harnack inequality in Theorem 2.1 we see that there exists $\tilde{\gamma} = \tilde{\gamma}(N,\gamma)$, $0 < \tilde{\gamma} \ll 1$, such that

$$\hat{v}((z,t)\circ(0,-\gamma\tilde{\varrho}^2)) \leq c\hat{v}(\tilde{z},\tilde{t}) \text{ whenever } (\tilde{z},\tilde{t}) \in Q_{M,\tilde{\gamma}\tilde{\varrho}}(z,t), \tag{6.57}$$

and

$$u(\tilde{z},\tilde{t}) \leq cu((z,t)\circ(0,\gamma\tilde{\varrho}^2)) \text{ whenever } (\tilde{z},\tilde{t}) \in Q_{M,\tilde{\gamma}\tilde{\varrho}}(z,t). \tag{6.58}$$

Moreover, using standard arguments based on Theorem 1.1 we see that

$$u((z,t)\circ(0,\gamma\tilde{\varrho}^2)) \leq cu((z,t)\circ(0,-\gamma\tilde{\varrho}^2)), \tag{6.59}$$

with the admissible dependency on c. Combining (6.55)-(6.59), we deduces that

$$\frac{1}{2} \leq \frac{\hat{v}((z,t)\circ(0,-\gamma\tilde{\varrho}^2))}{u((z,t)\circ(0,-\gamma\tilde{\varrho}^2))} \leq c\frac{\hat{v}(\tilde{z},\tilde{t})}{u(\tilde{z},\tilde{t})} \leq c, \tag{6.60}$$

whenever $(\tilde{z},\tilde{t}) \in Q_{M,\tilde{\gamma}\tilde{\varrho}}(z,t)$. Hence

$$\mathcal{O}_{\hat{v},u}(z,t,\tilde{\gamma}\tilde{\varrho}) \leq \theta, \tag{6.61}$$

where $\theta = 1 - 1/(2c) \in (0,1)$. Recalling the definition of $\hat{v}$, and rearranging (6.61) we can conclude that

$$\mathcal{O}_{v,u}(z,t,\tilde{\gamma}\tilde{\varrho}) \leq \theta\mathcal{O}_{v,u}(z,t,\tilde{\varrho}). \tag{6.62}$$

Assume now, on the contrary, that (6.56) does not hold and that instead

$$\frac{\hat{v}((z,t)\circ(0,-\gamma\tilde{\varrho}^2))}{u((z,t)\circ(0,-\gamma\tilde{\varrho}^2))} < \frac{1}{2}. \tag{6.63}$$

In this case let $\bar{v} = u - \hat{v}$. Then (6.55) and (6.56) hold with $\hat{v}$ replaced by $\bar{v}$. We can then first conclude that $\mathcal{O}_{\bar{v},u}(z,t,\tilde{\gamma}\tilde{\varrho}) \leq \theta$ and subsequently again that (6.62) holds. Next, iterating the estimate in (6.62) we deduce that

$$\mathcal{O}_{v,u}(z,t,\tilde{\varrho}) \leq \left(\frac{\tilde{\varrho}}{\tilde{\gamma}d}\right)^{\sigma_1}\mathcal{O}_{v,u}(z,t,d), \tag{6.64}$$

for some $\sigma_1 = \sigma_1(\theta) = \sigma_1(N,M,m^+/m^-) \in (0,1)$.

We next consider the case $\tilde{\varrho} > d$. Let $(z_0,t_0) \in \Delta_{f,2r_0}$ be such that

$$d = d_K((z,t),(z_0,t_0)).$$

Then $Q_{M,\tilde{\varrho}}(z,t) \subset Q_{M,2\bar{c}\tilde{\varrho}}(z_0,t_0)$ for some $\bar{c} = \bar{c}(N,M)$, $1 \leq \bar{c} < \infty$, and hence

$$\mathcal{O}_{v,u}(z,t,\tilde{\varrho}) \leq \mathcal{O}_{v,u}(z_0,t_0,2\bar{c}\tilde{\varrho}).$$

Let in the following $K := c$ where c is the constant appearing in Lemma 5.2. We first assume that $4K\bar c\tilde\varrho < \varrho/2$. Let now $\hat v$ be defined by

$$\hat v(\tilde z,\tilde t) = \big(\mathcal{O}_{v,u}(z_0,t_0,8K\bar c\tilde\varrho)\big)^{-1}\left(v(\tilde z,\tilde t)-\left(\inf_{\Omega_{f,2r_0}\cap Q_{M,8K\bar c\tilde\varrho}(z_0,t_0)} v/u\right)u(\tilde z,\tilde t)\right).$$

As in the interior case,

$$(i)\qquad 0\le \frac{\hat v(\tilde z,\tilde t)}{u(\tilde z,\tilde t)}\le 1,\ \text{whenever } (\tilde z,\tilde t)\in\Omega_{f,2r_0}\cap Q_{M,8K\bar c\tilde\varrho}(z_0,t_0),$$

$$(ii)\qquad \mathcal{O}_{\hat v,u}(z_0,t_0,8K\bar c\tilde\varrho)=1. \tag{6.65}$$

Now first assume that

$$\frac{\hat v\big(A^-_{4K\bar c\tilde\varrho,\Lambda}(z_0,t_0)\big)}{u\big(A^-_{4K\bar c\tilde\varrho,\Lambda}(z_0,t_0)\big)}\ge\frac12. \tag{6.66}$$

As $\hat v$ and u are solutions to $\mathcal{K}u=0$ on $\Omega_{f,2r_0}$, non-negative in $\Omega_{f,2r_0}\cap Q_{M,8K\bar c\tilde\varrho}(z_0,t_0)$, and $\hat v$ and u vanish continuously on $\Delta_{f,2r_0}$, it follows from Lemma 5.2 that

$$\frac{\hat v\big(A^-_{4K\bar c\tilde\varrho,\Lambda}(z_0,t_0)\big)}{u\big(A^+_{4K\bar c\tilde\varrho,\Lambda}(z_0,t_0)\big)}\le K\frac{\hat v(\tilde z,\tilde t)}{u(\tilde z,\tilde t)}\le K, \tag{6.67}$$

whenever $(\tilde z,\tilde t)\in\Omega_{f,2r_0}\cap Q_{M,2\bar c\tilde\varrho}(z_0,t_0)$. Again using Theorem 1.1, see (6.5) in particular, it follows that

$$\frac{\hat v\big(A^-_{4K\bar c\tilde\varrho,\Lambda}(z_0,t_0)\big)}{u\big(A^-_{4K\bar c\tilde\varrho,\Lambda}(z_0,t_0)\big)}\le c\frac{\hat v\big(A^-_{4K\bar c\tilde\varrho,\Lambda}(z_0,t_0)\big)}{u\big(A^+_{4K\bar c\tilde\varrho,\Lambda}(z_0,t_0)\big)}. \tag{6.68}$$

Hence, using (6.67), (6.68) and (6.66), we see that

$$\frac12\le\frac{\hat v(\tilde z,\tilde t)}{u(\tilde z,\tilde t)}\le cK,$$

whenever $(\tilde z,\tilde t)\in\Omega_{f,2r_0}\cap Q_{M,2\bar c\tilde\varrho}(z_0,t_0)$. Therefore

$$\mathcal{O}_{\hat v,u}(z_0,t_0,2\bar c\tilde\varrho)\le\theta, \tag{6.69}$$

where $\theta=1-1/(2cK)\in(0,1)$. Rewriting this expression we see that

$$\mathcal{O}_{v,u}(z,t,\tilde\varrho)\le\mathcal{O}_{v,u}(z_0,t_0,2\bar c\tilde\varrho)\le\theta\mathcal{O}_{v,u}(z_0,t_0,8K\bar c\tilde\varrho). \tag{6.70}$$

Assume now, on the contrary, that (6.66) does not hold and instead that

$$\frac{\hat v\big(A^-_{4K\bar c\tilde\varrho,\Lambda}(z_0,t_0)\big)}{u\big(A^-_{4K\bar c\tilde\varrho,\Lambda}(z_0,t_0)\big)}<\frac12. \tag{6.71}$$

In this case, let $\bar v=u-\hat v$. Then (6.65) and (6.66) hold with $\hat v$ replaced by $\bar v$. One can then first conclude that $\mathcal{O}_{\bar v,u}(z_0,t_0,2\bar c\tilde\varrho)\le\theta$ and subsequently again that (6.70) holds. Iterating (6.70) we have

$$\begin{aligned}\mathcal{O}_{v,u}(z,t,\tilde\varrho)&\le\theta\mathcal{O}_{v,u}(z_0,t_0,8K\bar c\tilde\varrho)\\&\le\left(\frac{8K\bar c\tilde\varrho}{\varrho}\right)^{\sigma_2}\mathcal{O}_{v,u}(z_0,t_0,\varrho),\end{aligned} \tag{6.72}$$

for some $\sigma_2=\sigma_2(M,N,m^+/m^-)\in(0,1)$. One easily sees that this also holds if $4K\bar c\tilde\varrho\ge\bar\varrho/2$.

From (6.64) and (6.72) it follows that if $\tilde\varrho\le d<\varrho$, then

$$\begin{aligned}\mathcal{O}_{v,u}(z,t,\tilde\varrho)&\le\left(\frac{\tilde\varrho}{\gamma d}\right)^{\sigma_1}\mathcal{O}_{v,u}(z,t,d)\\&\le\left(\frac{\tilde\varrho}{\gamma d}\right)^{\sigma_1}\left(\frac{8K\bar cd}{\varrho}\right)^{\sigma_2}\mathcal{O}_{v,u}(z_0,t_0,\varrho).\end{aligned} \tag{6.73}$$

With $\sigma = \min\{\sigma_1, \sigma_2\}$, (6.73) implies that

$$\mathcal{O}_{v,u}(z,t,\tilde{\varrho}) \le c\biggl(\frac{\tilde{\varrho}}{\varrho}\biggr)^{\sigma}\mathcal{O}_{v,u}(z_0,t_0,\varrho), \tag{6.74}$$

for all $(z,t) \in \Omega_{f,\tilde{\varrho}}$, $\tilde{\varrho} \le \bar{\varrho}$, $\bar{\varrho} = \delta\varrho$. Now, finally, consider $(\tilde{z},\tilde{t}) \in \Omega_{f,\tilde{\varrho}}$ and let $\hat{\varrho} = d_K((z,t),(\tilde{z},\tilde{t}))$. It then follows from (6.74) and Lemma 5.1, in conjunction with Theorem 1.1, that if $\delta = \delta(N,M)$, $0 < \delta \ll 1$ is chosen small enough, then

$$\begin{aligned} \biggl|\frac{v(x,t)}{u(z,t)} - \frac{v(\tilde{z},\tilde{t})}{u(\tilde{z},\tilde{t})}\biggr| &\le \mathcal{O}_{v,u}(z,t,\hat{\varrho}) \\ &\le c\biggl(\frac{\hat{\varrho}}{\varrho}\biggr)^{\sigma}\mathcal{O}_{v,u}(0,0,\varrho) \le c\biggl(\frac{\hat{\varrho}}{\varrho}\biggr)^{\alpha}\frac{v\bigl(A_{\varrho,\Lambda}\bigr)}{u\bigl(A_{\varrho,\Lambda}\bigr)}. \end{aligned} \tag{6.75}$$

This completes the proof of Theorem 1.2. □

6.4. **Proof of Theorem 1.3.** As emphasized, it suffices to prove Theorem 1.3 in the case $(z_0,t_0) = (0,0)$. We let Λ, c_0, η, ϱ_0, ϱ_1, be as stated in Remark 3.8. Based on this we consider ϱ, $0 < \varrho < \varrho_1$, and we need to prove that there exist $c_2 = c_2(N,M)$, $1 \le c_2 < \infty$, and $c_3 = c_3(N,M)$, $1 \le c_3 < \infty$, such that

$$\begin{aligned} &\omega_K(A^+_{\varrho,\Lambda}, \Delta_{f,2r_0} \cap Q_{M,2\tilde{\varrho}}(\bar{z}_0,\bar{t}_0)) \\ &\le c_2\omega_K(A^+_{\varrho,\Lambda}, \Delta_{f,2r_0} \cap Q_{M,\tilde{\varrho}}(\bar{z}_0,\bar{t}_0)), \end{aligned} \tag{6.76}$$

whenever $(\bar{z}_0,\bar{t}_0) \in \Delta_{f,2r_0}$ and $Q_{M,\tilde{\varrho}}(\bar{z}_0,\bar{t}_0) \subset Q_{M,\varrho/c_3}$. In the following c_3 is a degree of freedom to be chosen. To start the proof of (6.76), we recall (4.13) in Remark 4.1 which states that

$$\omega_K(A^+_{\varrho,\Lambda}, \Delta_{f,2r_0} \cap Q_{M,2\tilde{\varrho}}(\bar{z}_0,\bar{t}_0)) \le c\tilde{\varrho}^{\mathbf{q}}G(A^+_{\varrho,\Lambda}, A^-_{2c\tilde{\varrho},\Lambda}(\bar{z}_0,\bar{t}_0)),$$

provided $(\bar{z}_0,\bar{t}_0) \in \Delta_{f,2r_0}$ and $Q_{M,\tilde{\varrho}}(\bar{z}_0,\bar{t}_0) \subset Q_{M,\varrho/c_3}$. Let

$$m^+ = G(A^+_{\varrho,\Lambda}, A^+_{\varrho/1000,\Lambda}),\ m^- = G(A^+_{\varrho,\Lambda}, A^-_{\varrho/1000,\Lambda}). \tag{6.77}$$

Recall that $G(A^+_{\varrho,\Lambda},\cdot)$ is the adjoint Green function for $\Omega_{f,2r_0}$ with pole at $A^+_{\varrho,\Lambda}$. By elementary estimates and the Harnack inequality we see that

$$\bar{c}^{-1} \le \varrho^{\mathbf{q}}m^+ \le \bar{c}, \qquad \varrho^{\mathbf{q}}m^- \le \bar{c}, \tag{6.78}$$

for some $\bar{c} = \bar{c}(N,M)$, $1 \le \bar{c} < \infty$. We need to establish the corresponding lower bound on $\varrho^{\mathbf{q}}m^-$. Using the adjoint version of Lemma 3.12 we see that there exists $c = c(N,M)$, $1 \le c < \infty$, such that

$$\sup_{(z,t)\in\Omega_{f,\varrho/c}(z_0,t_0)} G(A^+_{\varrho,\Lambda},(z,t)) \le cm^-. \tag{6.79}$$

However,

$$\sup_{(z,t)\in\Omega_{f,\varrho/c}(z_0,t_0)} G(A^+_{\varrho,\Lambda},(z,t)) \ge c^{-1}m^+. \tag{6.80}$$

In particular, (6.78)-(6.80) imply that $c^{-1} \le m^+/m^- \le c$, for some $c = c(N,M)$, $1 \le c < \infty$. Using this, the adjoint version of Theorem 1.1, and the scale invariance of Theorem 1.1, we can, using by now familiar arguments, conclude that there exist $\tilde{c} = \tilde{c}(N,M)$, $1 \le \tilde{c} < \infty$, and c_3 as stated above, such that

$$G(A^+_{\varrho,\Lambda}, A^-_{2c\tilde{\varrho},\Lambda}(\bar{z}_0,\bar{t}_0)) \le \tilde{c}G(A^+_{\varrho,\Lambda}, A^+_{2c\tilde{\varrho},\Lambda}(\bar{z}_0,\bar{t}_0)), \tag{6.81}$$

provided $(\bar{z}_0,\bar{t}_0) \in \Delta_{f,2r_0}$ and $Q_{M,\tilde{\varrho}}(\bar{z}_0,\bar{t}_0) \subset Q_{M,\varrho/c_3}$. Finally, using the Harnack inequality and Lemma 4.2 we see that

$$\begin{aligned} \tilde{\varrho}^{\mathbf{q}}G(A^+_{\varrho,\Lambda}, A^+_{2c\tilde{\varrho},\Lambda}(\bar{z}_0,\bar{t}_0)) &\le c\tilde{\varrho}^{\mathbf{q}}G(A^+_{\varrho,\Lambda}, A^+_{\tilde{\varrho},\Lambda}(\bar{z}_0,\bar{t}_0)) \\ &\le c^2\omega_K(A^+_{\varrho,\Lambda}, \Delta_{f,2r_0} \cap Q_{M,\tilde{\varrho}}(\bar{z}_0,\bar{t}_0)), \end{aligned} \tag{6.82}$$

for some $c = c(N, M)$, $1 \leq c < \infty$. Put together we can conclude the validity of (6.76). This completes the proof of Theorem 1.3. □

7. Further results: generalizations and extensions

In this section we briefly discuss, without giving the complete proofs, to the extent one can generalize Theorems 1.1, 1.2 and 1.3 to the context of a subset of the more general operators of Kolmogorov type considered in [CNP1], [CNP2] and [CNP3]. In [CNP1], [CNP2] and [CNP3] we considered Kolmogorov operators of the form

$$\mathcal{L} = \sum_{i,j=1}^{m} a_{i,j}(z,t)\partial_{z_i z_j} + \sum_{i=1}^{m} a_i(z,t)\partial_{z_i} + \sum_{i,j=1}^{N} b_{i,j} z_i \partial_{z_j} - \partial_t, \tag{7.1}$$

where $(z, t) \in \mathbb{R}^N \times \mathbb{R}$, $1 \leq m \leq N$. The coefficients $a_{i,j}$ and a_i are bounded continuous functions and $B = (b_{i,j})_{i,j=1,\dots,N}$ is a matrix of real constants. Following [CNP1], [CNP2] and [CNP3] we here impose the structural assumptions **[H.1]**-**[H.4]** stated below.

[H.1] The matrix $A_0(z,t) = (a_{i,j}(z,t))_{i,j=1,\dots,m}$ is symmetric and uniformly positive definite in $\mathbb{R}^m$: there exists a positive constant λ such that

$$\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{m} a_{i,j}(z,t)\xi_i\xi_j \leq \lambda|\xi|^2, \qquad \forall\, \xi \in \mathbb{R}^m,\ (z,t) \in \mathbb{R}^{N+1}.$$

The matrix $B = (b_{i,j})_{i,j=1,\dots,N}$ has real constant entries.

[H.2] For any $(z_0, t_0) \in \mathbb{R}^{N+1}$ fixed, the constant coefficient operator

$$\mathcal{K} = \sum_{i,j=1}^{m} a_{i,j}(z_0,t_0)\partial_{z_i z_j} + \sum_{i,j=1}^{N} b_{i,j} z_i \partial_{z_j} - \partial_t \tag{7.2}$$

is hypoelliptic.

[H.3] The coefficients $a_{i,j}(z,t)$ and $a_i(z,t)$ are bounded functions belonging to the Hölder space $C_K^{0,\alpha}(\mathbb{R}^{N+1})$, $\alpha \in (0,1]$, defined with respect to the appropriate metric associated to $\mathcal{L}$.

Note that by a change of variables we can choose A_0 in **[H.2]** as the m-dimensional identity matrix. We also note that the operator $\mathcal{K}$ can be written as

$$\mathcal{K} = \sum_{i=1}^{m} X_i^2 + Y,$$

where

$$X_i = \sum_{j=1}^{m} \bar{a}_{i,j}\partial_{z_j}, \quad i = 1,\dots,m, \qquad Y = \langle z, B\nabla\rangle - \partial_t, \tag{7.3}$$

and where $\bar{a}_{i,j}$'s are the entries of the unique matrix $\bar{A}_0$ such that $A_0 = \bar{A}_0^2$. Again the hypothesis **[H.2]** is equivalent to the Hörmander condition

$$\text{rank Lie}\,(X_1,\dots,X_m,Y)\,(z,t) = N+1, \qquad \forall\,(z,t) \in \mathbb{R}^{N+1}. \tag{7.4}$$

The relevant Lie group related to the operator $\mathcal{K}$ in (7.2) is defined using the group law

$$(\tilde{z},\tilde{t}) \circ (z,t) = (z + \exp(-tB^*)\tilde{z}, \tilde{t} + t), \qquad (\tilde{z},\tilde{t}), (z,t) \in \mathbb{R}^{N+1}. \tag{7.5}$$

In particular, the vector fields $X_1, \ldots, X_m$ and Y are left-invariant, with respect to the group law (7.5). Furthermore, see [LP], **[H.2]** is equivalent to the following structural assumption on B: there exists a basis for $\mathbb{R}^{N+1}$ such that the matrix B has the form

$$\begin{pmatrix} * & B_1 & 0 & \cdots & 0 \\ * & * & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & B_\kappa \\ * & * & * & \cdots & * \end{pmatrix} \tag{7.6}$$

where B_j is a $m_{j-1} \times m_j$ matrix of rank m_j for $j \in \{1, \ldots, \kappa\}$, $1 \le m_\kappa \le \ldots \le m_1 \le m_0 = m$ and $m + m_1 + \ldots + m_\kappa = N$, while $*$ represents arbitrary matrices with constant entries. Based on (7.6), we introduce the family of dilations $(\delta_r)_{r>0}$ on $\mathbb{R}^{N+1}$ defined by

$$\delta_r = (D_r, r^2) = \operatorname{diag}(rI_m, r^3 I_{m_1}, \ldots, r^{2\kappa+1} I_{m_\kappa}, r^2), \tag{7.7}$$

where I_k, $k \in \mathbb{N}$, is the k-dimensional unit matrix. In the sequel we will write the dilation (7.7) on the form

$$\delta_r = \operatorname{diag}(r^{\alpha_1}, \ldots, r^{\alpha_N}, r^2), \tag{7.8}$$

where we set $\alpha_1 = \ldots = \alpha_m = 1$, and $\alpha_{m+m_1+\cdots+m_{j-1}+1} = \ldots = \alpha_{m+m_1+\cdots+m_j+1} = 2j + 1$ for $j = 1, \ldots, \kappa$. According to (7.7), we split the coordinate $z \in \mathbb{R}^N$ as

$$z = \big(z^{(0)}, z^{(1)}, \ldots, z^{(\kappa)}\big), \qquad z^{(0)} \in \mathbb{R}^m, \quad z^{(j)} \in \mathbb{R}^{m_j}, \quad j \in \{1, \ldots, \kappa\}, \tag{7.9}$$

and we define

$$|z|_K = \sum_{j=0}^{\kappa} \big|z^{(j)}\big|^{\frac{1}{2j+1}}, \qquad \|(z,t)\|_K = |z|_K + |t|^{\frac{1}{2}}.$$

Note that $\|\delta_r(z,t)\|_K = r\|(z,t)\|_K$ for every $r > 0$ and $(z,t) \in \mathbb{R}^{N+1}$. In line with [CNP1], [CNP2] and [CNP3] we also assume:

[H.4] The operator $\mathcal{K}$ in (7.2) is δ_r-homogeneous of degree two, i.e.

$$\mathcal{K} \circ \delta_r = r^2(\delta_r \circ \mathcal{K}), \qquad \forall\, r > 0.$$

Following [LP] we have that **[H.4]** is satisfied if (and only if) all the blocks denoted by $*$ in (7.6) are null. Building on $\mathcal{L}$ we next construct a new operator $\bar{\mathcal{L}}$ of Kolmogorov type by adding variables. Let $\bar{m} = \kappa$, where $\kappa \ge 1$ is an integer, and let $\bar{N} = N + \bar{m} + 1$. We now add the variables $\bar{z} = (\bar{z}_1, ..., \bar{z}_{\bar{m}+1})$ and form the operator

$$\bar{\mathcal{L}} = \partial_{\bar{z}_1 \bar{z}_1} + \sum_{i=1}^{\bar{m}} \bar{z}_i \partial_{\bar{z}_{i+1}} + \mathcal{L} \tag{7.10}$$

which we consider in $\mathbb{R}^{\bar{m}+1} \times \mathbb{R}^N \times \mathbb{R} = \mathbb{R}^{\bar{N}} \times \mathbb{R}$. We emphasize that the operator $\mathcal{L}$ is independent of the variables $(\bar{z}_1, ..., \bar{z}_{\bar{m}+1})$. Furthermore, both $\mathcal{L}$ and $\bar{\mathcal{L}}$ are operators of Kolmogorov type in the sense outlined above satisfying the structural assumptions **[H.1]**-**[H.4]**.

We claim that appropriate versions of Theorems 1.1, 1.2 and 1.3 can be established for non-negative solutions to $\bar{\mathcal{L}}u = 0$ in Lipschitz type domains of the form $\bar{z}_1 > f(z,t)$, i.e., in Lipschitz type domains defined by a function f which is independent of $(\bar{z}_2, ..., \bar{z}_{\bar{m}+1})$. To be more precise, let $\bar{z} = (\bar{z}_1, \bar{z}') := (\bar{z}_1, \bar{z}_2, ..., \bar{z}_{\bar{m}+1})$. Given positive numbers r_1, r_2, we now let

$$\Box_{r_1,r_2} = \{(z,t) \in \mathbb{R}^N \times \mathbb{R} \mid |z_i| < r_1^{\alpha_i} \text{ for } i \in \{1, \ldots, N\},\ |t| < r_2^2\}. \tag{7.11}$$

Given $\square_{r_1,r_2} \subset \mathbb{R}^N \times \mathbb{R}$, we say that a function f, $f : \square_{r_1,r_2} \to \mathbb{R}$, is a Lip_K-function, with respect to coordinate direction $\bar{z}_1$, independent of $\bar{z}'$ and with constant $M \geq 0$, if $\bar{z}_1 = f(z,t)$ and

$$\big|f(z,t) - f(\tilde{z},\tilde{t})\big| \leq M \left\|(z - \exp((\tilde{t}-t)B^*), t - \tilde{t})\right\|_K, \tag{7.12}$$

whenever $(z,t), (\tilde{z},\tilde{t}) \in \square_{r_1,r_2}$. In addition, given positive numbers r_1, r_2, r_3, we let

$$Q_{r_1,r_2,r_3} = \{(\bar{z}_1, z, t) \in \mathbb{R}^{\bar{N}+2} \mid (z,t) \in \square_{r_1,r_2},\ |\bar{z}_1| < r_3\},$$

for $i \in \{2, .., \bar{m}+1\}$. Furthermore, for any positive M and r, and we let $Q_{M,r} = Q_{r,\sqrt{2}r,4Mr}$. Finally, given f as above, with $f(0,0) = 0$ and $M, r > 0$, we define

$$\begin{aligned}
\bar{\Omega}_{f,r} &= \{(\bar{z}_1, \bar{z}', z, t) \mid (\bar{z}_1, z, t) \in Q_{M,r},\ \bar{z}_1 > f(z,t),\ |\bar{z}_i| < r^{2i-1}\},\\
\bar{\Delta}_{f,r} &= \{(\bar{z}_1, \bar{z}', z, t) \mid (\bar{z}_1, z, t) \in Q_{M,r},\ \bar{z}_1 = f(z,t),\ |\bar{z}_i| < r^{2i-1}\},
\end{aligned}$$

where in these definitions $i = 2, ..., \bar{m}+1$.

Definition 6. Given M, r_0, we say that $\bar{\Omega}_{f,2r_0}$ is an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Similar we refer to $\bar{\Delta}_{f,2r_0}$ as an admissible local Lip_K-surface with Lip_K-constants M, r_0.

Next, given $\varrho > 0$ and $\Lambda > 0$ we define the points $\bar{z}_\varrho^{\Lambda,+}, \bar{z}_\varrho^{\Lambda,-} \in \mathbb{R}^{\bar{m}+1}$ as follows. We let

$$\begin{aligned}
&\bar{z}_{1,\varrho}^{\Lambda,+} = \varrho\Lambda, \qquad \bar{z}_{i,\varrho}^{\Lambda,+} = -\varrho^2 \frac{2}{2i+1} \bar{z}_{i-1,\varrho}^{\Lambda,+}, \qquad i = 2, \ldots, \bar{m}+1,\\
&\bar{z}_{1,\varrho}^{\Lambda,-} = \varrho\Lambda, \qquad \bar{z}_{i,\varrho}^{\Lambda,-} = \varrho^2 \frac{2}{2i+1} \bar{z}_{i-1,\varrho}^{\Lambda,-}, \qquad i = 2, \ldots, \bar{m}+1.
\end{aligned} \tag{7.13}$$

Using this notation we introduce the following reference points.

Definition 7. Given $\varrho > 0$ and $\Lambda > 0$ we let

$$\begin{aligned}
\bar{A}^+_{\varrho,\Lambda} &= (\bar{z}_\varrho^{\Lambda,+}, 0, \varrho^2) \in \mathbb{R}^{\bar{m}+1} \times \mathbb{R}^N \times \mathbb{R},\\
\bar{A}_{\varrho,\Lambda} &= (\Lambda\varrho, 0, 0, 0) \in \mathbb{R} \times \mathbb{R}^{\bar{m}} \times \mathbb{R}^N \times \mathbb{R},\\
\bar{A}^-_{\varrho,\Lambda} &= (\bar{z}_\varrho^{\Lambda,-}, 0, -\varrho^2) \in \mathbb{R}^{\bar{m}+1} \times \mathbb{R}^N \times \mathbb{R}.
\end{aligned} \tag{7.14}$$

When we in the following state that a constant c depends on $\bar{\mathcal{L}}$, $c = c(\bar{\mathcal{L}})$, then c depends on $\bar{\mathcal{L}}$ through $\mathcal{L}$ and hence c depends on λ, B, and the constants describing the Hölder continuity of the coefficients $a_{i,j}$ and a_i. We claim that the following theorems are true. Let $Q_{M,r}(z_0,t_0) = (z_0,t_0) \circ Q_{M,r}$, $\bar{A}_{\varrho,\Lambda}(z_0,t_0) = (z_0,t_0) \circ \bar{A}_{\varrho,\Lambda}$. In Theorem 7.2, d_K is now defined relative the structure of $\bar{\mathcal{L}}$.

Theorem 7.1. *Let $\bar{\Omega}_{f,2r_0}$ be an admissible local Lip_K-domain, with Lip_K-constants M, r_0. Then there exist $\Lambda = \Lambda(N,M,\bar{\mathcal{L}})$, $1 \leq \Lambda < \infty$, and $c_0 = c_0(N,M,\bar{\mathcal{L}})$, $1 \leq c_0 < \infty$, such that the following is true. Assume that u is a non-negative solution to $\mathcal{K}u = 0$ in $\bar{\Omega}_{f,2r_0}$ and that u vanishes continuously on $\bar{\Delta}_{f,2r_0}$. Let $\varrho_0 = r_0/c_0$, introduce*

$$m^+ = u(\bar{A}^+_{\varrho_0,\Lambda}),\ m^- = u(\bar{A}^-_{\varrho_0,\Lambda}), \tag{7.15}$$

and assume that $m^- > 0$. Then there exist constants $c_1 = c_1(N,M,\bar{\mathcal{L}})$, $1 \leq c_1 < \infty$, $c_2 = c_2(N,M,\bar{\mathcal{L}},m^+/m^-)$, $1 \leq c_2 < \infty$, such that if we let $\varrho_1 = \varrho_0/c_1$, then

$$u(z,t) \leq c_2 u(\bar{A}_{\varrho,\Lambda}(z_0,t_0)),$$

whenever $(z,t) \in \bar{\Omega}_{f,2r_0} \cap Q_{M,\varrho/c_1}(z_0,t_0)$, for some $0 < \varrho < \varrho_1$ and $(z_0,t_0) \in \bar{\Delta}_{f,\varrho_1}$.

Theorem 7.2. *Let* $\bar\Omega_{f,2r_0}$ *be an admissible local* Lip_K*-domain, with* Lip_K*-constants* M, r_0. *Then there exist* $\Lambda = \Lambda(N,M,\bar{\mathcal{L}})$, $1 \le \Lambda < \infty$, *and* $c_0 = c_0(N,M,\bar{\mathcal{L}})$, $1 \le c_0 < \infty$, *such that the following is true. Assume that* u *and* v *are non-negative solutions to* $\mathcal{K}u = 0$ *in* $\bar\Omega_{f,2r_0}$ *and that* v *and* u *vanish continuously on* $\bar\Delta_{f,2r_0}$. *Let* $\varrho_0 = r_0/c_0$, *introduce*

$$\begin{aligned} m_1^+ &= v(\bar A^+_{\varrho_0,\Lambda}),\ m_1^- = v(\bar A^-_{\varrho_0,\Lambda}), \\ m_2^+ &= u(\bar A^+_{\varrho_0,\Lambda}),\ m_2^- = u(\bar A^-_{\varrho_0,\Lambda}), \end{aligned} \tag{7.16}$$

and assume that $m_1^- > 0, m_2^- > 0$. *Then there exist constants* $c_1 = c_1(N,M,\bar{\mathcal{L}})$, $c_2 = c_2(N,M,\bar{\mathcal{L}},m_1^+/m_1^-,m_2^+/m_2^-)$, $\sigma = \sigma(N,M,\bar{\mathcal{L}},m_1^+/m_1^-,m_2^+/m_2^-)$, $1 \le c_1, c_2 < \infty$, $\sigma \in (0,1)$, *such that if we let* $\varrho_1 = \varrho_0/c_1$, *then*

$$\left|\frac{v(z,t)}{u(z,t)} - \frac{v(\tilde z,\tilde t)}{u(\tilde z,\tilde t)}\right| \le c_2\left(\frac{d_K((z,t),(\tilde z,\tilde t))}{\varrho}\right)^\sigma \frac{v(\bar A_{\varrho,\Lambda}(z_0,t_0))}{u(\bar A_{\varrho,\Lambda}(z_0,t_0))},$$

whenever $(z,t),(\tilde z,\tilde t) \in \bar\Omega_{f,2r_0} \cap Q_{M,\varrho/c_1}(z_0,t_0)$, *for some* $0 < \varrho < \varrho_1$ *and* $(z_0,t_0) \in \bar\Delta_{f,\varrho_1}$.

Remark 7.1. Note that the operator $\bar{\mathcal{L}}$ is an operator in non-divergence form and as the coefficients $a_{i,j}$ and a_i are only assumed to be Hölder continuous, the definition of the Green function may be somewhat problematic. Hence the proofs of Theorem 7.1 and Theorem 7.2 should be done without introducing the Green function. By the same reasons we here do not formulate a version of Theorem 1.3 for the operator $\bar{\mathcal{L}}$. In the end, the proofs of Theorem 7.1 and Theorem 7.2 will appear elsewhere.

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