Semilinear equations on nilpotent Lie groups: global existence and blow-up of solutions.

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Abstract

In this note we consider a semilinear Cauchy problem on a nilpotent Lie group. We extend a classical result by Fujita about the global existence and the blow-up of solutions.

1. Introduction. The aim of this note is to extend to the case of nilpotent Lie groups some classical results [4], [15] concerning global existence and blow-up of solutions to a semilinear Cauchy problem. We assume that $(R^N, \circ)$ is a Lie group with stratified Lie algebra $\mathcal{G} = \bigoplus_{j=1}^{s_0} \mathcal{G}_j$. Let $\{X_1, \dots, X_m\}$ be a basis of $\mathcal{G}_1$ and let L be the second order differential operator in $\mathbb{R}^{N+1}$

$$L = \sum_{j=1}^{m} X_j^2 - \partial_t. \tag{1.1}$$

We refer to Section 2 where more precise hypotheses and additional definitions and notations are given. We stress that L is an hypoelliptic operator since the vector fields $X_1, \dots, X_m$ verify Hörmander condition (2.1). In this paper, we consider the following semilinear Cauchy problem

$$\begin{cases} Lu = -u^p & \text{in } \mathbb{R}^N \times]0, T[\\ u(x, 0) = a(x) & x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

Here $p > 1$, (x, t) denotes the point in $\mathbb{R}^N \times \mathbb{R}$ and the initial data a is a continuous, bounded, non-negative and non identically zero function. We study problem (1.2) via the integral equation

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;y,0)a(y)dy + \int_0^t \int_{R^N} \Gamma(x,t;y,s)u^p(y,s)dyds \equiv u_0(x,t) + \Phi u(x,t), \tag{1.3}$$

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where $\Gamma(\cdot,\cdot;y,s)$ denotes the fundamental solution of L with pole in (y,s). More precisely, we call a solution of (1.2) a function $u \in C \cap L^\infty(\mathbb{R}^N \times [0,T[;[0,+\infty[)$ which solves the integral equation (1.3) in the strip $\mathbb{R}^N \times [0,T[$. A function u which is a solution of (1.2) for every positive T is said a global solution of (1.2). We remark that, actually, a solution of (1.2) is a smooth positive function in $\mathbb{R}^N \times]0,T[$ and it is a solution of (1.2) in the classical sense.

In the following statement Q denotes the homogeneous dimension of $(\mathbb{R}^N, \circ)$ (see (2.3)). Our main result reads

Theorem 1.1 *Let $p^* = 1 + \frac{2}{Q}$. If $1 < p \le p^*$ then no global solution of (1.2) exists for any initial data. If $p > p^*$ and if the initial data a is suitably small (see (3.13)) then there exists a unique global solution to (1.2).*

The simplest example of nilpotent stratified Lie group is $(\mathbb{R}^N, +)$. In this case L in (1.1) is the heat operator and the theorem is a classical result by Fujita [4] for $p \ne p^*$ and by Hayakawa [5] for $p = p^*$. Actually, our method is closely inspired by the papers [4], [15] and it relies on the remarkable global estimates (2.7) and (2.8) of the fundamental solution given by Varopoulos [13].

A classical example of non-abelian stratified Lie group is the Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ)$. The group law in $\mathbb{H}^n$ is given by

$$(x,y,s)\circ(x',y',s') = \left(x + x', y + y', s + s' + \frac{1}{2}(x\cdot y' - x'\cdot y)\right),$$

for every $(x,y,s),(x',y',s') \in \mathbb{R}^{2n+1}$. In this case

$$L = \left(\partial_x - \frac{y}{2}\partial_s\right)^2 + \left(\partial_y + \frac{x}{2}\partial_s\right)^2 - \partial_t$$

is a degenerate parabolic operator in $\mathbb{R}^{2n+2}$. The homogeneous dimension of $\mathbb{H}^n$ is $Q = 2n+2$.

Fujita's results have been extended in several directions over the past years. In [7] and [12] a wide survey of the related literature is presented. Recently, in [9], the author has considered problem (1.2) with L in a class of Kolmogorov-Fokker-Planck type operators.

The paper is organized as follows. In Section 2 we present the necessary background material concerning homogeneous structures on nilpotent stratified Lie groups. In Section 3 we prove Theorem 1.1.

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2. Notations and preliminary results. By reader's convenience and in order to present a reasonably self-contained exposition, in this section we briefly recall some known facts about stratified Lie groups. More details about this topic can be found, for example, in [10] and in [2], [11], [14].

Let $(\mathbb{R}^N, \circ)$ be a Lie group and let $(\mathcal{G}, [,])$ denote the Lie algebra of the $\circ$-left-invariant vector fields with the usual Lie bracket. We assume the two following hypotheses:

(H_1) *$\mathcal{G}$ is nilpotent of step s_0, i.e. $\mathcal{G}^{(s_0)} \neq \{0\}$ and $\mathcal{G}^{(s_0+1)} = \{0\}$, where $\mathcal{G}^{(1)} \equiv \mathcal{G}$ and $\mathcal{G}^{(k+1)} \equiv [\mathcal{G}, \mathcal{G}^{(k)}]$ for $k \geq 1$;*

(H_2) *$\mathcal{G}$ is stratified, i.e. $\mathcal{G}$ admits a direct sum decomposition*

$$\mathcal{G} = \bigoplus_{k=1}^{s_0} \mathcal{G}_k$$

such that $\mathcal{G}_{k+1} = [\mathcal{G}_1, \mathcal{G}_k]$ for $1 \leq k < s_0$;

For a fixed basis $\{X_1, \dots, X_m\}$ of $\mathcal{G}_1$, we consider the differential operator

$$L = \sum_{j=1}^{m} X_j^2 - \partial_t.$$

L is an hypoelliptic operator since, by (H_2), the vector fields $X_1, \dots, X_m$ verify the classical Hörmander condition (see [6])

$$\text{rank } \mathcal{L}(X_1, \dots, X_m)(x) = N, \qquad \forall x \in \mathbb{R}^N, \tag{2.1}$$

where $\mathcal{L}(X_1, \dots, X_m)$ denotes the Lie algebra generated by $X_1, \dots, X_m$.

We define in $\mathbb{R}^N$ a distance d suitable for the study of L: the Carnot-Caratheodory (or control) distance. Condition (2.1) allows to prove that it is always possible to join two points $x, y \in \mathbb{R}^N$ by a curve that stays tangent to the fields $X_1, \dots, X_m$. Let us denote by $C(T)$ the class of all absolutely continuous curves $\gamma : [0, T] \longrightarrow \mathbb{R}^N$ that almost everywhere satisfy

$$\gamma'(t) = \sum_{j=1}^{m} a_j(t) X_j(\gamma(t))$$

for some a_j, $1 \leq j \leq m$, a.e. continuous functions such that

$$\sum_{j=1}^{m} a_j^2(t) \leq 1, \qquad \forall t \in [0, T].$$

Then, for every $x, y \in \mathbb{R}^N$, we define

$$d(x, y) = \inf\{T \mid \exists \gamma \in C(T) \text{ s.t. } \gamma(0) = x,\ \gamma(T) = y\}.$$

It was proved in [1] that $d(x,y) < \infty$ for every $x, y \in \mathbb{R}^N$ and that, actually, d is a distance. Moreover, for every compact subset K of $\mathbb{R}^N$, there exist constants C_1, C_2 such that

$$C_1\|x-y\| \le d(x,y) \le C_2\|x-y\|^{\frac{1}{s_0}}, \qquad \forall x, y \in K,$$

where $\|\cdot\|$ denotes the Euclidean norm. More details about distances associated to vector fields can be found in [8].

We define a family of dilations $(\delta_\lambda)_{\lambda>0}$ of $\mathbb{R}^N$ by setting, for $1 \le j \le s_0$ and $\lambda > 0$,

$$\delta_\lambda(v) = \lambda^j v, \qquad \forall v \in \exp(\mathcal{G}_j).$$

For every $\lambda > 0$, δ_λ is a Lie automorphism of $(\mathbb{R}^N, \circ)$. In particular, we have

$$\delta_\lambda(x \circ y) = \delta_\lambda(x) \circ \delta_\lambda(y), \qquad \forall x, y \in \mathbb{R}^N,\ \lambda > 0.$$

Moreover, the differential $d\delta_\lambda \equiv D_\lambda$ defines a family of Lie automorphisms of $\mathcal{G}$ adapted to its stratification, in the sense that

$$D_\lambda(X)(h) = \lambda^j X(\delta_\lambda(h)) \qquad \forall X \in \mathcal{G}_j,\ h \in \mathbb{R}^N,\ \lambda > 0. \tag{2.2}$$

In particular the principal part of L is homogeneous of degree two w.r.t. $(D_\lambda)_{\lambda>0}$.

For every $\lambda > 0$, the Jacobian determinant of δ_λ equals λ^Q where

$$Q = \sum_{j=1}^{s_0} j \cdot \dim\mathcal{G}_j. \tag{2.3}$$

Therefore it seems natural to call Q *homogeneous dimension of* $\mathbb{R}^N$ *w.r.t.* $(\delta_\lambda)_{\lambda>0}$. Clearly $Q \ge N$.

There is a remarkable link between the control distance d and the homogeneous Lie group structure on $\mathbb{R}^N$. Indeed, we have

$$d(h \circ x, h \circ y) = d(x,y) \qquad \forall x, y, h \in \mathbb{R}^N,$$
$$d(0, \delta_\lambda(x)) = \lambda d(0,x) \qquad \forall x \in \mathbb{R}^N,\ \lambda > 0.$$

By setting

$$|x| = d(0,x), \qquad x \in \mathbb{R}^N, \tag{2.4}$$

we define a homogeneous norm on $\mathbb{R}^N$, i.e. a function $|\cdot| \in C(\mathbb{R}^N; [0, +\infty[)$ such that

i) $|x| = 0$ if and only if $x = 0$;

ii) $|x| = |x^{-1}|$;

iii) $|\delta_\lambda(x)| = \lambda|x|$.

Moreover $|\cdot|$ satisfies the triangle inequality

$$|x \circ y| \le |x| + |y| \qquad \forall x, y \in R^N. \tag{2.5}$$

We denote by

$$B_d(x,r) = \{y \mid d(x,y) < r\}$$

the d-ball of center x and radius $r > 0$. Since the Lebesgue measure is a Haar measure in $(\mathbb{R}^N, \circ)$, we have that

$$|B_d(x,r)| = r^Q |B_d(0,1)|. \tag{2.6}$$

The following polar coordinates formula holds:

$$\int_{B_d(0,r)} f(|x|)dx = Q|B_d(0,1)| \int_0^r f(\rho)\rho^{Q-1}d\rho,$$

for every measurable function f.

Let $\Gamma(\cdot,\cdot) = \Gamma(\cdot,\cdot;0,0)$ denote the fundamental solution to the operator L in (1.1) with pole in $(0,0)$. Let us recall that Γ is a positive solution of $Lu = 0$ in $\mathbb{R}^N \times]0,+\infty[$, $\Gamma(\cdot,t) = 0$ for $t \leq 0$ and $\|\Gamma(\cdot,t)\|_1 = 1$ for every $t > 0$.

The following remarkable global estimates of Γ and $X_j\Gamma$ hold (see [13]): there exists a positive constant C such that

$$\frac{1}{Ct^{\frac{Q}{2}}} \exp\left(-C\frac{|x|^2}{t}\right) \leq \Gamma(x,t) \leq \frac{C}{t^{\frac{Q}{2}}} \exp\left(-\frac{|x|^2}{Ct}\right), \tag{2.7}$$

and

$$|X_j\Gamma(x,t)| \leq \frac{C}{t^{\frac{Q+1}{2}}} \exp\left(-\frac{|x|^2}{Ct}\right), \tag{2.8}$$

for every $x \in \mathbb{R}^N$ and $t > 0$.

3. Proof of Theorem 1.1. The aim of this section is the proof of Theorem 1.1. The following estimate of the solutions of (1.2) is the key step in proving the non-existence part of Theorem 1.1.

Lemma 3.1 *If u is a solution to (1.2) then*

$$tu_0^{p-1}(0,t) \leq \frac{1}{p-1}, \qquad \forall t \in [0,T[, \tag{3.1}$$

where u_0 is defined by

$$u_0(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;y,0)a(y)dy.$$

Proof. For fixed t, $0 < t < T$, and $\varepsilon > 0$, we set

$$V_\varepsilon(x,s) = \Gamma(0,t+\varepsilon;x,s), \qquad (x,s) \in \mathbb{R}^N \times [0,t],$$

and

$$J_\varepsilon(s) = \int_{\mathbb{R}^N} V_\varepsilon(x,s)u(x,s)dx \qquad s \in [0,t].$$

We claim that

$$\frac{d}{ds}J_\varepsilon(s) = \int_{\mathbb{R}^N} V_\varepsilon(x,s)u^p(x,s)dx, \qquad s \in]0,t[. \tag{3.2}$$

Let us prove (3.2). We first observe that $V_\varepsilon \in C^\infty(\mathbb{R}^N \times [0,t];]0,+\infty[)$ and $\|V_\varepsilon(\cdot,s)\|_1 = 1$ for every $s \in [0,t]$. We also remark that $X_j^* = -X_j$, $1 \le j \le m$, and $\Gamma^*(x,t;y,s) = \Gamma(y,s;x,t)$ is a fundamental solution of L^*, formal adjoint of L. Therefore

$$L^* V_\varepsilon = \left(\sum_{j=1}^m X_j^2 + \partial_s\right) V_\varepsilon = 0 \tag{3.3}$$

in $\mathbb{R}^N \times]0,t[$.

Now, we consider a cut-off function $\rho \in C_0^\infty(\mathbb{R};[0,1])$ such that $\rho(\tau) = 1$ for $|\tau| \le 1$ and $\rho(\tau) = 0$ for $|\tau| \ge 2$. We set, for $n \in \mathbb{N}$,

$$\chi_n(x) = \rho\left(\frac{|x|}{n}\right) = \rho\left(\left|\delta_{\frac{1}{n}}(x)\right|\right), \qquad x \in \mathbb{R}^N, \tag{3.4}$$

and

$$J_\varepsilon^{(n)}(s) = \int_{\mathbb{R}^N} V_\varepsilon(x,s)u(x,s)\chi_n(x)dx, \qquad s \in [0,t].$$

In (3.4), $|\cdot|$ denotes the homogeneous norm defined in (2.4). By the monotone convergence theorem, we have

$$\lim_{n\to\infty} J_\varepsilon^{(n)}(s) = J_\varepsilon(s), \qquad \forall s \in [0,t].$$

Next, we prove that $\frac{d}{ds}J_\varepsilon^{(n)}$ converges uniformly in $[0,t]$ to the right hand side of (3.2) as n goes to infinity. Indeed, we have

$$\begin{aligned}
\frac{d}{ds}J_\varepsilon^{(n)} &= \int_{\mathbb{R}^N} (u\partial_s V_\varepsilon + V_\varepsilon \partial_s u)\,\chi_n dx \\
&= \int_{\mathbb{R}^N} V_\varepsilon u^p \chi_n dx + \int_{\mathbb{R}^N} \left(u\chi_n \partial_s V_\varepsilon + V_\varepsilon \chi_n \sum_{j=1}^m X_j^2 u\right) dx \\
&\equiv I_1^{(n)} + I_2^{(n)}.
\end{aligned}$$

Using the upper estimate of the fundamental solution, we obtain that, for some positive constant C,

$$0 \leq \int\limits_{\mathbb{R}^N} V_\varepsilon(x,s) u^p(x,s) dx - I_1^{(n)}(s) \tag{3.5}$$

$$\leq \|u\|_\infty^p \int\limits_{\mathbb{R}^N} \frac{C}{(t+\varepsilon-s)^{\frac{Q}{2}}} \exp\left(-C\frac{|x|^2}{t+\varepsilon-s}\right)(1-\chi_n(x))dx \longrightarrow 0 \tag{3.6}$$

by the dominated convergence theorem, as n tends to infinity uniformly in s.

Concerning $I_2^{(n)}$, by (3.3) and by some integration by parts, we get

$$I_2^{(n)} = \sum_{j=1}^{m} \int\limits_{\mathbb{R}^N} (uV_\varepsilon X_j^2 \chi_n + 2uX_j V_\varepsilon X_j \chi_n) dx.$$

We observe that, by (2.2),

$$X_j \chi_n(x) = \frac{1}{n}(X_j\chi_1)(\delta_{\frac{1}{n}}(x)). \tag{3.7}$$

Thus, by the estimate (2.8), for some positive constant C, we have

$$\left| \int\limits_{\mathbb{R}^N} u(x,s) X_j V_\varepsilon(x,s) X_j\chi_n(x) dx \right| \leq \|u\|_\infty \int\limits_{\mathbb{R}^N} \frac{C}{\varepsilon^{\frac{Q+1}{2}}} \exp\left(-\frac{|x|^2}{C(t+\varepsilon)}\right) |X_j\chi_n(x)|\, dx =$$

(by (3.7) and by changing variable of integration $y = \delta_{\frac{1}{n}}(x)$)

$$= \frac{Cn^{Q-1}\|u\|_\infty}{\varepsilon^{\frac{Q+1}{2}}} \int\limits_{\mathbb{R}^N} \exp\left(-\frac{|\delta_n(y)|^2}{C(t+\varepsilon)}\right) |(X_j\chi_1)(y)| dy$$

$$\leq \frac{Cn^{Q-1}\|u\|_\infty \|X_j\chi_1\|_\infty}{\varepsilon^{\frac{Q+1}{2}}} \int\limits_{\mathbb{R}^N} \exp\left(-\frac{|\delta_n(y)|^2}{C(t+\varepsilon)}\right) dy$$

(by changing the variable of integration $x = \delta_n(y)$)

$$= \frac{1}{n}\left(\frac{C\|u\|_\infty \|X_j\chi_1\|_\infty}{\varepsilon^{\frac{Q+1}{2}}} \int\limits_{\mathbb{R}^N} \exp\left(-\frac{|x|^2}{C(t+\varepsilon)}\right) dx \right).$$

In the same way one can handle the term

$$\int\limits_{\mathbb{R}^N} uV_\varepsilon X_j^2 \chi_n dx,$$

in order to show that $I_2^{(n)}$ converges uniformly to 0 as n goes to infinity. This concludes the proof of (3.2).

By means of Hölder's inequality, from (3.2) we get

$$\frac{d}{ds} J_\varepsilon \geq (J_\varepsilon)^p. \tag{3.8}$$

Finally, integrating (3.8) on $[0, t]$ and letting ε go to zero, we obtain (3.1).

$\square$

Proof of Theorem 1.1.
[The case $1 < p < p^*$**]** Let u be a solution of (1.2). It is non-restrictive to consider the case $a(0) > 0$, so we can choose $a_0, \delta > 0$ such that $a(x) \geq a_0$ for $x \in B_d(0, \delta)$. Thus, for every $t \in [\delta, T[$, we have

$$u_0(0, t) \geq a_0 \int\limits_{B_d(0,\delta)} \Gamma(y^{-1}, t) dy$$

(by the lower estimate of the fundamental solution)

$$\geq \frac{a_0}{C_1 t^{\frac{Q}{2}}} \int\limits_{B_d(0,\delta)} \exp\left(-C_1 \frac{|y|^2}{\delta}\right) dy = \frac{C_2}{t^{\frac{Q}{2}}}, \tag{3.9}$$

for some positive constants C_1, C_2 depending only on L. Combining (3.1) with (3.9), one has that, if $1 < p < p^*$, then u cannot be a global solution.

[The case $p = p^*$**]** By contradiction, we suppose that there exists a global solution u to (1.2). From (3.1) for $p = p^*$, by the lower estimate of the fundamental solution, we get

$$\int\limits_{\mathbb{R}^N} \exp\left(-C_1 \frac{|y^{-1} \circ x|^2}{t}\right) a(y) dy \leq C_1 t^{\frac{Q}{2}} u_0(x, t) \leq C_2, \tag{3.10}$$

for some positive constants C_1, C_2 depending only on L. Thus, as t tends to infinity in (3.10), by the monotone convergence theorem, we obtain

$$\|a\|_1 \leq C_2.$$

Regarding $u(\cdot, t)$ as initial value, we have

$$\|u(\cdot, t)\|_1 \leq C_2, \qquad \forall t \geq 0. \tag{3.11}$$

For fixed $\alpha > 0$, we set $v(\cdot, t) = u(\cdot, t + \alpha)$. Once more using the estimates of the fundamental solution, it is not difficult to verify that v dominates a Gaussian function. Precisely, there exists a positive constant C_3 such that

$$v(x, t) \geq \frac{1}{C_3 (t + \alpha)^{\frac{Q}{2}}} \exp\left(-\frac{C_3 |x|^2}{t + \alpha}\right), \qquad \forall (x, t) \in \mathbb{R}^N \times]0, +\infty[. \tag{3.12}$$

Since v is a solution to the integral equation (1.3), by (3.12), we have

$$\|v(\cdot,t)\|_1 \geq \int_{\mathbb{R}^N} \int_0^t \int_{\mathbb{R}^N} \Gamma(x,t;y,s) \left(\frac{1}{C_3 (s+\alpha)^{\frac{Q}{2}}} \exp\left(-\frac{C_3 |y|^2}{s+\alpha} \right) \right)^{1+\frac{2}{Q}} dydsdx$$

(by Tonelli's theorem and since $\|\Gamma(\cdot, t-s)\|_1 = 1$ for $t > s$)

$$= C_3^{-1-\frac{2}{Q}} \int_0^t (s+\alpha)^{-\frac{Q}{2}-1} \int_{\mathbb{R}^N} \exp\left(-\left(1+\frac{2}{Q}\right) \frac{C_3|y|^2}{s+\alpha} \right) dyds$$

(performing the change of variable $\xi = \delta_{(s+\alpha)^{-\frac{1}{2}}}(y)$ and by a straightforward computation)

$$= C_4 \log\left(\frac{t+\alpha}{\alpha} \right),$$

for some $C_4 > 0$. On the other hand, obviously, estimate (3.11) also holds for the function v. Thus we have a contradiction.

[**The case** $p > p^*$] We are looking for a solution in the class of bounded, continuous functions. Therefore the uniqueness of the solution follows from standard arguments. We refer, for example, to [3], Chap.2.

Concerning the existence, we claim that there exist some constants $\delta_0, \alpha > 0$ such that, if the following estimate of the initial data holds

$$a \leq \delta_0 \Gamma(\cdot, \alpha) \tag{3.13}$$

then a global solution u to (1.2) exists. Moreover

$$u(x,t) \leq M\Gamma(x, t+\alpha), \qquad \forall (x,t) \in \mathbb{R}^N \times [0, +\infty[,$$

for some constant $M > 0$.

Indeed, we first observe that, if (3.13) holds, then for every $(x,t) \in \mathbb{R}^N \times [0, +\infty[$,

$$u_0(x,t) \leq \delta_0 \int_{\mathbb{R}^N} \Gamma(y^{-1} \circ x, t) \Gamma(y, \alpha) dy = \delta_0 \Gamma(x, t+\alpha). \tag{3.14}$$

Here we have used the so-called reproduction property of the fundamental solution. Next, we define the recurrent sequence $(u_n)_{n\in\mathbb{N}}$ as follows

$$u_{n+1} = u_0 + \Phi u_n,$$

where Φu is as in (1.3). The sequence $(u_n)_{n\in\mathbb{N}}$ is monotone increasing as it can be easily verified by induction. We want to show that it is possible to choose $\delta_0, \alpha > 0$ in (3.13) in such a way that

$$u_n(x,t) \leq M\Gamma(x, t+\alpha), \qquad \forall (x,t) \in \mathbb{R}^N \times [0, +\infty[, \ n \in \mathbb{N}, \tag{3.15}$$

for some positive constant M. If (3.15) holds then, by the monotone convergence theorem, $u = \sup_{n\in\mathbb{N}} u_n$ is the global solution of (1.2).

In order to prove (3.15), we set $\delta_{n+1} = \delta_0 + \delta_n^p$, $n \in \mathbb{N}$, and we claim that, for suitable α,

$$u_n(x,t) \leq \delta_n \Gamma(x, t+\alpha), \qquad (x,t) \in \mathbb{R}^N \times [0, +\infty[,\ n \in \mathbb{N}. \tag{3.16}$$

Now, for small δ_0, $(\delta_n)_{n\in\mathbb{N}}$ is convergent. Therefore (3.15) follows from (3.16). We prove (3.16) by induction.

For $n = 1$, we have

$$\begin{aligned} u_1(x,t) \leq& \delta_0 \Gamma(x,t+\alpha) + \delta_0^p \int_0^t \int_{\mathbb{R}^N} \Gamma(x,t;y,s)\Gamma^p(y,s+\alpha)dyds \\ \leq& \delta_1 \Gamma(x,t+\alpha), \end{aligned}$$

since, by estimate (2.7) of the fundamental solution,

$$\begin{aligned} \Phi\Gamma(x,t+\alpha) =& \int_0^t \int_{\mathbb{R}^N} \Gamma(x,t;y,s)\Gamma^p(y,s+\alpha)dyds \\ \leq& \int_0^t \int_{\mathbb{R}^N} \Gamma(x,t;y,s)\Gamma(y,s+\alpha) \left(\frac{C}{(s+\alpha)^{\frac{Q}{2}}} \right)^{p-1} \exp\left(-\frac{(p-1)|y|^2}{C(s+\alpha)} \right) dyds \end{aligned}$$

(by the reproduction property of Γ)

$$\leq \Gamma(x,t+\alpha) \int_0^{+\infty} \left(\frac{C}{(s+\alpha)^{\frac{Q}{2}}} \right)^{p-1} ds \leq \Gamma(x,t+\alpha), \tag{3.17}$$

since $p > p^*$ and by choosing α sufficiently great.

Finally, supposing that (3.16) holds for a fixed $n \in \mathbb{N}$, we have

$$\begin{aligned} u_{n+1}(x,t) &= u_0(x,t) + \Phi u_n(x,t) \\ &\leq \delta_0 \Gamma(x,t+\alpha) + \delta_n^p \Phi\Gamma(x,t+\alpha) \end{aligned}$$

(by (3.17))

$$\leq \delta_{n+1}\Gamma(x,t+\alpha).$$

This proves (3.16) and thus concludes the proof of the theorem.

□

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