On the fundamental solution for hypoelliptic second order partial differential equations with non-negative characteristic form

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Abstract

We consider a wide class of second order hypoelliptic partial differential operators with non-negative characteristic form. We prove the existence and some basic properties of a global fundamental solution.

1. Introduction. We are concerned with a second order partial differential operator of the following type

$$L = \sum_{i,j=1}^{N} a_{ij}(z)\partial_{x_i}\partial_{x_j} + \sum_{j=1}^{N} b_j(z)\partial_{x_j} - \partial_t \tag{1.1}$$

where $z = (x,t)$ is the point of $\mathbb{R}^{N+1}$, $A = (a_{ij})$ is a $N \times N$ symmetric and positive semidefinite matrix and the coefficients a_{ij}, b_j, $1 \le i,j \le N$, are smooth functions. We also assume the following hypotheses:

(H.1) *L is hypoelliptic*;

(H.2) $a_{11}(z) \neq 0$ *for every* $z \in \mathbb{R}^{N+1}$;

(H.3) *L is the heat operator out of a compact subset F_0 of* $\mathbb{R}^{N+1}$.

The aim of this paper is to prove that the operator L has a global fundamental solution Γ in $\mathbb{R}^{N+1}$ satisfying some basic qualitative properties of particular interest in potential theory. Before presenting our main results, we would like to briefly comment our hypotheses.

A sufficient condition for (H.1) is the following classical Hörmander's condition (see [6] and [10])

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$$\text{(H)} \qquad \operatorname{rank} \mathcal{L}(X_1, \ldots, X_N, Y - \partial_t)(z) = N + 1 \qquad \forall z \in \mathbb{R}^{N+1}.$$

In (H), $\mathcal{L}(X_1, \ldots, X_N, Y - \partial_t)$ denotes the Lie algebra generated by the vector fields

$$X_i = \sum_{j=1}^{N} a_{ij} \partial_{x_j}, \ i = 1, \ldots, N \qquad \text{and} \qquad Y - \partial_t = \sum_{j=1}^{N} b_i \partial_{x_j} - \partial_t.$$

It is well-known ([4], [10], [8], [1]) that, in general, Hörmander's condition (H) is not necessary for hypoellipticity. For instance, the operator

$$L_p = \partial_{x_1}^2 + \exp(-|x_1|^{\frac{p}{2}}) \partial_{x_2}^2 - \partial_t, \qquad (x_1, x_2, t) \in \mathbb{R}^3, \ -1 < p < 0,$$

is hypoelliptic (for example, as an immediate consequence of Theorem 1.1 of [1]) although (H) fails for $x_1 = 0$. Condition (H.2) simply ensures that L is uniformly non-totally degenerate. Finally, (H.3) yields an exponential decay of Γ at infinity. We explicitly remark that this hypothesis does not affect the analysis of the local properties of L.

Our first result is the following theorem which will be proved in Section 2.

Theorem 1.1 *There exists a fundamental solution Γ of L having the following properties:*

(i) Γ is a non-negative function which is smooth away from the diagonal of $\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$;

(ii) for every fixed $z \in \mathbb{R}^{N+1}$, $\Gamma(\cdot; z)$ and $\Gamma(z; \cdot)$ are locally integrable;

(iii) for every test function φ, the following identities hold:

$$L \int_{\mathbb{R}^{N+1}} \Gamma(\cdot; \zeta) \varphi(\zeta) d\zeta = -\varphi, \tag{1.2}$$

$$\int_{\mathbb{R}^{N+1}} \Gamma(\cdot; \zeta) L\varphi(\zeta) d\zeta = -\varphi; \tag{1.3}$$

(iv) $\Gamma(x, t; \xi, \tau) = 0$ if $t \leq \tau$;

(v) for every $\zeta \in \mathbb{R}^{N+1}$, $L\Gamma(\cdot; \zeta) = -\delta_\zeta$, where δ_ζ denotes the Dirac measure supported in $\{\zeta\}$;

(vi) if we define

$$\Gamma^*(z; \zeta) := \Gamma(\zeta; z), \qquad \forall z, \zeta \in \mathbb{R}^{N+1},$$

then Γ^* is a fundamental solution of L^*, the formal adjoint of L, satisfying the dual statements of (iii)-(v).

In Section 3, by using hypothesis (H.3) and by suitably modifying some classical results about caloric functions, we prove the following asymptotic behavior of Γ.

Theorem 1.2 *For every $\zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$ and for every $\varepsilon > 0$ there exists a compact set $F \subseteq \mathbb{R}^{N+1}$ and a positive constant C such that*

$$\Gamma(z; \zeta) \leq C K(z; \zeta_\varepsilon) \qquad \forall z \in F, \tag{1.4}$$

where $\zeta_\varepsilon = (\xi, \tau - \varepsilon)$ and K denotes the fundamental solution of the heat operator H in $\mathbb{R}^{N+1}$.

An analogous result clearly holds for the fundamental solution Γ^* of L^*. Therefore, and in view of Theorem 1.1-(vi), we can exchange the role of z and ζ in Theorem 1.2. From the proof, it will also result that if z belongs to a fixed compact set M, then the constant in (1.4) can be chosen so as to depend only on M. We stress that this result, although not unexpected, requires several non-trivial modifications of classical uniqueness results for the heat equation.

As a byproduct of these results, in Section 3, we also prove a uniqueness theorem for solutions to the Cauchy problem

$$\begin{cases} Lu = 0 & \text{in } \ \mathbb{R}^N \times]0, \infty[\\ u(\cdot, 0) = 0 & \text{in } \ \mathbb{R}^N. \end{cases} \tag{1.5}$$

Theorem 1.3 *Let $u \in C^\infty(\mathbb{R}^N \times]0, +\infty[) \cap C(\mathbb{R}^N \times [0, +\infty[)$ be a solution to the Cauchy problem (1.5). If one of the following conditions holds:*
(i) u is non-negative;
(ii) for every $T > 0$ there exists $\gamma > 0$ such that

$$\int_0^T \int_{\mathbb{R}^N} \exp(-\gamma |x|^2) |u(x, t)| dx dt < \infty;$$

then u vanishes identically.

The last section of the paper is devoted to the proof of some further classical properties of the fundamental solution Γ. The main results of this section are contained in the following theorem.

Theorem 1.4 *For every $\zeta \in \mathbb{R}^{N+1}$*

$$\limsup_{z \to \zeta} \Gamma(z; \zeta) = \infty. \tag{1.6}$$

Moreover, for every $x \in \mathbb{R}^N$ and $t > \tau$, we have

$$\int_{\mathbb{R}^N} \Gamma(x, t; \xi, \tau) d\xi = 1. \tag{1.7}$$

The following corollary is a straightforward consequence of Theorem 1.4.

Corollary 1.5 *Let $\varphi \in C_0(\mathbb{R}^N)$. A classical solution to the problem*

$$\begin{cases} Lu = 0 & \text{in } \mathbb{R}^N \times]0, \infty[\\ \lim\limits_{t \to 0^+} u(x, t) = \varphi(x) & \forall x \in \mathbb{R}^N, \end{cases}$$

is given by

$$u(x, t) = \int_{\mathbb{R}^N} \Gamma(x, t; \xi, 0) \varphi(\xi) d\xi \qquad (x, t) \in \mathbb{R}^N \times]0, +\infty[.$$

In the forthcoming paper [9], we shall use the results of this note to obtain a monotonic approximation theorem and a representation formula for L-superparabolic functions.

Acknowledgments. The main results of this paper were announced in [11].

2. Existence of the fundamental solution. In this section we prove Theorem 1.1. We begin by giving two simple maximum principle results.

Proposition 2.1 *(Weak maximum principle)*
Let L be an operator of type (1.1). Let Ω be a bounded open subset of $\mathbb{R}^{N+1}$ and $u \in C^2(\Omega)$ such that $Lu \geq 0$ and $\limsup_{z \to \zeta} u \leq 0$ for every $\zeta \in \partial\Omega$. Then $u \leq 0$ in Ω.

Proof. It is an immediate consequence of Picone's theorem. Indeed, if we set

$$w(z) = e^t, \qquad z \in \Omega,$$

then $w \in C^2(\Omega)$, $w > 0$ and $Lw < 0$ in Ω. ■

Given a cylinder $Q = O \times]a, b[$, where O is an open subset of $\mathbb{R}^N$ and $a < b$, we set

$$\partial_r Q = (O \times \{a\}) \cup (\partial O \times [a, b]). \tag{2.1}$$

We call $\partial_r Q$ the parabolic boundary of Q.

Proposition 2.2 *(Maximum principle on cylindrical domains)*
Let $u \in C^2(Q)$, where Q is an open cylinder in $\mathbb{R}^{N+1}$, and $Lu \geq 0$ on Q. If $\limsup_{z \to \zeta} u \leq 0$ for every $\zeta \in \partial_r \Omega$, then $u \leq 0$ on Q.

Proof. Let ε and δ be suitably small positive constants. We consider the function

$$u_\varepsilon(z) = u(z) + \varepsilon e^{-t}, \qquad z \in Q_\delta =: \Omega \times]a, b - \delta].$$

We show that u_ε has no maximum in Q_δ. Indeed

$$Lu_\varepsilon(z) = Lu(z) + \varepsilon e^{-t} > 0, \qquad \forall z \in Q_\delta. \tag{2.2}$$

By contradiction, if $\bar{z} \in Q_\delta$ is a maximum, then, for A positive semi-definite, $Lu_\varepsilon(\bar{z}) \leq 0$, but this contradicts (2.2). Hence, for every $z \in Q_\delta$ and $\varepsilon > 0$, we have

$$u(z) \leq u_\varepsilon(z) \leq \limsup_{\partial_r Q_\delta} u_\varepsilon = \limsup_{\partial_r Q_\delta} u + \varepsilon e^{-a} \leq \varepsilon e^{-a}.$$

Since ε is arbitrary, we obtain

$$u(z) \leq 0, \qquad \forall z \in Q_\delta.$$

We conclude by letting δ go to 0. ■

The following definition, as well as Theorems 2.5 and 2.7 below, are strongly inspired by the classical paper [2].

Definition 2.3 *Let O be an open subset of $\mathbb{R}^N$. The point $x_0 \in \partial O$ is strongly L-regular if there exists a L-non-characteristic outer normal to O in x_0 , i.e. a vector $\nu \neq 0$ such that $B(x_0 + \nu, |\nu|) \cap O = \emptyset$ and $\langle A(x_0, t)\nu, \nu\rangle > 0$ for every $t \in \mathbb{R}$.*

In the preceding definition, we have denoted by $B(x_0, r)$ the Euclidean ball in $\mathbb{R}^N$ centered at x_0, with radius $r > 0$.

Our first step in the proof of Theorem 1.1 is the construction of an open covering of $\mathbb{R}^N$ whose elements are sets with strongly L-regular boundary. For every $\varepsilon > 0$ and $n \in \mathbb{N}$, we let

$$O_n = B(ne_1, n + \varepsilon n) \cap B(-ne_1, n + \varepsilon n), \tag{2.3}$$

where $e_1 = (1, 0, \dots, 0)$ is the first versor of the canonical basis of $\mathbb{R}^N$.

Proposition 2.4 *There exists $\varepsilon > 0$ such that $(O_n)_{n\in\mathbb{N}}$ is an increasing sequence of open sets with strongly L-regular boundary and such that $\bigcup\limits_{n\in\mathbb{N}} O_n = \mathbb{R}^N$.*

Proof. By (H.2) and (H.3), there exists $\delta > 0$ such that $\langle A(z)\nu, \nu\rangle > 0$ for every $z \in \mathbb{R}^{N+1}$ and for every vector $\nu \in \mathbb{R}^N$ such that $|\nu - e_1| < \delta$. For fixed $n \in \mathbb{N}$, let $x \in \partial O_n$ such that $x_1 = 0$. Then

$$\nu = \frac{x + ne_1}{|x + ne_1|}$$

is an outer normal to O_n in x. In order to prove that ∂O_n is strongly L-regular, it suffices to verify that $|\nu - e_1| < \delta$:

$$\begin{aligned} |\nu - e_1| &= 2\sin\left(\frac{\widehat{\nu e_1}}{2}\right) = 2\sqrt{\frac{1 - \cos\widehat{\nu e_1}}{2}} = \sqrt{2(1-\nu_1)} \\ &= \sqrt{2\left(1 - \frac{n}{n + \varepsilon n}\right)} = \sqrt{\frac{2\varepsilon}{1+\varepsilon}} < \delta \end{aligned}$$

if ε is suitably small. ■

We set

$$U_{n,R} = O_n \times]-R, R[, \qquad n \in \mathbb{N},\ R > 0. \tag{2.4}$$

In order to simplify the notation, for fixed $n \in \mathbb{N}$ and $R > 0$, in the next theorem we shall denote by U the set $U_{n,R}$ in (2.4).

Theorem 2.5 *If $f \in C(U \cup \partial_r U)$, there exists a unique solution $u \in C(U \cup \partial_r U)$ to the Dirichlet problem*

$$\text{(PD)} \qquad \begin{cases} Lu = -f & \text{(in the distribution sense)} \\ u|_{\partial_r U} = 0. \end{cases}$$

Proof. The uniqueness of the solution immediately follows from the maximum principle on cylindrical domains. We recall that, being L hypoelliptic, if $Lw = 0$ in the sense of the distributions, then $w \in C^\infty$.
To prove the existence of the solution, we first assume that $f \in C^\infty(U)$ and use a viscosity argument. For every $\varepsilon > 0$, we consider the parabolic operator

$$L_\varepsilon = L + \varepsilon \triangle_{\mathbb{R}^N}$$

and denote by u_ε the solution to the problem (PD) related to L_ε and f. For every $\varepsilon > 0$, we have

$$\|u_\varepsilon\|_\infty \leq 2R\|f\|_\infty. \tag{2.5}$$

Indeed, if

$$w(z) = -(t + R)\|f\|_\infty, \qquad z \in U,$$

then $w \leq 0$ on U and $L_\varepsilon w = \|f\|_\infty \geq 0$. Therefore

$$L_\varepsilon(u_\varepsilon + w) = -f + \|f\|_\infty \geq 0 \qquad \text{in } U,$$

and $u_\varepsilon + w \leq 0$ on $\partial_r U$. From Proposition 2.2 we obtain

$$u_\varepsilon(z) \leq (t + R)\|f\|_\infty \leq 2R\|f\|_\infty \qquad \forall z \in U.$$

So (2.5) is proved. Therefore $\{u_\varepsilon \mid \varepsilon > 0\}$ is a bounded subset of $L^\infty(U)$. Hence there exists a sequence $(u_{\varepsilon_n})_{n\in\mathbb{N}}$ that converges in the weak dual topology to a function $u \in L^\infty(U)$ such that

$$\|u\|_\infty \leq 2R\|f\|_\infty. \tag{2.6}$$

Besides, for every $\varphi \in C_0^\infty(U)$, we have

$$-\langle f, \varphi\rangle = \langle L_{\varepsilon_n} u_{\varepsilon_n}, \varphi\rangle = \langle u_{\varepsilon_n}, L^*\varphi\rangle + \varepsilon_n\langle u_{\varepsilon_n}, \triangle\varphi\rangle, \qquad n \in \mathbb{N}. \tag{2.7}$$

We observe that

$$|\langle u_{\varepsilon_n}, \triangle\varphi\rangle| \leq c, \qquad n \in \mathbb{N},$$

where c is a suitable constant. Thus, letting n go to infinity in (2.7), by using the hypoellipticity of L, we have that $Lu = -f$ in U in the classical sense.

We next show that u assumes the boundary data. Making use of Proposition 2.4, we construct a barrier function ω at every point of the parabolic boundary of U, as follows:
(i) if $z_0 \in \partial O_n \times [-R, R]$, we set

$$\omega(z) = e^{-\lambda|x-(x_0+\nu)|^2} - e^{-\lambda},$$

where λ is a positive parameter and ν is an outer normal vector to O_n in x_0;
(ii) if $z_0 \in (O_n \times \{-R\})$, we set

$$\omega(z) = -t - R.$$

If λ is suitably large, it is possible to determine a barrier function ω for U at $z_0 \in \partial_r U$, such that $L\omega \geq 1$ in a neighborhood V of z_0. Furthermore, there exists $M \geq \|f\|_\infty$ such that

$$M\omega \leq -2R\|f\|_\infty \qquad \text{in } \; U \setminus V.$$

Hence

$$L_\varepsilon(M\omega \pm u_\varepsilon) = ML_\varepsilon\omega \mp f \geq \|f\|_\infty \mp f \geq 0 \qquad \text{in } \; U \cap V,$$

and, from (2.5),

$$M\omega \pm u_\varepsilon \leq 0 \qquad \text{in } \; \partial(U \cap V).$$

Thus, by the weak maximum principle,

$$|u_\varepsilon| \leq M|\omega| \qquad \text{in } \; U \cap V,$$

and, letting ε go to zero,

$$|u| \leq M|\omega| \qquad \text{in } \; U \cap V. \tag{2.8}$$

In particular, (2.8) implies that

$$\lim_{z \to z_0} u(z) = 0.$$

This proves the solvability of (PD) when f is smooth. If f is merely continuous, we consider a sequence $(f_n)_{n\in\mathbb{N}}$ of smooth functions which converges uniformly to f. If $(u_n)_{n\in\mathbb{N}}$ denotes the sequence of the corresponding solutions to the Dirichlet problem, from (2.5), we have

$$\|u_n - u_m\|_\infty \leq 2R\|f_n - f_m\|_\infty, \qquad n, m \in \mathbb{N}.$$

Thus $(u_n)_{n\in\mathbb{N}}$ converges uniformly to a continuous function u which is the desired solution. ■

We next prove the existence of a fundamental solution of L. As above, for fixed $n \in \mathbb{N}$ and $R > 0$, we shall write U instead of $U_{n,R}$ (see (2.4)).

Definition 2.6 *The linear positive operator*

$$\mathcal{G} \; : \; C(U \cup \partial_r U) \longrightarrow C(U \cup \partial_r U)$$

which maps $f \in C(U \cup \partial_r U)$ to $u = \mathcal{G}f$, the unique solution of (PD), is called Green's operator of L with respect to U.

Theorem 2.7 *There exists a non-negative smooth function G, defined out of the diagonal of $U \times U$, such that, for every $f \in C(U \cup \partial_r U)$,*

$$\mathcal{G}f(z) = \int_U G(z;\zeta)f(\zeta)d\zeta \qquad z \in U \cup \partial_r U.$$

Moreover G has the following properties:

(i) $G(\cdot;\zeta)|_{\partial_r U}=0$, for every $\zeta\in U$;

(ii) $G(x,t;\xi,\tau)=0$, if $t\le\tau$;

(iii) if G^* denotes the corresponding function for L^* then

$$G^*(z;\zeta)=G(\zeta;z),\qquad \forall z,\zeta\in U.$$

Definition 2.8 *We call G the Green's function of L with respect to U.*

Proof. We only prove (ii). We refer to Theorem 6.1 in [2] for the other proofs.
Fixed $z_0,\zeta_0\in U$ such that $t_0<\tau_0$, we set $r=\frac{\tau_0-t_0}{3}$ and we consider $\varphi\in C_0^\infty(U)$ such that $\mathrm{supp}(\varphi)\subseteq B(\zeta_0,r)$. Let $\mathcal{G}_\varepsilon$, $\varepsilon>0$, denote the Green's operator of $L_\varepsilon=L+\varepsilon\triangle$. We have shown in the proof of Theorem 2.5, that there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that

$$\mathcal{G}_{\varepsilon_n}\varphi\longrightarrow\mathcal{G}\varphi,\qquad \text{as } n\to\infty,$$

weakly in $L^\infty(U)$. Let $\psi\in C_0^\infty(U)$ be such that $\mathrm{supp}(\psi)\subseteq B(z_0,r)$. We have

$$0=\int_U(\mathcal{G}_{\varepsilon_n}\varphi)(z)\psi(z)dz\longrightarrow\int_U(\mathcal{G}\varphi(z))\psi(z)dz,\qquad \text{as } n\to\infty,\tag{2.9}$$

since $G_\varepsilon(z;\zeta)=0$ for $z\in B(z_0,r)$, $\zeta\in B(\zeta_0,r)$. From (2.9), $\mathcal{G}\varphi=0$ a.e. in $B(z_0,r)$. Moreover $\mathcal{G}\varphi\in C^\infty(U)$ implies that $\mathcal{G}\varphi(z_0)=0$ for every $\varphi\in C_0^\infty(U)$ with $\mathrm{supp}(\varphi)\subseteq B(\zeta_0,r)$. Therefore $G(z_0,\zeta_0)=0$. ■

Proof of Theorem 1.1.
We split the proof in four steps.

(1) We set

$$U_R=\mathbb{R}^N\times]-R,R[=\bigcup_{n\in\mathbb{N}}U_{n,R}\qquad R>0$$

where $U_{n,R}$ is defined in (2.3) and (2.4). We denote by $g_{n,R}(z;\zeta)$ the function defined in $\overline{U}_R\times\overline{U}_R$, equal to the Green's function of L with respect to $U_{n,R}$ if $z,\zeta\in\overline{U}_{n,R}$ and vanishing if $z\in\overline{U_R\setminus U_{n,R}}$ or if $\zeta\in\overline{U_R\setminus U_{n,R}}$.
We show that

$$g_{n,R}\le g_{n+1,R},\qquad \forall n\in\mathbb{N},\ R>0.\tag{2.10}$$

(2.10) is obvious if z or ζ are in $U_R\setminus U_{n,R}$. If $z,\zeta\in U_{n,R}$, we set

$$w=g_{n+1,R}(\cdot;\zeta)-g_{n,R}(\cdot;\zeta)\qquad \text{in } \overline{U}_{n,R}.$$

We first observe that, for every $\varphi\in C_0^\infty$,

$$\mathcal{G}_n^*(L^*\varphi)=-\varphi\qquad \text{in } U_{n,R},\ n\in\mathbb{N}.\tag{2.11}$$

Indeed the functions appearing in (2.11) are solutions to the Dirichlet problem

$$\begin{cases} L^*u = -L^*\varphi \\ u|_{\partial_r U_{n,R}} = 0 \end{cases}$$

so that (2.11) is a consequence of Theorem 2.5.
If we denote by $\mathcal{G}_{n,R}$ the Green's operator of L with respect to $U_{n,R}$, for every $\varphi \in C_0^\infty(U_{n,R})$, (2.11) yields

$$\begin{aligned} -\varphi(\zeta) = \mathcal{G}^*_{n,R}(L^*\varphi)(\zeta) &= \int G^*_{n,R}(\zeta; z)(L^*\varphi)(z)dz \\ &= \langle g_{n,R}(\cdot;\zeta), L^*\varphi\rangle \qquad \forall \zeta \in U_{n,R}. \end{aligned} \tag{2.12}$$

(2.12) implies that $Lw = 0$ in $U_{n,R}$ and, since $w \geq 0$ on $\partial_r U_{n,R}$, by the maximum principle on cylindrical domains, (2.10) follows.

(2) We set

$$G_R = \lim_{n\to\infty} g_{n,R} \qquad \text{in } \overline{U}_R \times \overline{U}_R.$$

As in the proof of Theorem 2.5, we show that for every $\varphi \in C_0^\infty(U_R)$ we have

$$\|\mathcal{G}_{n,R}\varphi\|_\infty \leq 2R\|\varphi\|_\infty, \qquad \forall n \in \mathbb{N}. \tag{2.13}$$

Let $\Phi \in C_0^\infty(U_R)$ be such that $\min\{\Phi, \varphi + \Phi\} \geq 0$ in U_R. Then

$$\begin{aligned} \mathcal{G}_R\varphi(z) &:= \int G_R(z;\zeta)\varphi(\zeta)d\zeta \\ &= \int G_R(z;\zeta)(\Phi(\zeta) + \varphi(\zeta))d\zeta - \int G_R(z;\zeta)\Phi(\zeta)d\zeta \end{aligned}$$

(by Beppo-Levi's theorem)

$$= \lim_{n\to\infty} \mathcal{G}_{n,R}\varphi(z) \qquad \forall z \in U_R.$$

Thus, (2.13) yields

$$\|\mathcal{G}_R\varphi\|_\infty \leq 2R\|\varphi\|_\infty. \tag{2.14}$$

An analogous result holds for the adjoint operator $\mathcal{G}^*_R$. For every $\psi \in C_0^\infty(U_R)$ and $n \in \mathbb{N}$ large enough, we have

$$-\int_{U_R} \varphi(z)\psi(z)dz = \int_{U_R} L(\mathcal{G}_{n,R}\varphi)(z)\psi(z)dz = \int_{U_R} (\mathcal{G}_{n,R}\varphi)(z)L^*\psi(z)dz$$

(from (2.14) and the dominated convergence theorem)

$$\longrightarrow \int_{U_R} (\mathcal{G}_R\varphi)(z)L^*\psi(z)dz = \langle L(\mathcal{G}_R\varphi), \psi\rangle \qquad \text{as } n \to \infty.$$

By the hypoellipticity of L, $\mathcal{G}_R\varphi$ is smooth and

$$L(\mathcal{G}_R\varphi) = -\varphi \qquad \text{in } \ U_R$$

in the classical sense.
On the other hand, in order to show that

$$\mathcal{G}_R(L\varphi) = -\varphi \qquad \text{in } \ U_R,$$

it suffices to observe that, by (2.11),

$$-\varphi(z) = \mathcal{G}_{n,R}(L\varphi)(z) = \int G_{n,R}(z;\zeta)L\varphi(\zeta)d\zeta$$

so that, as n goes to infinity,

$$-\varphi(z) \longrightarrow \int G_R(z;\zeta)L\varphi(\zeta)d\zeta.$$

(3) In this step, we show that, for every $R > 0$, G_R is smooth out of the diagonal of $U_R \times U_R$. We first verify that $G_R(\cdot;\zeta) \in C^\infty(U_R \setminus \{\zeta\})$ for every $\zeta \in U_R$. Then, it is sufficient to proceed as in Theorem 6.1 in [2], making use of Schwartz's kernel theorem (see, for example Theorem 5.2.6 in [7]).

Let $z_0, \zeta_0 \in U_R$, with $z_0 \neq \zeta_0$, and $\varphi \in C_0^\infty(U_R)$ with $\mathrm{supp}(\varphi) \subseteq B\left(z_0, \frac{|z_0-\zeta_0|}{2}\right)$. Then we have

$$\begin{aligned}\langle L_z G_R(\cdot;\zeta_0), \varphi\rangle &= \int G_R(z;\zeta_0)L^*\varphi(z)dz \\ &= \lim_{n\to\infty} \int G^*_{n,R}(\zeta_0;z)L^*\varphi(z)dz\end{aligned}$$

(by (2.12))

$$= -\varphi(\zeta_0) = 0.$$

By the hypoellipticity of L, $G_R(\cdot;\zeta_0)$ is a smooth function in a neighborhood of z_0 and $L_z G_R(z_0;\zeta_0) = 0$.

(4) For every $z, \zeta \in \mathbb{R}^{N+1}$, we define the fundamental solution of L as

$$\Gamma(z;\zeta) = G_R(z;\zeta)$$

where R is a positive number such that $z, \zeta \in U_R$. This definition is well-posed: indeed, let $R, R' > 0$ be such that, for example, $R < R'$ and $z, \zeta \in U_{n,R} \subseteq U_{n,R'}$ for some $n \in \mathbb{N}$. We consider

$$w = G_{n,R}(\cdot;\zeta) - G_{n,R'}(\cdot;\zeta)|_{\overline{U}_{n,R}}.$$

We have

$$w|_{\partial_r U_{n,R}} = 0 \qquad \text{and} \qquad Lw = 0 \qquad \text{in } \ U_{n,R},$$

therefore, by Proposition 2.2, $w = 0$ in $U_{n,R}$. As a consequence

$$G_R = G_{R'}|_{U \cup \partial_r U}.$$

From the construction of Γ, by steps (1)-(3) of this proof and by Theorem 2.7, the properties (i)-(iv) and (vi) immediately follow. ■

As a straightforward consequence of Theorem 1.1, we also have the following corollary.

Corollary 2.9 *For every non-negative measure μ with compact support, we have*

$$L \int_{\mathbb{R}^{N+1}} \Gamma(\cdot;\zeta) d\mu(\zeta) = -\mu \qquad \textit{(in the distribution sense).}$$

Proof. We observe that the potential

$$\int_{\mathbb{R}^{N+1}} \Gamma(\cdot;\zeta) d\mu(\zeta)$$

is a distribution. Indeed, for every non-negative function $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$, we have

$$\int_{\mathbb{R}^{N+1}} \varphi(z) \int_{\mathbb{R}^{N+1}} \Gamma(z;\zeta) d\mu(\zeta) dz = \int_{\mathbb{R}^{N+1}} \int_{\mathbb{R}^{N+1}} \Gamma(z;\zeta) \varphi(z) dz d\mu(\zeta) < \infty$$

since, by (1.2),

$$\int_{\mathbb{R}^{N+1}} \Gamma(z;\cdot) \varphi(z) dz \in C^\infty(\mathbb{R}^{N+1})$$

and μ is compactly supported. Thus

$$\int_{\mathbb{R}^{N+1}} \Gamma(\cdot;\zeta) d\mu(\zeta) \in L^1_{\text{loc}}(\mathbb{R}^{N+1}).$$

If we still fix $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$, we have

$$\begin{aligned} \langle L\Gamma_\mu, \varphi \rangle &= \int \int \Gamma(z;\zeta) d\mu(\zeta) L^* \varphi(z) dz \\ &= \int \int \Gamma^*(\zeta; z) L^* \varphi(z) dz d\mu(\zeta) \\ &= - \int \varphi(\zeta) d\mu(\zeta), \end{aligned}$$

where the last equality follows from Theorem 1.1-(vi). ■

3. Some estimates of the fundamental solution. The aim of this section is the proof of Theorem 1.2 stated in the introduction. We first introduce some notations which we shall systematically use in the sequel:

(N.1) $\pi_1 = \{x \in \mathbb{R}^N \mid \ x = (x_1, \ldots, x_N),\ x_1 > 0\}$;

(N.2) $\widetilde{x} = (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}$, for every $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$;

(N.3) for every positive constants R and c,

$$S_{R,c} =: Q_R \times]0, c[,$$

where

$$Q_R = \{x \in \mathbb{R}^N \mid \ |x| > R\}.$$

The first step of the proof of Theorem 1.2 is the following uniqueness result for the Dirichlet problem related to the heat operator H in $S_{R,c}$.

Theorem 3.1 *Let $u \in C^2(S_{R,c}) \cap C(S_{R,c} \cup \partial_r S_{R,c})$ be a solution of the Dirichlet problem*

$$\begin{cases} Hu = 0 & \text{in} \ \ S_{R,c} \\ u|_{\partial_r S_{R,c}} = 0. \end{cases} \tag{3.1}$$

If there exists $\gamma > 0$ such that

$$\int_0^c \int_{Q_R} \exp(-\gamma |x|^2) |u(x,t)| dx dt < \infty \tag{3.2}$$

then $u \equiv 0$ in $S_{R,c}$.

Proof. We first prove that, if (3.2) holds, then there exists $\delta = \delta(\gamma)$ such that u is a bounded function in $S_{R,\delta}$. Let $\delta > 0$ fixed as we shall specify in the sequel and $\bar{z} = (\bar{x}, \bar{t}) \in S_{R,\delta}$. For every $\rho > 0$ we set

$$B_\rho = B(\bar{x}, \rho) \cap Q_R,$$

where $B(\bar{x}, \rho) = \{x \in \mathbb{R}^N \mid \ |x - \bar{x}| < \rho\}$. Let h_ρ be a function such that $0 \le h_\rho \le 1$, $\text{supp}(h_\rho) \subseteq B(\bar{x}, \rho + 1)$, $h_\rho(x) = 1$ if $x \in B(\bar{x}, \rho)$ and with first and second derivatives continuous and bounded by a constant independent of ρ. Let $\varepsilon > 0$ and ρ such that $\overline{B(0, R)} \subseteq B(\bar{x}, \rho)$.

We explicitly remark that, by the results of [5], Ch. 3, $u \in C^1(\overline{S}_{R,\delta})$. We set $v = h_\rho K(\bar{z}; \cdot)$ and we integrate the Green's identity

$$vHu - uH^* v = \text{div}(v \nabla u - u \nabla v) - \partial_t (uv)$$

on the region $B_{\rho+1}\times]0,\bar{t}-\varepsilon[$. Keeping in mind that $u|_{\partial_r S_{R,c}}=0$, we obtain, as ε goes to zero,

$$
\begin{aligned}
u(\bar{z}) &= \lim_{\tau\to\bar{t}^-}\int\limits_{B_{\rho+1}} u(x,\tau)h(x)K(\bar{z};x,\tau)dx \\
&= \int\limits_0^{\bar{t}}\int\limits_{B_{\rho+1}} u(x,t)H^*v(x,t)dxdt + \int\limits_0^{\bar{t}}\int\limits_{|x|=R} \langle v(x,t)\nabla u(x,t),\nu(x)\rangle d\sigma(x)dt \qquad (3.3)
\end{aligned}
$$

where $\nu(x)$ denotes the outer normal to Q_R in x. Since $H^*v=0$ in B_ρ, (3.3) yields

$$
\begin{aligned}
u(\bar{z}) = &\int\limits_0^{\bar{t}}\int\limits_{B_{\rho+1}\setminus B_\rho} u(x,t)H^*v(x,t)dxdt \\
&+\int\limits_0^{\bar{t}}\int\limits_{|x|=R} \langle v(x,t)\nabla u(x,t),\nu(x)\rangle d\sigma(x)dt. \qquad (3.4)
\end{aligned}
$$

Moreover

$$
\begin{aligned}
|H^*v(z)| &= |2\langle\nabla h(x),\nabla K(\bar{z}-z)\rangle + K(\bar{z}-z)\triangle h(x)| \\
&\le \frac{c}{(\bar{t}-t)^{\frac{n+1}{2}}}\exp\left(-\frac{|\bar{x}-x|^2}{4(\bar{t}-t)}\right) \qquad (3.5)
\end{aligned}
$$

for some positive constant c. By using (3.2), it is easy to show that there exists $\delta=\delta(\gamma)$ such that

$$
\lim_{\rho\to\infty}\left|\int\limits_0^{\bar{t}}\int\limits_{B_{\rho+1}\setminus B_\rho} u(z)H^*v(z)dz\right| = 0.
$$

On the other hand, if $|\bar{x}|\ge 2R$, there exists a constant $L=L(R)>0$ such that, for every $(x,t)\in\partial B(0,R)\times[0,\delta]$, $0\le v(\bar{x},\bar{t};x,t)\le L$. Thus, from (3.4) and (3.5), we obtain

$$
|u(\bar{z})| = \left|\int\limits_0^{\bar{t}}\int\limits_{|x|=R} v(z)\langle\nabla u(z),\nu(x)\rangle d\sigma(x)dt\right| < \infty \qquad (3.6)
$$

for $\bar{z}\in S_{2R,\delta}$. From (3.6) it immediately follows that u is bounded in $S_{R,\delta}$.

Thanks to the boundedness of u, we now prove that

$$
\lim_{|z|\to\infty,\ z\in S_{R,\delta}} u(z) = 0. \qquad (3.7)
$$

The thesis will follow from (3.7). Indeed an immediate consequence of the maximum principle shows that $u \equiv 0$ on $S_{R,\delta}$. Repeating this process for finitely many times, we deduce that $u \equiv 0$ on the strip $S_{R,c}$.

We extend the function u by defining $u(x,t) = 0$ for every $(x,t) \in Q_R \times]-\infty, 0]$. In this way $Hu = 0$ in $Q_R \times]-\infty, \delta[$. We denote by

$$\Omega_\rho^H(z) = \{\zeta \in \mathbb{R}^{N+1} \mid \ K(z;\zeta) \geq (4\pi\rho)^{-\frac{N}{2}}\}$$

the H-parabolic ball centered at z and with radius $\rho > 0$. To every $z \in S_{R,\delta}$ we associate a radius $\rho(z) > 0$ such that

$$\Omega_{\rho(z)}^H(z) \subseteq Q_R \times]-\infty, \delta[$$

and $\lim\limits_{|z|\to\infty} \rho(z) = \infty$.

The following mean value formula holds (see [12])

$$\begin{aligned}
|u(z)| &= \left| \frac{1}{(4\pi\rho(z))^{\frac{N}{2}}} \int\limits_{\Omega_{\rho(z)}^H(z)} u(\zeta) \frac{|x-\xi|^2}{4(t-\tau)^2} d\zeta \right| \\
&\leq \frac{\sup\limits_{S_{R,\delta}} |u|}{(4\pi\rho(z))^{\frac{N}{2}}} \int\limits_{\Omega_{\rho(z)}^H(z) \cap \{0<\tau<t\}} \frac{|x-\xi|^2}{4(t-\tau)^2} d\zeta \\
&\leq \frac{\sup\limits_{S_{R,\delta}} |u|}{(4\pi)^{\frac{N}{2}}} \int\limits_{\Omega_1^H(0) \cap \{0<s<\frac{\delta}{\rho(z)}\}} \frac{|y|^2}{4s^2} dy ds \longrightarrow 0 \qquad \text{as } |z| \to \infty.
\end{aligned}$$

■

The result of Theorem 3.1 also holds if we relax condition (3.2) by requiring that u is non-negative. Indeed we have:

Theorem 3.2 *Let $u \in C^2(S_{R,c}) \cap C(S_{R,c} \cup \partial_r S_{R,c})$ be a non-negative solution of the Dirichlet problem (3.1), then $u \equiv 0$.*

The proof of Theorem 3.2 is based on the following two lemmas.

Lemma 3.3 *Let u be a non-negative caloric function in $\pi_1 \times]0, c[$, for some positive constant c. Then we have*

$$0 \leq \int\limits_{\pi_1} [K(x-y,t) - K(x_1+y_1, \widetilde{x} - \widetilde{y}, t)] u(y + \delta e_1, \delta) dy \leq u(x + \delta e_1, t + \delta) \qquad (3.8)$$

for every $0 < \delta < c$ and $(x,t) \in \pi_1 \times]0, c - \delta[$.

Lemma 3.4 *In the same hypotheses of the preceding lemma, if* $0 < \delta < \frac{c}{2}$ *then there exists* $\gamma > 0$ *such that*

$$\int_0^\delta \int_{x_1 \geq \frac{c}{2}} \exp(-\gamma|x|^2) u(x,t) dx dt < \infty.$$

Proof of Lemma 3.3. Since $K(x-y,t) - K(x_1+y_1, \widetilde{x}-\widetilde{y}, t) \geq 0$ (see (3.11)), the first inequality in (3.8) holds. For fixed $R > 0$, we set

$$\pi_1^R = \{x \in \pi_1 \mid \ x_1 < R \ \text{ and } \ |\widetilde{x}| < R\}$$

and

$$F(x,t) = \int_{\pi_1^R} [K(x-y,t) - K(x_1+y_1, \widetilde{x}-\widetilde{y}, t)] u(y+\delta e_1, \delta) dy,$$

for every $(x,t) \in \pi_1 \times]0, c-\delta[$. Then

$$\lim_{t \to 0^+} F(x,t) = u(x+\delta e_1, \delta)$$

uniformly on compact subsets of π_1^R and

$$\lim_{t \to 0^+} F(x,t) = 0$$

uniformly on compact subsets of $\pi_1 \setminus \overline{\pi_1^R}$.

We fix $\bar{x} \in \partial \pi_1^R$. By sake of continuity, for every $\varepsilon > 0$ there exists $\eta > 0$ such that

$$u(x+\delta e_1, \delta) \leq u(\bar{x}+\delta e_1, \delta) + \varepsilon \qquad \text{if } \ |x-\bar{x}| < \eta.$$

Therefore

$$\begin{aligned}
&\limsup_{(x,t) \to (\bar{x},0)} F(x,t) \\
&\qquad = \limsup_{(x,t) \to (\bar{x},0)} \int_{\pi_1^R \cap B(\bar{x},\eta)} [K(x-y,t) - K(x_1+y_1, \widetilde{x}-\widetilde{y}, t)] u(y+\delta e_1, \delta) dy \\
&\qquad \leq (u(\bar{x}+\delta e_1, \delta) + \varepsilon) \limsup_{(x,t) \to (\bar{x},0)} \int_{\pi_1^R \cap B(\bar{x},\eta)} [K(x-y,t) - K(x_1+y_1, \widetilde{x}-\widetilde{y}, t)] dy \\
&\qquad = u(\bar{x}+\delta e_1, \delta) + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we get

$$\limsup_{(x,t) \to (\bar{x},0)} F(x,t) = u(\bar{x}+\delta e_1, \delta).$$

For $S > R$, we now consider

$$v(x,t) = u(x + \delta e_1, t + \delta) - F(x,t), \qquad (x,t) \in \overline{\pi_1^S} \times]0, c-\delta[.$$

For what we have seen above $\liminf_{\overline{\pi_1^S} \times \{0\}} v(x,t) \geq 0$, besides $v(0, \widetilde{x}, t) = u(\delta, \widetilde{x}, t + \delta) \geq 0$. On the other hand, since $\lim_{|z|\to\infty} F(z) = 0$, for every $\varepsilon > 0$ there exists a suitable $S = S(\varepsilon)$, such that

$$v(z) \geq -\varepsilon \qquad z \in (\partial\pi_1^S \setminus \{x_1 = 0\}) \times [0, c-\delta[.$$

Thus, by the maximum principle, we obtain $v \geq -\varepsilon$, in $\pi_1^S \times]0, c-\delta[$. As $\varepsilon \to 0^+$, we obtain

$$v(z) \geq 0, \qquad z \in \pi_1 \times]0, c-\delta[.$$

More explicitly, for every $(x,t) \in \pi_1 \times]0, c-\delta[$ the following inequality holds

$$\int_{\pi_1^R} [K(x-y,t) - K(x_1 + y_1, \widetilde{x} - \widetilde{y}, t)] u(y + \delta e_1, \delta) dy \leq u(x + \delta e_1, t + \delta). \tag{3.9}$$

We conclude by letting R go to infinity in (3.9). ■

Proof of Lemma 3.4.
With the change of variable

$$x' = x + \delta e_1, \qquad y' = y + \delta e_1,$$

from (3.8) we obtain

$$\int_{y_1>\delta} [K(x-y,t) - K(x_1 + y_1 - 2\delta, \widetilde{x} - \widetilde{y}, t)] u(y,\delta) dy \leq u(x, t+\delta)$$

for every $(x,t) \in \{x_1 > \delta\} \times]0, c-\delta[$. In particular, for $x = (1+\delta)e_1$,

$$u((1+\delta)e_1, t+\delta) \geq \int_{y_1>\delta} [K((1+\delta)e_1 - y, t) - K(1 - \delta + y_1, -\widetilde{y}, t)] u(y,\delta) dy \tag{3.10}$$

(observing that

$$K(1 - \delta + y_1, -\widetilde{y}, t) = \exp\left(\frac{\delta - y_1}{t}\right) K((1+\delta)e_1 - y, t) \tag{3.11}$$

and, as a consequence, that the integrand in (3.10) is non-negative)

$$\geq \left(1 - \exp\left(\frac{\delta - \frac{c}{2}}{c - \delta}\right)\right) \int_{y_1 > \frac{c}{2}} K((1+\delta)e_1 - y, t) u(y,\delta) dy.$$

Integrating with respect to the variable δ, we obtain

$$\int_0^\delta \int_{y_1 > \frac{c}{2}} K((1+s)e_1 - y, t) u(y,s) dy ds \leq \int_0^\delta \frac{u((1+s)e_1, t+s)}{1 - \exp\left(\frac{s - \frac{c}{2}}{c-s}\right)} ds < \infty \tag{3.12}$$

for every $t \in]0, c-\delta[$. We conclude by putting $t = \delta$ in (3.12) and by observing that there exists $\gamma > 0$ such that

$$K((1+s)e_1 - y, \delta) \geq \frac{1}{(4\pi\delta)^{\frac{N}{2}}} \exp(-\gamma |y|^2) \qquad \forall (y,s) \in \left\{ y_1 > \frac{c}{2} \right\} \times]0, \delta[.$$

■

We are now in position to prove Theorem 3.2.

Proof of Theorem 3.2.
By the preceding lemmas, there exists $\gamma > 0$ such that

$$\int_0^{\frac{c}{3}} \int_{Q_{R+\frac{c}{2}}} \exp(-\gamma |x|^2) u(x,t) dx dt < \infty.$$

We also have

$$\int_0^{\frac{c}{3}} \int_{Q_R} \exp(-\gamma |x|^2) u(x,t) dx dt < \infty.$$

Therefore Theorem 3.1 ensures that $u = 0$ in $S_{R,\frac{c}{3}}$. We next consider the Dirichlet problem

$$\begin{cases} Hu = 0 & \text{in } \ S_{R,c} \setminus S_{R,\frac{c}{3}} \\ u|_{\partial_r (S_{R,c} \setminus S_{R,\frac{c}{3}})} = 0. \end{cases}$$

As above, we prove that $u = 0$ in $S_{R,\frac{c}{3}+\frac{2c}{9}}$. By repeating this procedure, the assertion follows. ■

Finally, we prove Theorem 1.2.

Proof of Theorem 1.2.
We fix $R > 0$ such that $F_0 \cup \{\zeta\} \subseteq B(0,R) \times]-R, R[$, where F_0 is the compact set of hypothesis (H.3). For every $n \in \mathbb{N}$, we set

$$\varphi = \Gamma(\cdot;\zeta)|_{\partial_r (Q_R \times]-R,R[)},$$

$$\varphi_n(z) = \begin{cases} \Gamma(z;\zeta) & z \in (\partial_r(Q_R \times]-R,R[)) \setminus (Q_{R+n} \times \{-R\}) \\ 0 & z \in (\partial_r(Q_{R+n} \times]-R,R[)) \setminus (Q_{R+n} \times \{-R\}). \end{cases}$$

We denote by u_n, $n \in \mathbb{N}$, the solution to the Dirichlet problem

$$\begin{cases} Hu = 0 & \text{in } (Q_R \setminus Q_{R+n}) \times]-R, R[\\ u = \varphi_n & \text{in } \partial_r((Q_R \setminus Q_{R+n}) \times]-R, R[). \end{cases}$$

By the maximum principle, $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence of non-negative functions. For fixed $\varepsilon > 0$, if $\zeta_\varepsilon = (\xi, \tau - \varepsilon)$ there exists $T = T(\varepsilon)$ such that

$$\varphi \leq TK(\cdot; \zeta_\varepsilon). \tag{3.13}$$

Using the maximum principle, from (3.13) we get

$$u_n \leq TK(\cdot; \zeta_\varepsilon) \qquad \text{in } (Q_R \setminus Q_{R+n}) \times]-R, R[, \quad \forall n \in \mathbb{N} \tag{3.14}$$

from which it follows that $(u_n)_{n\in\mathbb{N}}$ is bounded. Let

$$u = \sup_{n\in\mathbb{N}} u_n \qquad \text{in } Q_R \times]-R, R[.$$

By means of the Harnack's theorem, we see that u is a non-negative solution of

$$\begin{cases} Hu = 0 & \text{in } Q_R \times]-R, R[\\ u = \varphi & \text{in } \partial_r(Q_R \times]-R, R[). \end{cases}$$

Moreover, from Theorem 3.2 u is equal to $\Gamma(\cdot; \zeta)$, and letting $n \to \infty$ in (3.14), we obtain

$$\Gamma(\cdot; \zeta) \leq TK(\cdot; \zeta_\varepsilon) \qquad \text{in } Q_R \times]-R, R[. \tag{3.15}$$

In order to prove the estimate in the semispace $\mathbb{R}^N \times]R, \infty[$, we consider the Cauchy problem

$$\text{(CP)} \qquad \begin{cases} Hu = 0 & \text{in } \mathbb{R}^N \times]R, \infty[\\ u(x, R) = \Gamma(x, R; \zeta). \end{cases}$$

Since $\Gamma(\cdot; \zeta)$ is a solution of (CP), we have

$$\Gamma(z; \zeta) = \int_{\mathbb{R}^N} K(z; y, R)\Gamma(y, R; \zeta) dy$$

(modifying T in (3.15), if necessary, so that $\Gamma(\cdot, R; \zeta) \leq TK(\cdot, R; \zeta_\varepsilon)$)

$$\leq T \int_{\mathbb{R}^N} K(z; y, R)K(y, R; \zeta_\varepsilon) dy = TK(z; \zeta_\varepsilon)$$

by the reproduction property of K. ■

We close this section by proving the uniqueness theorem for solutions to the Cauchy problem (1.5) stated in the introduction.

Proof of Theorem 1.3. We fix $R > 0$ such that

$$F_0 \subseteq B_{\mathbb{R}^N}(0, R) \times \mathbb{R},$$

where F_0 is defined in (H.3). From Lemma 3.4, for every $c > 0$, if any of (i) and (ii) holds then there exist $\gamma_0 > 0$ and $\delta > 0$ such that

$$\int_0^c \int_{|x|>R+\delta} \exp(-\gamma_0 |x|^2)|u(x,t)| dxdt < \infty.$$

Thus, since $L = \triangle - \partial_t$ in $S_{R,c}$, proceeding as in the proof of Theorem 3.1, (see (3.7)), we have

$$\lim_{|z|\to\infty,\ \ z\in S_{R+\delta,c}} u(z) = 0.$$

As an obvious application of the maximum principle shows, u is identically zero in $\mathbb{R}^N \times [0, c]$. Moreover we can proceed analogously to show that $u \equiv 0$ in every strip $\mathbb{R}^N \times [nc, (n + 1)c]$, $n \in \mathbb{N}$. ■

4. Some properties of the fundamental solution. In this section we prove Theorem 1.4 and Corollary 1.5.

Proof of Theorem 1.4. We begin by proving (1.6). Without loss of generality we can assume that $\zeta = 0$. We want to show that $\Gamma_0 := \Gamma(\cdot; 0)$ is unbounded near the origin. By contradiction, we suppose that Γ_0 is bounded.

We take for granted, for a moment, the existence of a non-negative function p with the following property: there exist two positive constants T and c, only dependent on L, such that, for every fixed $\zeta \in \mathbb{R}^{N+1}$, $p(\cdot; \zeta)$ is a smooth L-superparabolic function in Q_T and

$$cK_\zeta(x_1 - \xi_1, t - \tau) \leq p(x_1, t; \zeta) \qquad \text{in } \ Q_T. \tag{4.1}$$

Here $Q_T = \mathbb{R}\times]\tau, \tau + T[$. For the definition of K_ζ, see (4.3) and (4.4).

For every $\varepsilon \in]0, 1[$ there clearly exists $\delta = \delta(\varepsilon) \in]0, T/2[$ such that, if

$$v_\varepsilon(z) = \varepsilon p(x_1, t; 0, -\delta) - \Gamma_0(z), \qquad z \in \mathbb{R}^N\times]0, T/2[,$$

then

$$\liminf_{z\to 0} v_\varepsilon(z) \geq 0.$$

Moreover we can choose $\delta(\varepsilon)$ in such a way that $\delta(\varepsilon)$ tends to 0 as $\varepsilon \to 0^+$. We recall Theorem 1.2 and observe that $v_\varepsilon(x, 0)$ is non-negative for every $x \in \mathbb{R}^N$. Thus by applying the maximum principle to the function v_ε in the strip $\mathbb{R}^N\times]0, T/2[$, we have

$$\Gamma_0(z) \leq \varepsilon p(x_1, t; 0, -\delta(\varepsilon)). \tag{4.2}$$

We remark that, for every fixed $z = (x,t) \in \mathbb{R}^N \times]0, T/2[$, $p(x_1, t; 0, -\delta(\varepsilon))$ is a bounded function of ε in $]0,1[$. Therefore, as $\varepsilon \to 0^+$, from (4.2) we deduce that $\Gamma_0 \equiv 0$ in $\mathbb{R}^N \times]0, T/2[$. This is an obvious contradiction.

We now prove the existence of p. We imitate the proof of Th. 2.2 in [3]. In order to avoid repetitions, we shall only sketch the proof.

Let ω be a bounded non-decreasing function verifying (1.3) and (2.22) of [3], i.e. ω is a modulus of continuity of the coefficients of L. The notation is as in (2.22) of [3]. For $z = (x,t) = (x_1, \dots, x_N, t), \zeta = (\xi, \tau) = (\xi_1, \dots, \xi_N, \tau) \in \mathbb{R}^{N+1}$, we set

$$Q_\zeta(x_1) = \frac{x_1^2}{a_{11}(\zeta)} \tag{4.3}$$

and

$$P_\zeta(x_1, t) = Q_\zeta(x_1 - \xi_1 + (t-\tau)b_1(\zeta)).$$

By using (H.2) and (H.3), it is possible to choose a positive number μ such that, for every $z, \zeta \in \mathbb{R}^{N+1}$, μ satisfies the following constraints analogous to (2.22)-(2.27) of [3]:

$$a_{11}(z)(\partial_{x_1} P_\zeta(x_1,t))^2 \geq \mu^{-1}|x_1 - \xi_1 + (t-\tau)b_1(\zeta)|^2$$

$$(\partial_{x_1} P_\zeta(x_1,t))^2 \leq \mu|x_1 - \xi_1 + (t-\tau)b_1(\zeta)|^2$$

$$|\partial_{x_1}^2 P_\zeta(x_1,t)| \leq 4\mu$$

$$|\partial_{x_1} P_\zeta(x_1,t)| \leq 4\mu|x_1 - \xi_1 + (t-\tau)b_1(\zeta)|^2$$

$$\mu^{-1}|x_1 - \xi_1 + (t-\tau)b_1(\zeta)|^2 \leq P_\zeta(x_1,t) \leq \mu|x_1 - \xi_1 + (t-\tau)b_1(\zeta)|^2.$$

For every fixed $\zeta \in \mathbb{R}^{N+1}$, we consider the parabolic operator with constant coefficients in $\mathbb{R}^2$

$$L_\zeta = a_{11}(\zeta)\partial_{x_1}^2 + b_1(\zeta)\partial_{x_1} - \partial_t.$$

The fundamental solution of L_ζ, with pole in $(0,0)$, is given by

$$K_\zeta(x_1,t) = \begin{cases} (4\pi t)^{-1}\exp\left(-\frac{Q_\zeta(x_1+tb_1(\zeta))}{4t}\right) & t > 0 \\ 0 & t \leq 0. \end{cases} \tag{4.4}$$

We restrict the choice of μ so that analogous conditions to (2.28)-(2.30) of [3] hold, that is

$$\sup_{\zeta \in \mathbb{R}^{N+1}} \sup_{(x_1,t)\in\mathbb{R}^2} (|x_1 + tb_1(\zeta)|^2 + |t|)K_\zeta(x_1,t) \leq \frac{\mu}{2}$$

and

$$\omega\left(\frac{\mu}{2}\right) > \frac{3}{4}m$$

where

$$m = \sup_{z \in \mathbb{R}^{N+1}} \omega(|z|).$$

Now, if f and h are as in (2.31), (2.34) of [3], we let

$$\check{e}(t) = \exp\left(\int_0^1 f(s)ds\right)$$

$$\check{g}(u) = \int_0^u \exp\left(-\int_1^v h(w)dw\right) dv \qquad u \geq 0$$

and define

$$p(x_1, t; \zeta) = \check{g}(\check{e}(t)K_\zeta(x_1 - \xi_1, t - \tau)).$$

By following the proof of Th. 2.2 in [3], we show that there exist $T = T(L) > 0$ and $c = c(L) > 0$ such that

$$Lp(z) = (a_{11}(z)\partial_{x_1}^2 + b_1(z)\partial_{x_1} - \partial_t)p(x_1, t; \zeta) \leq 0$$

in $\mathbb{R}^N \times]\tau, \tau + T[$ so that (4.1) holds. We stress that neither T or c depend on ζ. This completes the proof of (1.6).

We proceed by proving (1.7). We give two preliminary lemmas.

Lemma 4.1 *Let $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ be such that $0 \leq \varphi \leq 1$ and*

$$\sup_{(x,t),(\xi,\tau) \in supp(\varphi)} |t - \tau| \leq \varepsilon.$$

Then

$$u(z) = \int_{\mathbb{R}^{N+1}} \Gamma(z; \zeta)\varphi(\zeta)d\zeta \leq \varepsilon \qquad \forall z \in \mathbb{R}^{N+1}.$$

Proof. Since $\Gamma(x, t; \xi, \tau) = 0$ for $t \leq \tau$, then $u(x, t) = 0$ for $t \leq t_0 =: \min\{s \in \mathbb{R} \mid (y, s) \in \mathrm{supp}(\varphi)$ for some $y \in \mathbb{R}^N\}$. It is non-restrictive to suppose $t_0 = 0$.
We define

$$v(x, t) = t - u(x, t) \qquad x \in \mathbb{R}^N, \ t \geq 0.$$

Then, from Theorem 1.1, we have that $v \in C^\infty$ and

$$Lv(z) = -1 + \varphi(z) \leq 0.$$

Moreover $v(x, 0) = -u(x, 0) = 0$. From Theorem 1.2,

$$\lim_{|x| \to \infty} u(x, t) = 0,$$

therefore

$$\lim_{|x|\to\infty} v(x,t) = t \qquad \forall t \geq 0.$$

Thus, by using the maximum principle on cylindrical domains, $v(x,t) \geq 0$ for $t \geq 0$, that is

$$u(x,t) \leq t \qquad \forall t \geq 0. \tag{4.5}$$

In particular $u(x,t) \leq \varepsilon$ for $0 < t \leq \varepsilon$. On the other hand, by assumption,

$$\sup_{(x,t),(\xi,\tau)\in \mathrm{supp}(\varphi)} |t-\tau| \leq \varepsilon,$$

then

$$Lu(z) = 0 \qquad \forall t > \varepsilon,$$

and

$$\lim_{|x|\to\infty} u(x,t) = 0 \qquad \forall t > \varepsilon.$$

The maximum principle once more gives

$$u(x,t) \leq \varepsilon \qquad \forall t > \varepsilon$$

which completes the proof of the lemma. □

Lemma 4.2 *For every $x \in \mathbb{R}^N$ and $t, \tau \in \mathbb{R}$ we have*

$$\int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)d\xi \leq 1. \tag{4.6}$$

Proof. The left hand side of (4.6) vanishes if $t \leq \tau$. Thus it is enough to prove the lemma in the case $t > \tau$.
Let $\varepsilon > 0$ and let $(\varphi_n)_{n\in\mathbb{N}}$ be an increasing sequence of smooth functions with compact support $\mathrm{supp}(\varphi_n) \subseteq]\tau-\varepsilon, \tau[$ and such that φ_n pointwise converges to $\chi_{]\tau-\varepsilon,\tau[}$. From Lemma 4.1, we have

$$\int_{\mathbb{R}^{N+1}} \Gamma(z;\xi,s)\varphi_n(\xi,s)d\xi ds \leq \varepsilon$$

hence, as n goes to infinity,

$$\frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau}\int_{\mathbb{R}^N} \Gamma(x,t;\xi,s)d\xi ds \leq 1. \tag{4.7}$$

From this inequality, as ε goes to 0, (4.6) follows. □

Finally, we are in position to conclude the proof of Theorem 1.4. Without loss of generality we can suppose $\tau > 0$. We set

$$u(x,t) = \int_0^{+\infty} \int_{\mathbb{R}^N} \Gamma(x,t;\xi,s)d\xi ds \qquad x \in \mathbb{R}^N,\ t > 0.$$

Integrating inequality (4.6) with respect to τ on the interval $[0,t]$, we obtain

$$u(x,t) = \int_0^{t} \int_{\mathbb{R}^N} \Gamma(x,t;\xi,s)d\xi ds \le t \qquad \forall x \in \mathbb{R}^N,\ \forall t > 0.$$

We want to show that $u \in C^\infty(\mathbb{R}^N \times]0,+\infty[)$ and $Lu = -1$. We fix $(x,t) \in \mathbb{R}^N \times]0,+\infty[$ and consider a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ with $\mathrm{supp}(\varphi) \subseteq B_{\mathbb{R}^{N+1}}(0,t)$ and such that $\varphi \equiv 1$ in a neighborhood V of (x,t). We have

$$\begin{aligned} u(z) = & \int_{\mathbb{R}^{N+1}} \Gamma(x,t;\xi,s)\varphi(\xi,s)d\xi ds \\ & + \int_0^{+\infty} \int_{\mathbb{R}^N} \Gamma(x,t;\xi,s)(1-\varphi(\xi,s))d\xi ds \equiv I_1(z) + I_2(z). \end{aligned}$$

From Theorem 1.1, we obtain

$$LI_1(z) = -\varphi(z) = -1.$$

Moreover

$$LI_2 = \int_0^{+\infty} \int_{\mathbb{R}^N} L\Gamma(\cdot;\xi,s)(1-\varphi(\xi,s))d\xi ds = 0$$

in V. Hence $v(x,t) \equiv t - u(x,t)$ is a non-negative function such that

$$Lv(x,t) = 0 \qquad \forall (x,t) \in \mathbb{R}^N \times]0,+\infty[,$$

and $v(x,0) = 0$. Using Theorem 1.3, we see that v is identically zero, that is

$$\int_0^t \left(1 - \int_{\mathbb{R}^N} \Gamma(x,t;\xi,s)d\xi \right) ds = 0 \tag{4.8}$$

From this identity, keeping in mind Lemma 4.2, we obtain

$$\int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)d\xi = 1.$$

■

We close this section by proving the Corollary 1.5.

Proof of Corollary 1.5. It is clear that $Lu = 0$ in $\mathbb{R}^N \times]0, \infty[$. We only have to study the boundary behavior of u. Let us fix $x \in \mathbb{R}^N$. Using the continuity of φ, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{|x-\xi|\le\delta} |\varphi(\xi) - \varphi(x)| < \varepsilon.$$

From Theorem 1.4, we obtain

$$\begin{aligned} |u(x,t) - \varphi(x)| &= \left| \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)(\varphi(\xi) - \varphi(x)) d\xi \right| \\ &\le \varepsilon + 2\|\varphi\|_\infty \int_{|x-\xi|\ge\delta} \Gamma(x,t;\xi,0) d\xi. \end{aligned}$$

We observe that $\lim_{t\to 0^+} \Gamma(x,t;\xi,0) = 0$ if $x \ne \xi$. Therefore, using Theorem 1.2 and its remarks, from the previous inequality we finally obtain

$$\limsup_{t\to 0^+} |u(x,t) - \varphi(x)| \le \varepsilon \qquad \forall \varepsilon > 0.$$

This completes the proof. ■

References

[1] D.R. Bell, S-E.A. Mohammed, *An extension of Hörmander's theorem for infinitely degenerate second-order operators*, Duke Math. J., **78,3** (1995), 453–475.

[2] J.M. Bony, *Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier, Grenoble , **19,1** (1969), 277–304.

[3] E.B. Fabes, N. Garofalo, E. Lanconelli, *Wiener's criterion for divergence form parabolic operators with C^1-Dini continuous coefficients*, Duke Math. J., **59,1** (1989), 191–232.

[4] V.S. Fediĩ, *Estimates in $H_{(s)}$ norms and hypoellipticity*, Dokl. Akad. Nauk. SSSR, **193** (1970), 301–303 (transl. in Soviet Math. Dokl., **11** (1970), 940–942 MR42 #6419).

[5] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, N.J. (1964).

[6] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math., **119** (1967), 147–171.

[7] L. Hörmander, *The analysis of linear partial differential operators I*, Springer–Verlag, (1990).

[8] S. Kusuoka, D. Stroock, *Applications of the Malliavin calculus, Part II*, J. Fac. Sci. Univ. Tokyo Sect.IA Math., **32** (1985), 1–76.

[9] E. Lanconelli, A. Pascucci, *Superparabolic functions related to second order hypoelliptic operators,* preprint

[10] O.A. Olejnik, E.V. Radkevič, *Second order equations with non-negative characteristic form*, Providence, Amer. Math. Soc. (1973).

[11] A. Pascucci, *Soluzione fondamentale e teoria del potenziale per equazioni ipoellittiche del second'ordine*, Seminario di Analisi Matematica, Dip. di Mat. Univ. Bologna (A.A. 1996/97, Tecnoprint Bologna).

[12] N.A. Watson, *A theory of temperatures in several variables*, Proc. Lond. Math. Soc., **26,3** (1973), 385–417.