Decisions Econ Finan
DOI 10.1007/s10203-007-0076-6

Path dependent volatility

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Received: 30 November 2006 / Accepted: 19 September 2007

Abstract We propose a general class of non-constant volatility models with dependence on the past. The framework includes path-dependent volatility models such as that by Hobson and Rogers and also path dependent contracts such as options of Asian style. A key feature of the model is that market completeness is preserved. Some empirical analysis, based on the comparison with standard local volatility and Heston models, shows the effectiveness of the path dependent volatility. In particular, it turns out that, when large market movements occur, the tracking errors of Heston minimum-variance hedging are up to twice the hedging errors of a path dependent volatility model.

Keywords Option pricing · Kolmogorov equations · Volatility modeling

JEL Classification CO_2

Mathematics Subject Classification (2000) 35K65 · 91B28

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1 Introduction

In the Black–Merton–Scholes option pricing theory (Black and Scholes 1973; Merton 1973), the underlying asset is modeled as a geometric Brownian motion whose dynamic under the risk neutral measure is given by

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t. \tag{1.1}$$

In (1.1) r denotes the locally riskless interest rate and σ is the volatility. Under the assumption that both the parameters are constant, model (1.1) leads to closed formulas for plain vanilla options. Nowadays the Black and Scholes formula is widely used in practice, to the extent that prices of call and put options are usually quoted in terms of the so-called Black and Scholes implied volatility. However it is also apparent that the prices at which derivatives are traded are inconsistent with the assumption of a constant volatility: indeed especially after the market crash of 1987, the strong empirical evidences of the stochastic nature of the volatility stimulated the development of more realistic models. The overall aim of a non-constant volatility model is twofold: on one hand, to produce prices of plain vanilla options which agree with the observed volatility surfaces and to price exotic options consistently; on the other hand, to find the correct replicating strategy in order to improve the hedging performance.

The first task is usually not difficult to achieve: from a theoretical point of view, any model which depends on a sufficiently large number of parameters can be calibrated to fit (or at least approximate) market prices. But it should be emphasized that any calibration procedure depends on the quantity and quality of the available data: in particular, since only option prices corresponding to a finite number of maturities and strikes are quoted, generally the fitting of prices cannot usually be done in a unique way. Then the essential and hard problem is to determine the "correct" hedging strategy: indeed it is well-known that the hedge parameters are strongly model-dependent even for call and put options (cf. Davis 2004; Cont 2006).

In a local volatility (henceforth LV) model the volatility is supposed to be a deterministic function of the time and current price of the underlying asset. The main advantages are that the market is complete and in principle it is possible to specify the volatility function in such a way that option prices given by the model agree with market prices. On the other hand the empirical study by Dumas et al. (1998) shows that, for hedging purposes, the local volatility underperforms an ad-hoc use of the Black and Scholes model (which consequently should be preferable for its parsimony). The conclusion in Dumas et al. (1998) is that, as far as one aims to preserve market completeness, a volatility model depending on the whole past trajectory of the asset (instead of the current price alone) should be investigated.

The first results in this direction were obtained by Hobson and Rogers who proposed in 1998 a volatility model defined in terms of the difference between the current price and an exponentially weighted average of past prices. Precisely, in a Wiener space with one-dimensional Brownian motion W, we denote by S_t the stock price and by M_t and D_t, respectively, the *trend* and the *deviation* processes defined by

$$M_t = \lambda e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} Z_s ds, \quad \lambda > 0 \tag{1.2a}$$

and

$$D_t = Z_t - M_t, \tag{1.2b}$$

where $Z_t = \log(e^{-rt} S_t)$ is the log-discounted price process. The function $e^{\lambda s}$ in (1.2) is called *the average weight:* the parameter λ describes the rate at which past prices are discounted.

Hobson and Rogers assume that S_t is an Itô process, solution to the stochastic differential equation (SDE)

$$dS_t = \mu(D_t) S_t dt + \sigma(D_t) S_t dW_t. \tag{1.3}$$

In (1.3), μ and $\sigma > 0$ are deterministic functions satisfying usual hypotheses in order to guarantee that the system of SDEs (1.2)–(1.3) has a solution. A key feature of the model is that the process (S_t, D_t) is Markovian (cf. Lemma 3.1 in Hobson and Rogers 1998). Thus, the price U of an option with maturity T, is given by

$$U(S_t, t) = e^{-r(T-t)} K u(r(T-t) + \log(S_t/K), \quad M_t - \log K, T - t),$$

where $u = u(x, y, t)$ is the solution to the following Cauchy problem:

$$\frac{\sigma^2(x-y)}{2}(\partial_{xx} u - \partial_x u) + (x-y)\lambda \partial_y u - \partial_t u = 0, \quad \text{in } \mathbb{R}^2 \times [0, T], \tag{1.4}$$

$$u(x, y, 0) = (e^x - 1)^+ \quad \text{for } (x, y) \in \mathbb{R}^2. \tag{1.5}$$

Path dependent volatility models are supported by the empirical evidence about the dependence of the volatility with respect to the deviation D: Fig. 1 plots implied volatilities against adjusted log-moneyness for options on the S&P 500 index in the years 2003–2004. The implied volatilities are grouped by ranges of values of D. It is immediate to observe on the figure that implied volatilities increase as the D decreases (see also the empirical analysis in Platania and Rogers 2005, Sect. 2). This enlightens the well-known negative correlation between volatility and market prices.

We emphasize that no additional source of risk has been added in the Hobson and Rogers (henceforth, HR) model: therefore, unlike many other non-constant volatility models, the market is complete and the arbitrage argument which underlies the Black and Scholes theory is preserved. While keeping the market completeness, the HR model is able to approximate observed volatility surfaces (see the analysis in Hobson and Rogers 1998 and Di Francesco et al. 2006).

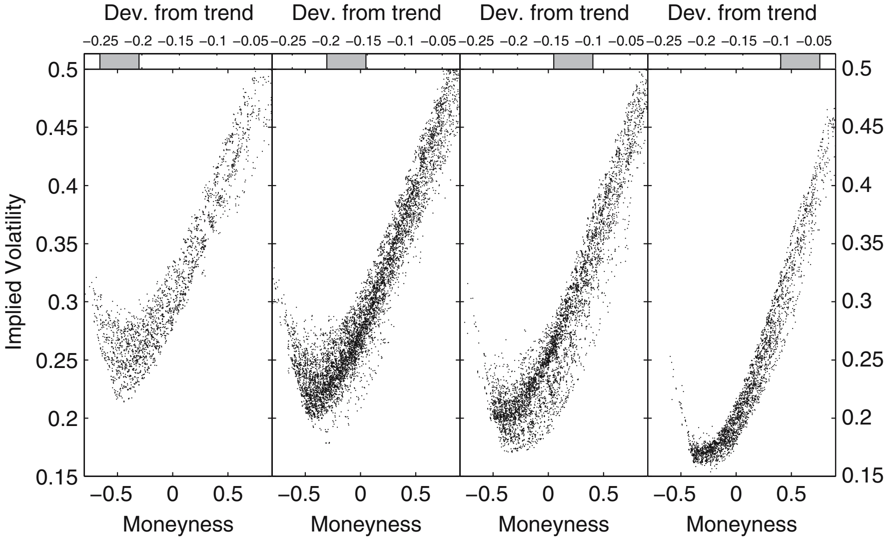


Fig. 1 Effects of the deviation from the trend on marked implied volatilities. The implied volatilities are plotted against adjusted log-moneyness $\log(e^{r(T-t)}S_t/K)/\sqrt{T-t}$ and grouped by different ranges of D_t as shown by the *bar* in the *top of each panel*. Data from the S&P 500 index options, years 2003–2004

We also remark that a path dependent volatility incorporates information on the past and then, once it is calibrated to the market, the model somehow "knows" the behaviour of investors in different market circumstances and can also keep into account of the positive or negative trend of the asset. For instance, unlike standard local or stochastic volatility models, in case of a sudden fall of the market a path dependent volatility model is designed to automatically increase the level of volatility in order to undertake the market dynamics in a more natural way. This is the reason why it seems that path dependent volatility models do not need to be continuously re-calibrated (which is a well-known disadvantage of local volatility models) and have better out-of-sample performances (see analysis in Foschi and Pascucci 2005).

Thanks to these fine features, the HR model raised some interest among academics and practitioners: the problem of parameters calibration (λ in the average weight and the volatility function σ) was studied by Platania and Rogers (2005), Figà-Talamanca and Guerra (2006). Di Francesco and one of the authors studied the numerics of the model: we explicitly remark that the PDE in (1.4) is not uniformly parabolic even if it is hypoelliptic by Hörmander's theorem (Hörmander 1967). In particular the convergence of finite difference schemes does not follow by standard arguments but it is proved in Di Francesco as a consequence of some a priori estimates for solutions to (1.4)–(1.5) provided in Di Francesco and Pascucci (2005).

Recently the free boundary and optimal stopping problems for American options in the HR model were studied in Pascucci (2007) and Di Francesco et al. (2007). An extension to the framework of term-structure modeling was given by Chiarella and Kwon (2000). Hahn et al. (2007) considered the HR dynamic in a portfolio optimization problem. Hubalek et al. (2004) proposed a generalization to better fit market smiles. The robustness of the HR model with respect to the data and parameters was studied by Blaka Hallulli and Vargiolu (2005). Trifi (2006) shows that the HR model is the continuous time limit of an ARCH-type model.

Next we mention some of the weak points of the HR model. As noted in Blaka Hallulli and Vargiolu (2005), some mathematical and economical concerns arise from the definition of the deviation process D in (1.2). Indeed D involves the path of the underlying asset on all its past $]-\infty, t[$. The requirement of an infinite horizon in the past obviously raises practical problems since only finite time series are available so that misspecifications in the model are unavoidable. To overcome this problem, in Blaka Hallulli and Vargiolu (2005) the following extension of the HR model has been proposed: the volatility is specified as

$$\sigma(S_t) = \sigma(S_t, Y_t, S_{t-\tau})$$

where

$$Y_t = \int_{t-\tau}^{t} \mathrm{e}^{-\lambda(t-v)} f(S_v) \mathrm{d}v$$

where f is a strictly monotone function and τ is a given delay parameter (see also Gushchin and Küchler 2004). Unfortunately the conclusion in Blaka Hallulli and Vargiolu (2005) is that if τ is finite then the previous model cannot admit a Markovian realization so that it loses any appeal.

As a further remark, the average weight $\lambda \mathrm{e}^{-\lambda t}$ in (1.2) could not be flexible enough to take into account of the special properties of the underlying process that may arise, for instance, from stagionality effects, fusions of stocks, capitalization changes. This will also be verified later in Sect. 4.

In this paper we focus on these problems: we propose a simple generalization of the HR model and introduce a new class of models for asset prices with volatility dependent on the past. Our idea is to consider a more flexible deviation process defined in terms of a *generic* average weight, possibly corresponding to a *finite* time horizon. We call this the *path dependent volatility* (henceforth PDV) model. The notion of PDV model is sufficiently general to include the HR model and also path dependent derivatives such as Asian style options.

The paper is organized as follows. In Sect. 2, we introduce the PDV volatility and prove some results about the absence of arbitrage and completeness of the market in the framework of PDEs and martingales theories. In Sect. 3, we analyze some suitable transformations of the pricing PDE which seem to be more convenient for the numerical approximation. In Sect. 4, the path dependent volatility model is validated against market data and the performance compared with those of standard HR, LV and Heston stochastic volatility models. Section 5 contains conclusions and further research directions.

2 Path dependent volatility

In order to introduce the PDV model, we consider an average weight φ which is a non-negative, piecewise continuous and integrable function on $]-\infty, T]$. We also assume that φ is strictly positive in $[0, T]$ and we set

$$\Phi(t) = \int_{-\infty}^{t} \varphi(s)\mathrm{d}s. \tag{2.1}$$

We remark explicitly that φ may have compact support: in that case the domain of integration in (2.1) is bounded. Moreover we denote by r the risk free rate and $B_t = \mathrm{e}^{rt}$.

Next we define the average process as

$$M_t = \frac{1}{\Phi(t)} \int_{-\infty}^{t} \varphi(s) Z_s \mathrm{d}s, \quad t \in]0, T],$$

or equivalently

$$\mathrm{d}M_t = \tfrac{\varphi(t)}{\Phi(t)} (Z_t - M_t)\, \mathrm{d}t, \tag{2.2}$$

where $Z_t = \log(\mathrm{e}^{-rt} S_t)$ denotes the log-discounted price process whose dynamics, under the physical measure, is assumed to be

$$\mathrm{d}Z_t = \mu(Z_t - M_t)\mathrm{d}t + \sigma(Z_t - M_t)\mathrm{d}W_t, \tag{2.3}$$

and μ, σ are bounded, Hölder continuous functions and σ is uniformly strictly positive. Under these assumptions, it is known that (2.3)–(2.2) has a unique weak solution and *the couples* (Z, M) *and* (Z, D) *are Markovian processes*. Typical specifications of the average weights are given by the following examples:

- $\varphi(t) = \mathrm{e}^{P(t)} \max\{Q(t), 0\}$ where P, Q are suitable polynomial functions: the HR model corresponds to $P(t) = \lambda t$ and $Q(t) = 1$;
- $\varphi(t) = 1$ for $t \in [0, T]$ and null elsewhere: this corresponds to the geometric average of an Asian option;
- φ piecewise linear function.

Next we consider a self-financing strategy

$$\mathrm{d}V_t = \alpha_t \mathrm{d}S_t + \beta_t \mathrm{d}B_t,$$

and prove that the market is arbitrage free and complete. Precisely in the sequel we restrict ourselves to a Markovian setting: we set

$$\alpha_t = \alpha(t, Z_t, M_t), \quad \beta_t = \beta(t, Z_t, M_t)$$

where α, β are suitably regular, deterministic functions and we denote by

$$V_t = \alpha(t, Z_t, M_t)S_t + \beta(t, Z_t, M_t)B_t \tag{2.4}$$

the value of the portfolio. Moreover we set

$$f(t, Z_t, M_t) = B_t^{-1} V_t \tag{2.5}$$

the discounted value of V. The next theorem characterizes the self-financing property.

Theorem 2.1 *The following conditions are equivalent*:

1. *the portfolio in* (2.4) *is self-financing*;
2. *the function f in* (2.5) *solves the partial differential equation*

$$\frac{\sigma^2(z-m)}{2}(\partial_{zz} f - \partial_z f) + \frac{\varphi(t)}{\Phi(t)}(z-m)\partial_m f + \partial_t f = 0 \tag{2.6}$$

in $]0, T[\times\mathbb{R}^2$, *and the following relations hold*

$$\alpha(t, z, m) = \mathrm{e}^{-z}\partial_z \mathrm{f(t, z, m)}, \quad \beta(\mathrm{t, z, m}) = \mathrm{f(t, z, m)} - \partial_z \mathrm{f(t, z, m)}. \tag{2.7}$$

Proof By the self-financing condition, it holds

$$\mathrm{d}f = -rf\mathrm{d}t + B_t^{-1}(\alpha \mathrm{d}S_t + \beta \mathrm{d}B_t) =$$

(since $\beta \mathrm{d}B_t = r\beta B_t \mathrm{d}t = r(V_t - \alpha S_t)\mathrm{d}t$)

$$= \alpha B_t^{-1}(\mathrm{d}S_t - rS_t\mathrm{d}t) =$$

(since, by Itô formula, $\mathrm{d}S_t = B_t \mathrm{e}^{Z_t}\left(\mathrm{d}Z_t + \left(r + \frac{\sigma^2}{2}\right)\mathrm{d}t\right)$)

$$= \alpha \mathrm{e}^{Z_t}\left(\mathrm{d}Z_t + \frac{\sigma^2}{2}\mathrm{d}t\right). \tag{2.8}$$

On the other hand, by Itô formula and (2.2), we get

$$\mathrm{d}f(t, Z_t, M_t) = \left(\partial_t f + \frac{\sigma^2}{2}\partial_{zz} f + \frac{\varphi}{\Phi}(Z_t - M_t)\partial_m f\right)\mathrm{d}t + \partial_z f \mathrm{d}Z_t. \tag{2.9}$$

Comparing (2.8) and (2.9), by the uniqueness of the representation of an Itô process, we infer

$$\partial_z f(t, Z_t, M_t) = \alpha(t, Z_t, M_t)\mathrm{e}^{Z_t}. \tag{2.10}$$

Now we recall that, since by assumption φ is strictly positive on $[0, T]$, the conditions of the classical Hörmander's theorem are satisfied and the process (Z_t, M_t) has a *strictly positive* density on $\mathbb{R}^2$ for $t > 0$ (we also refer to the paper Di Francesco and Pascucci (2005) by Di Francesco and one of the authors for a direct proof of this result). Then (2.7) readily follows from (2.10). Analogously, by equating the dt-parts of (2.8) and (2.9) and using (2.10), we obtain the PDE (2.6).

We do not prove the inverse implication which is straightforward. □

Next we prove that in the PDV model the market is arbitrage-free and complete.

Corollary 2.2 *For any contingent claim $H = H(S_T, M_T)$, with $H \in L^1_{\text{loc}}(\mathbb{R}^2)$ and $H \geq 0$, there exists a unique self-financing and admissible*[1] *strategy replicating H. The strategy is determined by formulas* (2.7) *where f is the unique solution of the Cauchy problem for Eq.* (2.6) *with final condition*

$$f(T, z, m) = \mathrm{e}^{-rT} H(\mathrm{e}^z, m). \tag{2.11}$$

In particular, the market is arbitrage-free and complete:

$$H_t := e^{rt} f(t, Z_t, M_t) \tag{2.12}$$

is the arbitrage price of the claim H.

Proof The thesis is a direct consequence of Theorem 2.1 and of the existence and uniqueness results for degenerate parabolic equations of Kolmogorov type [which include (2.6)] proved by Di Francesco and Pascucci (2005) and by Polidoro (1995). □

The previous results can also be proved by using the martingale theory. Note that the dynamic of the stock price S_t is given by

$$\mathrm{d}S_t = \left(r + \mu(D_t) + \frac{\sigma^2(D_t)}{2} S_t\right) \mathrm{d}t + \sigma(D_t) S_t \mathrm{d}W_t$$

where the deviation process $D_t = Z_t - M_t$ satisfies the SDE

$$\mathrm{d}D_t = \left(\mu(D_t) - \frac{\varphi}{\Phi} D_t\right) \mathrm{d}t + \sigma(D_t) \mathrm{d}W_t.$$

Then we set

$$\theta(D_t) = \frac{\sigma(D_t)}{2} + \frac{\mu(D_t)}{\sigma(D_t)}$$

and consider the process

$$\widetilde{W}_t = W_t + \int_0^t \theta(D_s) \mathrm{d}s.$$

[1] Such that the value of the portfolio is non-negative.

Under suitable conditions on the coefficients (cf. for instance the Appendix in Hobson and Rogers 1998) the position

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \exp\left(-\frac{1}{2}\int_0^t \theta^2(D_s)\mathrm{d}s - \int_0^t \theta(D_s)\mathrm{d}W_s\right)$$

defines a probability measure Q on the filtration $\mathcal{F}_t$ of W, which is equivalent to P and such that $\widetilde{W}$ is a Q-Brownian motion. Then under Q we have

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma(D_t)S_t\mathrm{d}\widetilde{W}_t$$

so that the discounted price $\mathrm{e}^{-rt}S_t$ is a Q-martingale and the arbitrage price in (2.12) of the contingent claim H can be written as

$$H_t = \mathrm{e}^{-r(T-t)}E^Q\left(H \mid \mathcal{F}_t\right).$$

3 Some convenient transformation

For an European call with strike K, the pricing PDE (2.6) is coupled with the final condition

$$f(T, z, m) = \mathrm{e}^{-rT}(\mathrm{e}^z - K)^+.$$

By the change of variables

$$f(t, z, m) = Ku(T - t, z - \log K, m - \log K) \tag{3.1}$$

we obtain the equivalent Cauchy problem

$$\frac{\sigma^2(x-y)}{2}(\partial_{xx}u - \partial_x u) + \frac{\varphi(T-\tau)}{\Phi(T-\tau)}(x-y)\partial_y u - \partial_\tau u = 0,$$

$$\tau \in]0, T[, \ (x, y) \in \mathbb{R}^2, \tag{3.2}$$

$$u(0, x, y) = \mathrm{e}^{-rT}\left(\mathrm{e}^x - 1\right)^+, \quad x \in \mathbb{R}. \tag{3.3}$$

Note that problem (3.2)–(3.3) is independent of K and therefore *it allows to price all call options with different strikes in a single run.*

In view of the numerical approximation, we also consider the following further change of variables:

$$t = g(\tau) := -\log\Phi(T-\tau).$$

If $u(\tau, x, y) = v(g(\tau), x, y)$ then

$$\partial_\tau u = \frac{\varphi(T-\tau)}{\Phi(T-\tau)}\partial_t v$$

and Eq. (3.2) is equivalent to

$$a(x-y,t)\,(\partial_{xx}v-\partial_x v)+(x-y)\partial_y v-\partial_t v=0, \tag{3.4}$$

for $t \in]-\log\Phi(T), -\log\Phi(0)[$ and $(x,y)\in\mathbb{R}^2$, where

$$a(x-y,t)=\frac{\sigma^2(x-y)}{2\frac{\varphi}{\Phi}(g^{-1}(t))}.$$

For instance, in the HR model, $\frac{\varphi}{\Phi}\equiv\lambda$ and $g(t)=\lambda(t-T)$ so that the PDE has to be solved for $t\in[-\lambda T,0]$.

It is clear that in the case of constant volatility function σ in (2.3), the model reduces to the classical Black and Scholes framework independently of φ. In the case of an Asian option, the following change of variables

$$f(t,x,\eta)=u\left(t,x,\frac{\eta}{\Phi(T-t)}\right),\quad y=\frac{\eta}{\Phi(T-t)}$$

seems to be convenient. Indeed we have

$$\partial_\eta f=\frac{1}{\Phi(T-t)}\partial_y u,\quad \partial_t f=\frac{\varphi(T-t)}{\Phi^2(T-t)}\eta\partial_y u+\partial_t u,$$

and therefore u is solution to (3.2) if and only if

$$\frac{\sigma^2}{2}(\partial_{xx}f-\partial_x f)+\varphi(T-\tau)x\partial_\eta f-\partial_t f=0,\quad t\in]0,T[,\ (x,y)\in\mathbb{R}^2. \tag{3.5}$$

Note that $\varphi\equiv 1$ for a geometric average Asian option. We also remark that the explicit expression of the fundamental solution to Eq. (3.5), even for a generic φ, is known (cf. Barucci et al. 2001).

4 Empirical tests

In this section the PDV model is calibrated to real market data and compared with some standard non-constant volatility models, namely: the standard HR (Hobson and Rogers 1998), Dupire LV (Dupire 1997) and Heston (Heston 1993) stochastic volatility models. For the calibration of each model we use a least squares approach.

We begin by defining the weight function $\varphi(t)$ in terms of $g'(t)=\varphi(T-t)/\Phi(T-t)$ (using the notation introduced in Sect. 3). In particular we choose $g'(t)$ as a piecewise linear function defined by

$$g'(t)=\sum_{i=0}^{K}\alpha_i s_i(t),$$

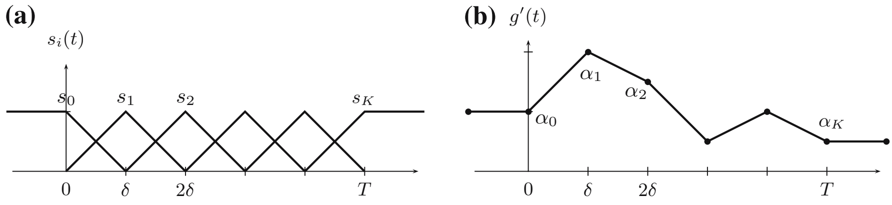
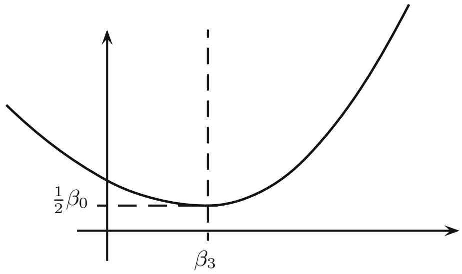


Fig. 2 The hat functions $s_i(t)$ for $i = 1, \ldots, K$ and $g'(t)$

Fig. 3 The σ^2 function for $\beta_0 = 1/2$, $\beta_1 = 1$, $\beta_2 = 1/2$ and $\beta_3 = 1$



where

$$
\begin{aligned}
s_0(t) &= \widetilde{s}(t/\delta)\chi_{[0,\delta)} + \chi_{(-\infty,0)}, \\
s_K(t) &= \widetilde{s}(t/\delta - K)\chi_{[T-\delta,T)} + \chi_{[T,\infty)}, \\
s_i(t) &= \widetilde{s}(t/\delta - i), \quad \text{for } i = 2, \ldots, K-1
\end{aligned}
$$

with $\delta = T/K$,

$$\widetilde{s}(t) = (t+1)\chi_{[-1,0)} + (1-t)\chi_{[0,1)},$$

and χ is the indicator function. That is, $s_i(t)(i = 2, \ldots, K-1)$ are the hat functions centered at $i\delta$ with support $[(i-1)\delta, (i+1)\delta]$, $s_1(t)$ and $s_K(t)$ are such that $g'(t)$ is constant outside $[0, T]$ as shown in Fig. 2.

The volatility function is defined as

$$\sigma^2(d) = \begin{cases} \frac{1}{2}\max(\sigma_{\min}^2 + \alpha_r(d-d_0)^2,\ 2\sigma_{\mathrm{Max}}^2), & \text{if } d \geq d_0 \\ \frac{1}{2}\max(\sigma_{\min}^2 + \alpha_l(d-d_0)^2,\ 2\sigma_{\mathrm{Max}}^2), & \text{if } d < d_0. \end{cases} \tag{4.1}$$

An example of σ is shown in Fig. 3. Overall, the number of parameters to be calibrated is $K+5$: $\sigma_{\min}^2$, α_l, α_r, d_0 to specify the volatility function σ, and $\alpha_0, \alpha_1, \ldots, \alpha_K$ to specify the weight φ.

The HR model used in the comparison is defined by the SDEs (1.2) and (1.3) where σ is specified in (4.1). The original HR model, which has $\alpha_l = \alpha_r$ and $d_0 = 0$, is slightly simpler. In this way we can compare the PDV and HR models on the same basis and test the added value of the new weighting scheme. Our previous investigations Foschi and Pascucci (2005) have shown that a more general specification of the volatility function does not considerably improve the fitting of the HR model to market prices.

In the LV model, S_t is solution of the SDE

$$dS_t = \mu_t S_t dt + \sigma(S_t, t) S_t dW_t. \tag{4.2}$$

As shown by Dupire (1997), the LV function $\sigma(S_t, t)$ can be theoretically computed by knowing the option price as a function of strike and maturity. In practice, we chose a piecewise linear function σ with knots at the observed strikes and maturities. We calibrate σ by a least squares approach.

In the stochastic volatility model by Heston, S_t and σ_t^2, the price and the squared volatility processes, respectively, are given, in the risk neutral measure, by the solution of the SDE

$$dS_t = r_t S_t dt + \sigma_t S_t d\hat{W}_t \tag{4.3}$$

$$d\sigma_t^2 = \kappa(\sigma_\infty^2 - \sigma_t^2)dt + \gamma\sigma_t \left(\rho d\hat{W}_t + \sqrt{1-\rho^2} d\hat{\hat{W}}_t\right) \tag{4.4}$$

where $d\hat{W}_t$ and $d\hat{\hat{W}}_t$ are two independent Brownian motions on the risk-neutral probability measure. The approach used to compute the prices of European options is the computational method of Carr and Madan which uses a Fourier inversion technique (Heston 1993; Carr and Madan 1999). In the experiments, the σ_∞, the long-term volatility, κ, the mean reversion speed, γ, the volatility of volatility, and ρ, the correlation, are inferred from market prices.

4.1 Empirical results

The dataset consists in closing prices of options on futures on the FTSE-100 index quoted at Euronext in the period March 22–May 19, 2006 and maturities on June, September, December 2006 and March, June, September and December 2007. For each day and each maturity the dataset contains the underlying future price [with values in the range 5,675–6,307], the Call and Put closing prices for strikes 4,025–6,725 and the corresponding implied volatilities. The underlying values have been corrected for dividends, in order to have a common underlying for all the expirations and then option prices are recomputed by using the dataset's implied volatilities. Thus, after the adjustment the underlying has a null drift in the equivalent martingale measure, that is the interest and dividend rates are null. An example of the implied volatility surface is shown in Fig. 4.

In the first set of experiments, the parameters of the four models are daily calibrated to market prices by a least squares fitting of market prices as in Dumas et al. (1998) and Foschi and Pascucci (2005). An example of the absolute pricing errors of the four models on a specific date is reported in Fig. 5.

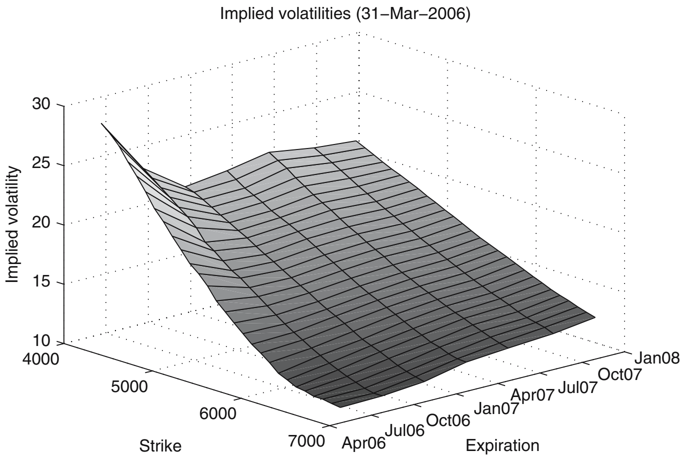


Fig. 4 Implied volatility surface for March 31, 2006

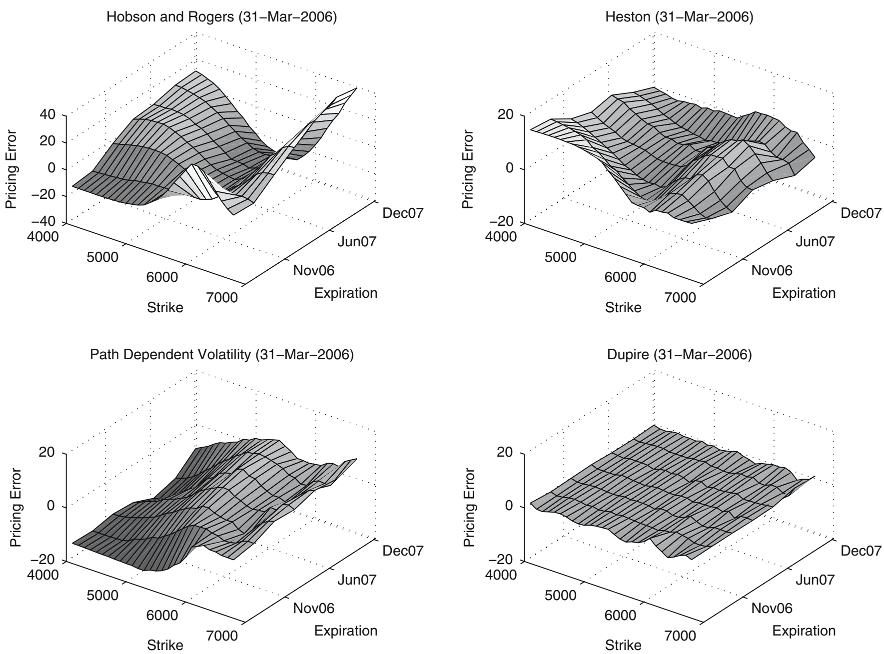


Fig. 5 Pricing error surfaces for HR, Heston, PDV and Dupire models on March 31, 2006. Value of the underlying is 5,964.5

A resume of the performances on each are reported in Figs. 6 and 7. These figures plot for each day and for each model the residual mean squared errors (RMSE) and residual mean squared percentage errors (RMSPE). In the computation of the RMSPE options with price smaller than 5 have been discarded. As can be seen from Fig. 6 the fitting of the LV model is, as expected, always almost perfect; that of standard HR

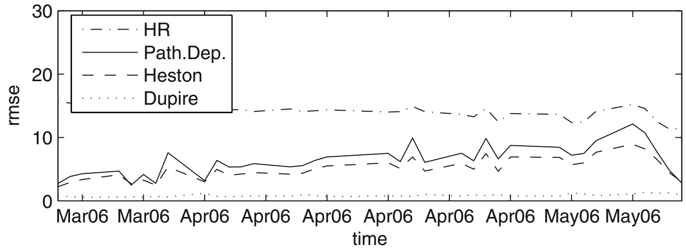


Fig. 6 Root mean squared pricing errors of for each day in the test period

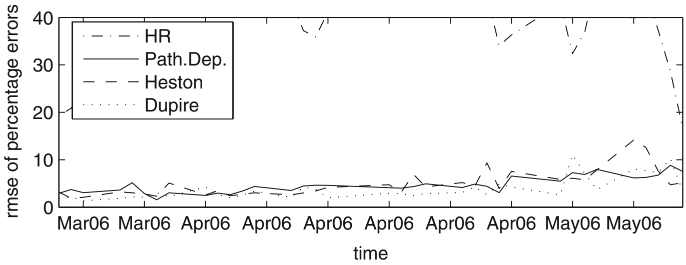


Fig. 7 Root mean square of percentage errors in pricing

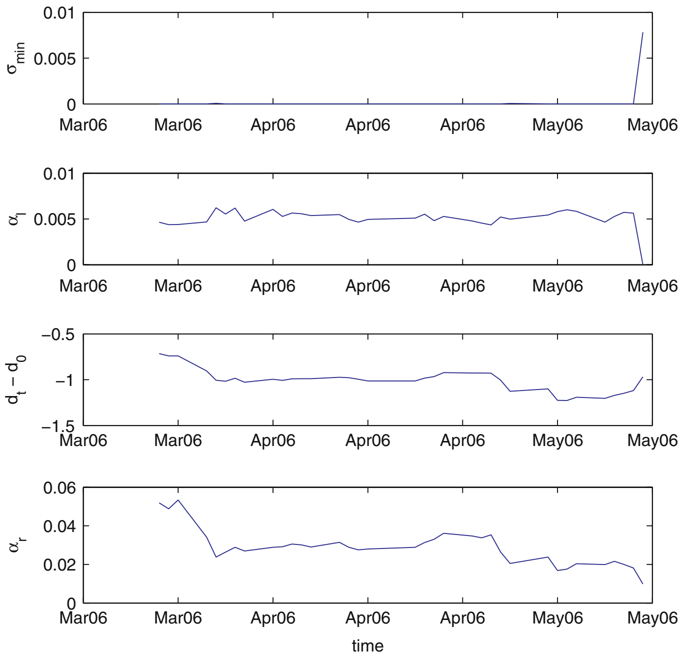


Fig. 8 Evolution of parameters for HR model, with daily calibration on the test period (22 March – 19 May, 2006)

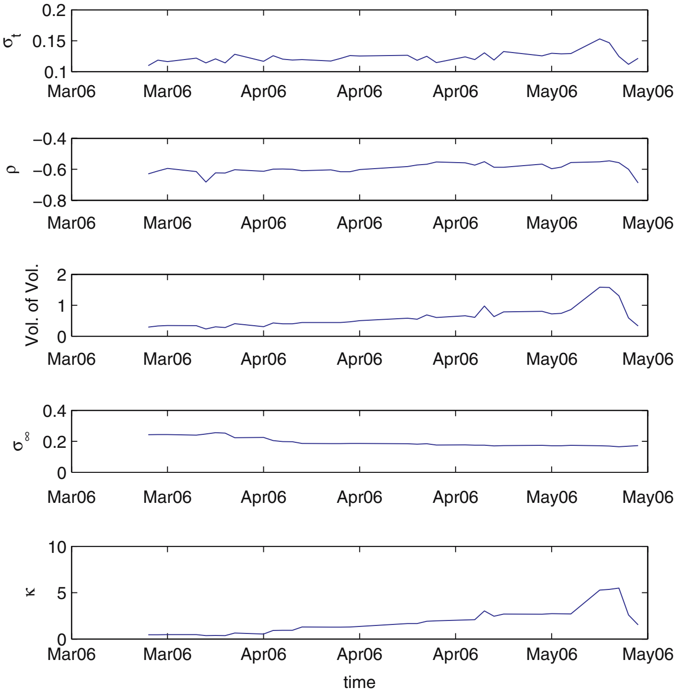


Fig. 9 Evolution of Heston parameters with daily calibration on the test period

model is at least twice that of the remaining two models. The Heston model is slightly better than the PDV; however the reverse happen when considering relative errors (cf. Fig. 7).

In order to study the stability of the parameters on different samples, their evolution is reported in Figs. 8, 9 and 10 for the HR, Heston and PDV models. Due to its large number we do not report the evolution of Dupire's local volatilities, but it is well known they are not stable due to its over-parametrization and ill-posedness (cf. Cont and BenHamida 2005). The parameters for the HR model are quite stable until near the end of the sample, where the model flattens to standard Black and Scholes, $\alpha_l, \alpha_r \simeq 0$. This change of regime happens also for the Heston and PDV models, but a bit more in advance. These two models show spikes in the series of parameters on the same day (cf. the graph of ρ in Fig. 9 and of α_r in Fig. 10).

This behaviour of the three models can be explained by looking at the time series of the underlying index which is shown in Fig. 11. Near the end of the sample, exactly on May 12, the index level drops significantly: consequently the option market reacts and the parameters of the three models try to adapt to market movements.

As a final and more significant experiment we compared the hedging performances of the methods by considering the tracking error of the replicating portfolio suggested by each model w.r.t. the evolution of each single call. Since a local volatility model is generally over-determined and can be calibrated in different ways (see, for instance,

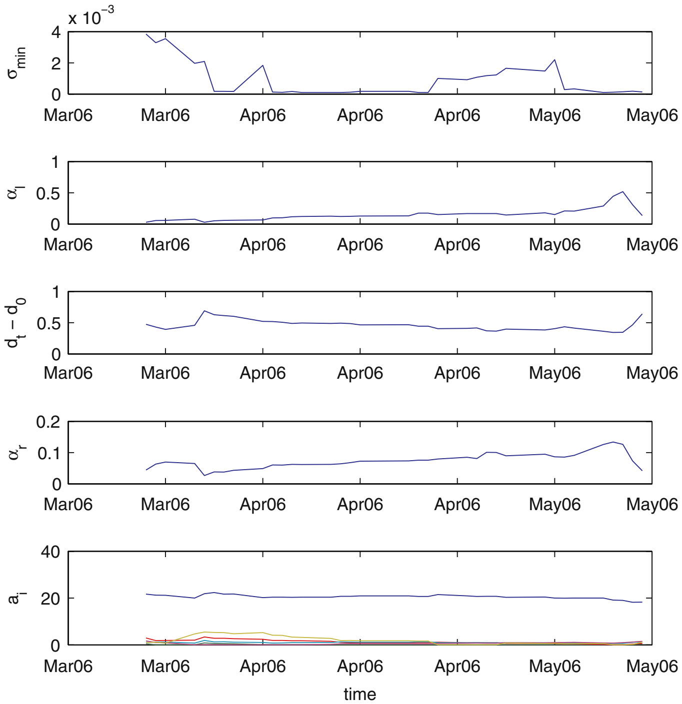
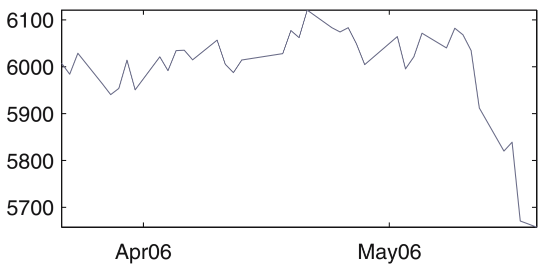


Fig. 10 Evolution of parameters for the PDV model, with daily calibration on the test period

Fig. 11 Closing prices for the FTSE-100 index on the period 22 March – 19 May, 2006



Cont and BenHamida 2005) each one leading to different hedge ratios, we do not consider the LV hedging performances that seem not to be significant.

As a first qualitative comparison, in Fig. 12 we plot the surface of the differences between the Delta of each model and the sticky Delta, that is the Black and Scholes' Delta computed at the corresponding implied volatility.

For each model we proceed as follows. At the ith day, time t_i, the model has been calibrated to the market cross-section of prices. Then, for a given expiry T and a given strike K we consider the portfolio composed by a short position on one Call C_{t_i} and a long on the replicating portfolio $V_{t_i} = \alpha_i S_{t_i} + \beta_i B_{t_i}$ as in (2.4). That is, the portfolio $\Pi_{t_i} = V_{t_i} - C_{t_i}$, which has null value in case of perfect replication. Then, the next day the portfolio has value given by $\alpha_i S_{t_{i+1}} + \beta_i B_{t_{i+1}} - C_{t_{i+1}}$, the corresponding profit

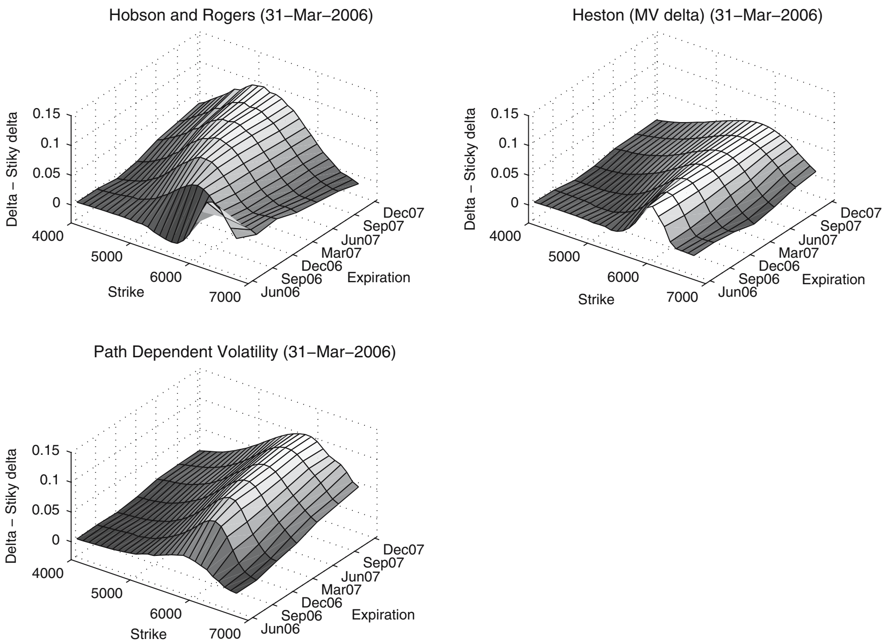


Fig. 12 Difference between Delta and sticky Delta for HR, Heston and PDV models on 31 March, 2006. Value of the underlying is 5,964.5

Fig. 13 Hedging error against strike for maturity 16 June, 2006. The errors are computed on the period 22 March–19 May, 2006

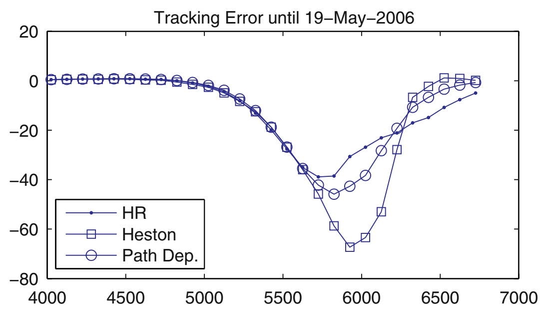
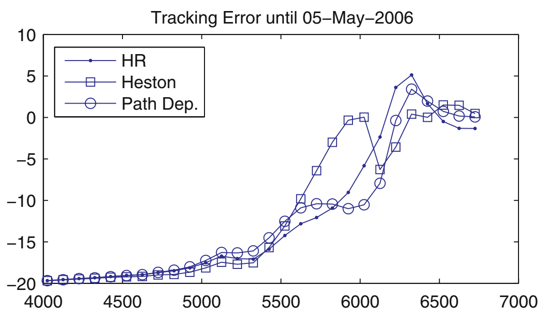


Fig. 14 Hedging error against strike for maturity date 16 June, 2006. The errors are computed on the period 22 March–5 May, 2006



and losses are accumulated and a new portfolio $\Pi_{t_{i+1}}$ is built. Recalling that we are working on a dividend and interest rate free setting, the total profits and losses on the period $[t_1, t_n]$ are given by

$$C_1 - C_n + \sum_{i=1}^{n-1} \alpha_i (S_{t_{i+1}} - S_{t_i}).$$

This procedure has been repeated for each model, for each strike, for the expirations 16 June, 2006 and 15 December, 2006 and for the periods 22 March–5 May and 22 March–19 May. The two periods correspond to quiet and nervous market situations, respectively (see Fig. 11).These performance results are reported on Figs. 13, 14, 15 and 16. Standard delta-hedging have been used for HR and PDV models, while minimum-variance delta hedging is used for Heston stochastic volatility model (Alexander and Nogueira 2006).

Figure 13 shows the replication error of the hedging strategies until about one month to expiration and after the fall of May 12. The HR model is the better for at-the-money options and the Heston model is superior for out-of-the-money options. The PDV model is between the two, but in both the cases is near to the best and the overall performance is thus preferable to the other two.

The performances in a quiet market scenario reported in Fig. 14 are a bit mixed. In this experiment the overall performances of Heston are slightly better than those of PDV and HR models.

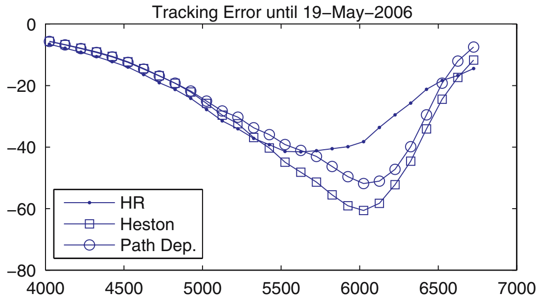


Fig. 15 Hedging error against strike for maturity 15 December, 2006. The errors are computed on the period 22 March–19 May, 2006

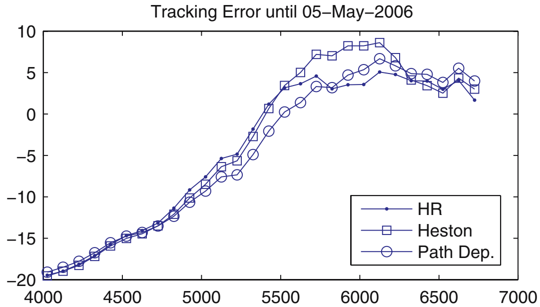


Fig. 16 Hedging error against strike for maturity 15 December, 2006. The errors are computed on the period 22 March–5 May, 2006

Similar, but less marked, results are reported in Figs. 15 and 16 where far from expiration options are considered. In the nervous scenario the HR model is more protective trying to limiting the losses, Heston model is not able to do that and the PDV model lies between the two. On the contrary, in the quiet scenario, Heston has a slightly better performance but the difference between the three models is not too strong.

5 Conclusions

In this paper we propose a generalization of the Hobson and Rogers model and introduce a new class of models for asset prices with path dependent volatility (PDV). Our idea is to consider a flexible weighting scheme, possibly corresponding to a *finite* time horizon in the past. The model is complete since no new sources of randomness are introduced and unique preference-independent prices for contingent claims are defined.

In order to evaluate the performance of the PDV hedging strategies compared with those produced by HR and Heston models, we test them in different market scenarios. We consider the market reversal in May 2006 and examine the behaviours in the quiet and nervous market scenarios before and after that reversal. It turns out that, regardless of option maturity, in the quiet scenario the Heston model has a slightly better performance, while in the nervous one, the HR model outperforms the others. In particular, after sudden price movements the tracking errors of Heston minimum-variance delta hedging are up to twice those of the HR model. The PDV model lies between the two, but it is always near the best one. Resuming, the PDV model results to be flexible enough to follows market movements without loosing the protective behaviour of the HR model.

As a final remark, we note that market completeness can also be considered as a drawback of HR and PDV models, since it can be barely considered a realistic assumption. Indeed in a PDV model options are in principle redundant in that they can be perfectly replicated by delta-hedging in the underlying asset. Then in this framework trading strategies that hedge against volatility risk (for instance, vega-hedging using traded options) are meaningless from a theoretical point of view. An obvious idea is to investigate jump-diffusion or truly stochastic volatility models in the framework of PDV and we aim to come back to this point in a forthcoming paper.

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