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Superparabolic Functions Related to Second Order Hypoelliptic Operators

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Abstract. In this paper, we consider a wide class of second order hypoelliptic partial differential operators with nonnegative characteristic form. We prove a monotone smoothing theorem and a representation formula for superparabolic functions.

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1. Introduction

In $\mathbb{R}^{N+1}$ we consider the second order partial differential operator

$$L = \sum_{i,j=1}^{N} a_{ij}(z)\partial_{x_i}\partial_{x_j} + \sum_{j=1}^{N} b_j(z)\partial_{x_j} - \partial_t, \tag{1.1}$$

where $z = (x, t)$ is the point of $\mathbb{R}^{N+1}$, $A = (a_{ij})$ is a $N \times N$ symmetric and positive semidefinite matrix and the coefficients a_{ij}, b_j, $1 \leqslant i, j \leqslant N$, are smooth functions. We also assume the following hypotheses:

(H.1) L is hypoelliptic;
(H.2) $a_{11}(z) \neq 0$ for every $z \in \mathbb{R}^{N+1}$;
(H.3) L is the heat operator out of a compact subset F_0 of $\mathbb{R}^{N+1}$.

We proved in a previous note [9] that the operator L has a global fundamental solution satisfying several classical properties. Thanks to the results in [9], in this paper we aim to show a monotone approximation theorem and an integral representation formula for L-superparabolic functions.

Before presenting our main results in details, we would like to briefly comment the hypotheses (H.1)–(H.3). As it is well-known, a sufficient condition for (H.1) is the following Hörmander's hypothesis (see [7] and [11])

$$\text{(H)}\quad \operatorname{rank} \mathcal{L}(X_1, \dots, X_N, Y_0 - \partial_t)(z) = N + 1, \quad \forall z \in \mathbb{R}^{N+1}.$$

In (H), $\mathcal{L}(X_1, \ldots, X_N, Y_0 - \partial_t)$ denotes the Lie algebra generated by the vector fields

$$X_i = \sum_{j=1}^{N} a_{ij} \partial_{x_j}, i = 1, \ldots, N \quad \text{and} \quad Y_0 - \partial_t = \sum_{j=1}^{N} b_i \partial_{x_j} - \partial_t.$$

We explicitly remark that (H) is nonequivalent to (H.3) (see [1]).

Condition (H.2) simply ensures that L is uniformly nontotally degenerate. Condition (H.3) yields an asymptotic estimate of Γ at infinity in terms of the fundamental solution of the heat equation. Due to the local nature of our results, (H.3) does not affect the generality.

Before proceding with the plan of the paper, we recall some well-known definitions.

DEFINITION 1.1. Let Ω be an open subset of $\mathbb{R}^{N+1}$. The sheaf of the L-parabolic functions is defined by

$$\mathcal{H}^L(\Omega) = \{u \in C^\infty(\Omega) | Lu = 0\}.$$

We say that an open set $V \subset \mathbb{R}^{N+1}$ is L-regular ($V \in \mathcal{T}_r$) if V is bounded and, for every $\varphi \in C(\partial V)$, there exists a unique function $u := H_\varphi^V \in \mathcal{H}^L(V) \cap C(\bar{V})$ such that $u|_{\partial V} = \varphi$. We say that

$$u \colon \Omega \to] - \infty, +\infty]$$

is L-superparabolic ($u \in S(\Omega)$) if u is lower semicontinuous, $u < \infty$ on a dense subset of Ω and

$$u \geqslant H_\varphi^V \quad \text{on} \quad V,$$

for every $V \in \mathcal{T}_r$, $\bar{V} \subseteq \Omega$, and for every $\varphi \in C(\partial V)$ such that $\varphi \leqslant u$.

Proceeding as in [10], Theorem 1, we can obtain the following characterization of L-superparabolic functions.

PROPOSITION 1.2. *Let*

$$u \colon \Omega \to] - \infty, +\infty]$$

be a lower semicontinuous function. The following statements are equivalent:

(i) $u \in S(\Omega)$;
(ii) $u \in L^1_{\text{loc}}(\Omega)$ *and* $Lu \leqslant 0$ *in the distribution sense.*

In Section 2 we are concerned with the problem of the monotone regularization of L-superparabolic functions. Our result reads:

THEOREM 1.3. *Let Ω be an open subset of $\mathbb{R}^{N+1}$ and $u \in S(\Omega)$. For every bounded open set $V \subseteq \bar{V} \subseteq \Omega$, there exists an increasing sequence $(u_n)_{n\in\mathbb{N}}$ of smooth superparabolic functions in $\mathbb{R}^{N+1}$ such that*

$$\lim_{n\to\infty} u_n(z) = u(z), \quad \forall z \in V.$$

The problem of monotone regularization of superparabolic functions has a long history. It is well-known that a supertemperature can be easily approximated by an increasing sequence of smooth supertemperatures. This can be attained by the device of the classic Friedrichs' mollification. This is also the case of an operator L of type (1.1) with constant coefficients. However, the general case of variable coefficients cannot be treated as above. Many authors have developed different strategies in order to construct ad hoc mollifiers for a certain class of differential operators.

In a paper dated 1963 [8], concerning uniformly elliptic equations, Littman proved a result analogous to Theorem 1.3. In that case, the approximation sequence of the superarmonic function was obtained by the convolution with a kernel constructed from the fundamental solution.

More recently, in [6], the case is treated of the heat equation on the Heisenberg group $\mathbb{H}^n$. In this situation, the particular algebraic structure of $\mathbb{H}^n \times \mathbb{R}$ naturally supplies some suitable mollifiers analogous to the classical ones.

In [3], the case is considered of parabolic operators in divergence form, with uniformly elliptic principal part. The main tools in [3] are some mean value operators on the level sets of the fundamental solution which are constructed through a process of superposition. Such an approach does not seem to work in our setting. Indeed the method used in [3] requires a sharp asymptotic estimate of the fundamental solution and of its derivatives of every order.

The main and easy idea in the proof of our Theorem 1.3 is to use the property of the fundamental solution of having the support contained in a halfspace (see Theorem 5.2(iv) in the Appendix).

In Section 3, we prove a representation formula for smooth functions on the level sets of the fundamental solution of L. For every $r > 0$ and $z \in \mathbb{R}^{N+1}$, we define

$$\Omega_r(z) = \left\{ \zeta \in \mathbb{R}^{N+1} | \Gamma(z;\zeta) > \frac{1}{r} \right\}.$$

We call $\Omega_r(z)$ the L-parabolic ball of center z and radius r. The properties of the L-parabolic balls stated in the next proposition straightforwardly follow from the properties of Γ proved in [9].

PROPOSITION 1.4. *For every $z \in \mathbb{R}^{N+1}$, the L-parabolic balls centered at z have the following properties:*

(i) *for every $r > 0$, $\Omega_r(z)$ is a bounded nonempty set.*

(ii) $\Omega_r(z)$ *shrinks to* $\{z\}$ *as* r *goes to* 0, *that is*

$$\bigcap_{r>0} \overline{\Omega_r(z)} = \{z\}.$$

(iii) *If we denote by* $|\Omega_r(z)|$ *the Lebesgue measure of* $\Omega_r(z)$, *then*

$$\lim_{r\to 0^+} \frac{|\Omega_r(z)|}{r} = 0.$$

(iv) *For almost every* $r > 0$, $\partial\,\Omega_r(z)$ *is a* N*-dimensional* C^∞ *manifold.*

The main result of Section 3 is the following theorem.

THEOREM 1.5. *Let* $u \in C^2(\mathbb{R}^{N+1})$. *For every* $z \in \mathbb{R}^{N+1}$ *and* $r > 0$, *we have*

$$\begin{aligned} u(z) &= u_r(z) - \Phi_r u(z) \qquad (1.2)\\ &:= \int_{\Omega_r(z)} u(\zeta) E_r(z;\zeta)\,\mathrm{d}\zeta - \frac{1}{r}\int_0^r \int_{\Omega_l(z)} \left(\Gamma(z;\zeta) - \frac{1}{l}\right) Lu(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}l, \end{aligned}$$

where

$$E_r(z;\zeta) = \frac{1}{r}\left\langle A(\zeta)\nabla_\xi\Gamma(z;\zeta), \frac{\nabla_\xi\Gamma(z;\zeta)}{\Gamma(z;\zeta)^2}\right\rangle + \frac{1}{r}\,\mathrm{div}\,Y(\zeta)\,\mathrm{lg}(r\Gamma(z;\zeta))$$

and the vector field Y *is defined in* (3.2). *In particular, a solution of* $Lu = 0$ *verifies the mean value formula* $u = u_r$.

Weighted representation formulas, involving level sets of the fundamental solution, have been established by several authors. We refer to [2] for a historical background on the subject. We only observe that a particular case of (1.2) is the classic mean value formula for the caloric functions proved in [13]. (1.2) also contains the mean formula proved in [5].

In Section 4 we prove our main theorem. We use the smoothing result of Theorem 1.3 to extend formula (1.2) to the class of L-superparabolic functions. We prove

THEOREM 1.6. *Let* Ω *be an open subset of* $\mathbb{R}^{N+1}$ *and* $u \in S(\Omega)$. *Let* $\mu = -Lu$. *For every* $z \in \Omega$ *and* $r > 0$ *such* $\overline{\Omega_r(z)} \subseteq \Omega$, *we have*

$$\begin{aligned} u(z) &= u_r(z) - \Phi_r u(z) \qquad (1.3)\\ &:= \int_{\Omega_r(z)} u(\zeta) E_r(z;\zeta)\,\mathrm{d}z + \frac{1}{r}\int_0^r \int_{\Omega_l(z)} \left(\Gamma(z,\zeta) - \frac{1}{l}\right) \mathrm{d}\mu(\zeta)\,\mathrm{d}l. \end{aligned}$$

As a straightforward consequence of Theorem 1.6, we prove a result about monotone approximation of L-superparabolic functions by means of the mean value operators introduced in the second paragraph.

COROLLARY 1.7. *Let* $u \in S(\mathbb{R}^{N+1})$. *For every* $z_0 \in \mathbb{R}^{N+1}$

(i) $u_\rho(z_0) \leqslant u_r(z_0)$, *for* $r \leqslant \rho$;
(ii) $\lim_{r\to 0^+} u_r(z_0) = u(z_0)$.

For greater convenience, in the Appendix we briefly recall the results of [9] that we shall systematically use in the sequel.

2. Smoothing of L-Superparabolic Functions

For every nonnegative measure μ with compact support in $\mathbb{R}^{N+1}$, we define the L-potential Γ_μ of μ by

$$\Gamma_\mu(z) = \int_{\mathbb{R}^{N+1}} \Gamma(z;\zeta)\,\mathrm{d}\mu(\zeta), \qquad z \in \mathbb{R}^{N+1}.$$

If Ω is an open subset of $\mathbb{R}^{N+1}$ and $u \in S(\Omega)$, then, by Proposition 1.2, $Lu \leqslant 0$ in the distribution sense. Hence $-Lu$ is a nonnegative measure in Ω. For every fixed bounded open set $V \subseteq \bar{V} \subseteq \Omega$, let

$$\mu = -(Lu)|_{\bar{V}} \quad \text{and} \quad u_V = \Gamma_\mu. \tag{2.1}$$

By Theorem 5.2(iii) in the Appendix and by the hypoellipticity of L, we have that

$$(u - u_V)|_V \in C^\infty(V) \quad \text{and} \quad L(u - u_V) = 0 \quad \text{on} \quad V.$$

Moreover u_V is a nonnegative function and

$$Lu_V = Lu \leqslant 0 \quad \text{on} \quad V, \qquad Lu_V = 0 \quad \text{on} \quad \mathbb{R}^{N+1}\setminus\bar{V}.$$

Thanks to these remarks, the following proposition holds.

PROPOSITION 2.1. *Let* $u \in S(\Omega)$. *For every bounded open set* $V \subseteq \bar{V} \subseteq \Omega$, *there exists a nonnegative function* $u_V \in S(\mathbb{R}^{N+1}) \cap \mathcal{H}^L(\mathbb{R}^{N+1}\setminus\bar{V})$ *such that*

$$(u - u_V)|_V \in \mathcal{H}^L(V). \tag{2.2}$$

Now we are in position to prove Theorem 1.3.

Proof of Theorem 1.3. In virtue of Proposition 2.1, it is sufficient to smooth the function u_V defined in (2.1).

We consider a cut-off function $\psi \in C_0^\infty([0,+\infty[,[0,1])$, such that $\psi(t) = 1$ for $t \geqslant 1$, $\psi(t) = 0$ for $t \leqslant \frac{1}{2}$ and $(\mathrm{d}/\mathrm{d}t)\psi(t) \geqslant 0$ for every $t > 0$. Let $\psi_n(t) = \psi(nt)$, $n \in \mathbb{N}$. For every $(x,t), (\xi,\tau) \in \mathbb{R}^{N+1}$, we set

$$\Gamma_n(x,t;\xi,\tau) = \Gamma(x,t;\xi,\tau)\psi_n(t-\tau), \qquad n \in \mathbb{N}$$

and

$$u_n(z) = \int \Gamma_n(z;\zeta)\,\mathrm{d}\mu(\zeta), \quad n \in \mathbb{N}.$$

It follows from Theorem 5.2(iv) that $(\Gamma_n)_{n\in\mathbb{N}}$ is an increasing sequence of smooth functions such that

$$\lim_{n\to\infty} \Gamma_n(z;\zeta) = \Gamma(z;\zeta), \quad \forall z,\zeta \in \mathbb{R}^{N+1}.$$

By the monotone convergence theorem, $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence of smooth functions that pointwise converges to u_V in $\mathbb{R}^{N+1}$ as n goes to infinity. Moreover, by Proposition 1.2, $u_n \in S(\mathbb{R}^{N+1})$ for every $n \in \mathbb{N}$, since

$$Lu_n(z) = -\int_{\mathbb{R}^{N+1}} \Gamma(z;\zeta)\psi_n'(t-s)\,\mathrm{d}\mu(\zeta) \leqslant 0, \qquad n \in \mathbb{N}, \quad z \in \mathbb{R}^N. \tag{2.3}$$

□

REMARK 2.2. Our proof is much simpler of those of the analogous results recalled in the Introduction. On the other hand, we stress that our smoothing method drastically modifies the parabolic support of u (see (2.3)).

3. Classical Representation Formulas

The aim of this section is to prove Theorem 1.5. Let us begin with proving Proposition 1.4.

Proof of Proposition 1.4. Property (i) immediately follows from Theorems 5.2 and 5.3, since, for every $z \in \mathbb{R}^{N+1}$,

$$\Gamma(z;\cdot) \in C^\infty(\mathbb{R}^{N+1}\backslash\{z\}) \quad \text{and} \quad \lim_{|\zeta|\to\infty} \Gamma(z;\zeta) = 0.$$

More precisely, we deduce that, for every compact neighborhood K of z, there exists a positive r such that $\Gamma(z;\zeta) \leqslant 1/r$ in $\mathbb{R}^{N+1}\backslash K$, that is

$$\Omega_r(z) \subseteq K.$$

Moreover, since by Theorem 5.4

$$\limsup_{\zeta\to z} \Gamma(z;\zeta) = \infty,$$

then $z \in \overline{\Omega_r(z)}$ for every $r > 0$. This is enough to prove (ii).

From (ii) and recalling that $\Gamma(z;\cdot) \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ (see Theorem 5.2(ii)), we obtain

$$\frac{|\Omega_r(z)|}{r} \leqslant \int_{\Omega_r(z)} \Gamma(z;\zeta)\,\mathrm{d}\zeta \to 0 \quad \text{as } r \to 0^+.$$

We conclude the proof, noting that, since $\Gamma(z;\cdot) \in C^\infty(\mathbb{R}^{N+1}\setminus\{z\})$, (iv) is a straightforward consequence of Sard's lemma. $\square$

In the proof of Theorem 1.5 we shall use the following result.

LEMMA 3.1. *For every fixed $\zeta \in \mathbb{R}^{N+1}$ and $t, r > 0$, we set*

$$I_r(t) = \left\{x \in \mathbb{R}^N | \Gamma(x,t;\zeta) \leqslant \frac{1}{r}\right\}.$$

Then

$$\lim_{t\to\tau^+} \int_{I_r(t)} \Gamma(x,t;\zeta)\,\mathrm{d}x = 0. \tag{3.1}$$

Proof. It is nonrestrictive to suppose that $\zeta = 0$. By Theorem 5.3, it is possible to choose a suitable compact subset M_0 of $\mathbb{R}^N$ such that for every $\varepsilon > 0$ small enough there exists $T = T(\varepsilon) > 0$ such that

$$\Gamma(z;0) \leqslant T K(z;0,-\varepsilon) \leqslant T K(x,\varepsilon;0),$$

for every $0 < t < \varepsilon$ and $x \in \mathbb{R}^N\setminus M_0$. Thus, since

$$\lim_{t\to 0^+} \Gamma(x,t;0) = 0, \quad \forall x \in \mathbb{R}^N\setminus M_0,$$

by the dominated convergence theorem, we have

$$\lim_{t\to 0^+} \int_{I_r(t)\setminus M_0} \Gamma(x,t;0)\,\mathrm{d}x = 0.$$

On the other hand, another application of Lebesgue's theorem gives

$$\lim_{t\to 0^+} \int_{I_r(r)\cap M_0} \Gamma(x,t;0)\,\mathrm{d}x = \lim_{t\to 0^+} \int_{M_0} \Gamma(x,t;0)\chi_{I_r(t)}(x)\,\mathrm{d}x = 0.$$

Indeed

$$0 \leqslant \Gamma(x,t;0)\chi_{I_r(t)}(x) \leqslant \frac{1}{r}$$

and

$$\lim_{t\to 0^+} \Gamma(x,t;0) = 0, \quad \forall x \in I_r(t) \cap M_0. \qquad \square$$

Proof of Theorem 1.5. With Theorem 5.4 and Lemma 3.1 in hand, formula (1.2) is a standard consequence of the following Green's identity (3.4).

We rewrite L in divergence form

$$L = \mathrm{div}(A\nabla) + Y - \partial_t,$$

where

$$Y = Y_0 - \sum_{i,j=1}^{N} (\partial_{x_i} a_{ij})\partial_{x_j} := \sum_{j=1}^{N} \beta_j \partial_{x_j}. \tag{3.2}$$

If Y^* denotes the adjoint of Y, then for every $u \in C^1\,(\mathbb{R}^{N+1})$ we have

$$Y^* u = -\sum_{j=1}^{N} \partial_{x_j}(\beta_j u) = -u\,\mathrm{div}\,Y - Yu \tag{3.3}$$

and the adjoint operator of L is given by

$$L^* = \mathrm{div}(A\nabla) + Y^* + \partial_t.$$

A standard computation yields

$$\begin{aligned} uL^*\Gamma(z;\cdot) - \Gamma(z;\cdot)Lu \;=\; & \mathrm{div}(uA\nabla\Gamma(z;\cdot) - \Gamma(z;\cdot)A\nabla u) \\ & -(Y-\partial_t)(u\Gamma(z;\cdot)) - u\Gamma(z;\cdot)\,\mathrm{div}\,Y. \end{aligned} \tag{3.4}$$

In order to remove the singularity of $\Gamma(z;\cdot)$, we cut the L-parabolic ball. For $\varepsilon < t$, we set

$$\Omega_r^{(\varepsilon)}(z) = \Omega_r(z) \cap \{\zeta = (\xi,\tau) \in \mathbb{R}^{N+1} | \tau < \varepsilon\}$$

and

$$I_r^{(\varepsilon)}(z) = \Omega_r(z) \cap \{\zeta = (\xi,\tau) \in \mathbb{R}^{N+1} | \tau = \varepsilon\}.$$

Integrating (3.4) on $\Omega_r^{(\varepsilon)}(z)$ and applying the divergence theorem, we obtain

$$\int_{\Omega_r^{(\varepsilon)}(z)} \Gamma(z;\zeta)Lu\,\mathrm{d}\zeta = \int_{\partial\Omega_r^{(\varepsilon)}(z)} \langle \Gamma(z;\cdot)A\nabla u - uA\nabla\Gamma(z;\cdot), \nu_\xi\rangle\,\mathrm{d}\sigma \tag{3.5}$$

$$+ \int_{\partial\Omega_r^{(\varepsilon)}(z)} u\Gamma(z;\cdot)\langle Y - \partial_t, \nu\rangle\,\mathrm{d}\sigma. \tag{3.6}$$

In (3.6), ν ad ν_x denote respectively the outer normal and the spatial component of the outer normal to the integration domain. We observe that $\nu = (\nu_x, \nu_t) = (0, 1)$

in $I_r^{(\varepsilon)}(z)$ and $\Gamma(z;\cdot) = 1/r$ in $\partial\Omega_r^{(\varepsilon)}(z)\backslash I_r^{(\varepsilon)}(z)$. Therefore (3.6) yields

$$\begin{aligned}&\int_{\Omega_r^{(\varepsilon)}(z)} \Gamma(z;\zeta) Lu \,\mathrm{d}\zeta \\ &\quad = -\int_{\partial\Omega_r^{(\varepsilon)}(z)\backslash I_r^{(\varepsilon)}(z)} u\langle A\nabla_\xi \Gamma(z;\cdot), \nu_\xi\rangle \,\mathrm{d}\sigma + \frac{1}{r}\int_{\partial\Omega_r^{(\varepsilon)}(z)\backslash I_r^{(\varepsilon)}(z)} (\langle A\nabla u, \nu_\xi\rangle \\ &\qquad + u\langle Y - \partial_t, \nu\rangle) \,\mathrm{d}\sigma - \int_{I_r^{(\varepsilon)}(z)} u\Gamma(z;\cdot)\,\mathrm{d}\sigma(\zeta). \end{aligned} \tag{3.7}$$

Now we examine the behavior of each term of (3.7) as ε goes to t. Since $\Gamma(z;\cdot)$ is locally integrable

$$\lim_{\varepsilon\to t^-} \int_{\Omega_r^{(\varepsilon)}(z)} \Gamma(z;\cdot) Lu \,\mathrm{d}\zeta = \int_{\Omega_r(z)} \Gamma(z;\cdot) Lu\,\mathrm{d}\zeta.$$

Moreover, by the divergence theorem

$$\begin{aligned}&\int_{\partial\Omega_r^{(\varepsilon)}(z)\backslash I_r^{(\varepsilon)}(z)} (\langle A\nabla u, \nu_\xi\rangle + u\langle Y - \partial_t, \nu\rangle)\mathrm{d}\sigma \\ &\quad = \int_{\partial\Omega_r^{(\varepsilon)}(z)} (Lu + u \operatorname{div} Y)\,\mathrm{d}\zeta + \int_{I_r^{(\varepsilon)}(z)} u\,\mathrm{d}\sigma \\ &\quad \to \int_{\Omega_r(z)} (Lu + u\operatorname{div} Y)\,\mathrm{d}\zeta \quad \text{as } \varepsilon\to t^-. \end{aligned} \tag{3.8}$$

Since, in $\partial\Omega_r^{(\varepsilon)}(z)\backslash I_r^{(\varepsilon)}(z)$,

$$\langle A\nabla_\xi\Gamma(z;\cdot), \nu_\xi\rangle = \left\langle A\nabla_\xi\Gamma(z;\cdot), \frac{\nabla_\xi\Gamma(z;\cdot)}{|\nabla_\zeta\Gamma(z;\cdot)|}\right\rangle \geqslant 0,$$

by the monotone convergence theorem, we have

$$\lim_{\varepsilon\to t^-} \int_{\partial\Omega_r^{(\varepsilon)}(z)\backslash I_r^{(\varepsilon)}(z)} u\langle A\nabla_\xi\Gamma(z;\cdot), \nu_\xi\rangle\,\mathrm{d}\sigma = \int_{\partial\Omega_r(z)} u\langle A\nabla_\xi\Gamma(z;\cdot), \nu_\xi\rangle\,\mathrm{d}\sigma.$$

At last, we use Theorem 5.4 and Lemma 3.1 to prove that

$$\lim_{\varepsilon\to t^-} \int_{I_r^{(\varepsilon)}(z)} u\Gamma(z;\cdot)\,\mathrm{d}\sigma = u(z).$$

Indeed, by (5.3),

$$\left|\int_{I_r^{(\varepsilon)}(z)} u(\zeta)\Gamma(z;\zeta)\,\mathrm{d}\sigma(\zeta) - u(z)\right|$$

$$
\begin{aligned}
&= \left| \int_{I_r^{(\varepsilon)}(z)} u(\zeta)\Gamma(z;\zeta)\,\mathrm{d}\sigma(\zeta) - u(z)\int_{\mathbb{R}^N\times\{\varepsilon\}} \Gamma(z;\zeta)\,\mathrm{d}\sigma(\zeta) \right| \\
&\leqslant \int_{I_r^{(\varepsilon)}(z)} |u(\zeta)-u(z)|\Gamma(z;\zeta)\,\mathrm{d}\sigma(\zeta) \\
&\quad + |u(z)| \int_{(\mathbb{R}^N\times\{\varepsilon\})\setminus I_r^{(\varepsilon)}(z)} \Gamma(z;\zeta)\,\mathrm{d}\sigma(\zeta) := J_1^{(\varepsilon)}(z) + J_2^{(\varepsilon)}(z).
\end{aligned}
$$

Now, since $u \in C^2(\mathbb{R}^{N+1})$,

$$
J_1^{(\varepsilon)}(z) \leqslant \sup_{\zeta\in I_r^{(\varepsilon)}(z)} |u(\zeta)-u(z)| \to 0 \quad \text{as } \varepsilon \to t^-.
$$

Moreover, by Lemma 3.1,

$$
\lim_{\varepsilon\to t^-} J_2^{(\varepsilon)}(z) = 0.
$$

Hence, letting ε go to t^- in (3.7), we derive the following representation formula

$$
u(z) = \int_{\partial\Omega_r(z)} u\langle A\nabla_\xi\Gamma(z;\cdot), \nu_\xi\rangle\,\mathrm{d}\sigma(\zeta) \tag{3.9}
$$

$$
+ \frac{1}{r}\int_{\Omega_r(z)} u \operatorname{div} Y\,\mathrm{d}\zeta - \int_{\Omega_r(z)} \left(\Gamma(z;\cdot) - \frac{1}{l}\right) Lu\,\mathrm{d}\zeta. \tag{3.10}
$$

Proceeding as in [4], Theorem 1.5, by means of Federer's co-area formula, form (3.10) we get

$$
u(z) = \frac{1}{r}\int_{\Omega_r(z)} u\left\langle A\nabla_\xi\Gamma(z;\cdot), \frac{\nabla_\xi\Gamma(z;\cdot)}{\Gamma(z;\cdot)^2}\right\rangle \mathrm{d}\zeta \tag{3.11}
$$

$$
+\frac{1}{r}\int_0^r \int_{\Omega_l(z)} u \operatorname{div} Y\,\mathrm{d}\zeta\,\frac{\mathrm{d}l}{l}
$$

$$
-\int_0^r \int_{\Omega_l(z)} \left(\Gamma(z;\cdot)-\frac{1}{l}\right) Lu\,\mathrm{d}\zeta\,\mathrm{d}l. \tag{3.12}
$$

Formula (1.2) follows immediately from (3.12) observing that, by Fubini's theorem,

$$
\begin{aligned}
\int_0^r \int_{\Omega_l(z)} u \operatorname{div} Y\,\mathrm{d}\zeta\,\frac{\mathrm{d}l}{l} &= \int_{\Omega_r(z)} \int_{1/\Gamma(z;\cdot)}^r u \operatorname{div} Y\,\frac{\mathrm{d}l}{l}\,\mathrm{d}\zeta \\
&= \int_{\Omega_r(z)} u \operatorname{div} Y \lg(r\Gamma(z;\cdot))\,\mathrm{d}\zeta. \qquad \square
\end{aligned}
$$

REMARK 3.2. If we choose $u \equiv 1$ in (1.2), we get

$$\int_{\Omega_r(z)} E_r(z;\zeta)\,\mathrm{d}\zeta = 1.$$

4. Representation Formulas for L-superparabolic Functions

The aim of this section is to prove Theorem 1.6. We begin with proving an expected property of the L-superparabolic functions.

PROPOSITION 4.1. *Let Ω be an open subset of $\mathbb{R}^{N+1}$ and $u \in S(\Omega)$. Then u satisfies the super-mean value property, that is, for every $z \in \Omega$ and $r > 0$ such that $\overline{\Omega_r(z)} \subseteq \Omega$,*

$$u(z) \geqslant u_r(z). \tag{4.1}$$

Proof. Let z and r as in the statement. By Theorem 1.3, there exists an increasing sequence $(u_n)_{n\in\mathbb{N}}$ of smooth superparabolic functions in $\mathbb{R}^{N+1}$ such that

$$\lim_{n\to\infty} u_n(\zeta) = u(\zeta), \qquad \forall \zeta \in \Omega_r(z) \cup \{z\}.$$

Moreover, by Proposition 1.2 and Theorem 1.5,

$$u_n(z) \geqslant (u_n)_r(z), \quad \forall n \in \mathbb{N}. \tag{4.2}$$

Thus, as n goes to infinity in (4.2), by the monotone convergence theorem, we obtain (4.1). □

REMARK 4.2. If E_r in (1.2) is positive (for example, if $Y = 0$) then it is not difficult to show that every lower semi-continuous function for which (4.1) holds, satisfies a minimum principle and then it is L-superparabolic.

LEMMA 4.3. *Let $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ such that $0 \leqslant \varphi \leqslant 1$ and*

$$\sup_{(x,t),(\xi,\tau)\in\,\mathrm{supp}(\varphi)} |t-\tau| \leqslant \varepsilon.$$

Then

$$u(z) = \int_{\mathbb{R}^{N+1}} \Gamma(z;\zeta)\varphi(\zeta)\,\mathrm{d}\zeta \leqslant \varepsilon, \quad \forall z \in \mathbb{R}^{N+1}.$$

Proof. Since $\Gamma(x,t;\xi,\tau) = 0$ for $t \leqslant \tau$, then $u(x,t) = 0$ for $t \leqslant t_0 := \min\{s \in \mathbb{R} | (y,s) \in \mathrm{supp}(\varphi)$ for some $y \in \mathbb{R}^N\}$. It is nonrestrictive to suppose that $t_0 = 0$. We define

$$v(x,t) = t - u(x,t), \qquad x \in \mathbb{R}^N, \quad t \geqslant 0.$$

Then, by Theorem 5.2, $v \in C^\infty$ and

$$Lv(z) = -1 - \varphi(z) \leqslant 0.$$

Moreover

$$v(x, 0) = -u(x, 0) = 0$$

and since by Theorem 5.3 $u(x, t) \to 0$ as $|x| \to \infty$, we have

$$\lim_{|x|\to\infty} v(x, t) = t, \qquad \forall t \geqslant 0.$$

Thus, by the maximum principle on cylindrical domains, Proposition 5.1, $v(x, t) \geqslant 0$ for $t \geqslant 0$, that is

$$u(x, t) \leqslant t, \quad \forall t \geqslant 0. \tag{4.3}$$

In particular $u(x, t) \leqslant \varepsilon$ for $0 < t \leqslant \varepsilon$. On the other hand, since by assumption

$$\sup_{(x,t),(\xi,\tau)\in \operatorname{supp}(\varphi)} |t - \tau| \leqslant \varepsilon,$$

then

$$Lu(z) = 0 \quad \text{for } t > \varepsilon$$

and

$$\lim_{|x|\to\infty} u(x, t) = 0, \quad \forall t > \varepsilon.$$

Thus, another application of the maximum principle gives

$$u(x, t) \leqslant \varepsilon, \quad \forall t > \varepsilon$$

and this completes the proof. □

PROPOSITION 4.4 (Weak representation formula). *Let $u \in S(\mathbb{R}^{N+1})$ and $\mu = -Lu$. For every $\varphi \in C_0(\mathbb{R}^{N+1})$ and $r > 0$, we have*

$$\begin{aligned}\int_{\mathbb{R}^{N+1}} & u(z)\varphi(z)\,\mathrm{d}z \\ &= \int_{\mathbb{R}^{N+1}} u_r(z)\varphi(z)\,\mathrm{d}z \\ &\quad + \frac{1}{r}\int_{\mathbb{R}^{N+1}} \varphi(z) \int_0^r \int_{\Omega_l(z)} \left(\Gamma(z;\zeta) - \frac{1}{l}\right) \mathrm{d}\mu(\zeta)\,\mathrm{d}l\,\mathrm{d}z.\end{aligned} \tag{4.4}$$

Proof. It is nonrestrictive to suppose that φ is nonnegative. In virtue of Proposition 2.1, we can also suppose that μ is compactly supported and $u = \Gamma_\mu$. By Theorem 1.3, there exists an increasing sequence $(u_n)_{n\in\mathbb{N}}$ of smooth superparabolic functions, which approximates u. We have

$$\int_{\mathbb{R}^{N+1}} \varphi(z)u_n(z)\,\mathrm{d}z = \int_{\mathbb{R}^{N+1}} \varphi(z)(u_n)_r(z)\,\mathrm{d}z - \int_{\mathbb{R}^{N+1}} \varphi(z)\Phi_r u_n(z)\,\mathrm{d}z, \quad \forall n \in \mathbb{N}.$$

Since, by Proposition 4.1,

$$0 \leqslant (u_n)_r \leqslant u_n \leqslant u, \quad \forall n \in \mathbb{N},$$

then, by Lebesgue theorem,

$$\lim_{n\to\infty} \int \varphi(z)u_n(z)\,\mathrm{d}z = \int \varphi(z)u(z)\,\mathrm{d}z$$

and

$$\lim_{n\to\infty} \int \varphi(z)(u_n)_r(z)\,\mathrm{d}z = \int \varphi(z)u_r(z)\,\mathrm{d}z.$$

Now we consider the last term of (4.4). We let

$$G(\zeta) = \frac{1}{r}\int_0^r \int_{\Gamma(z;\zeta)>1/l} \left(\Gamma(z;\zeta) - \frac{1}{l}\right)\varphi(z)\,\mathrm{d}z\,\mathrm{d}l, \quad \zeta \in \mathbb{R}^{N+1}.$$

We wish to show that $G \in C_0(\mathbb{R}^{N+1})$. From the estimate of Theorem 5.3, it follows that G is compactly supported.

Let $(\psi_n)_{n\in\mathbb{N}}$ be the sequence of cut-off functions defined in the proof of Theorem 1.3. It is easy to check that, for every $n \in \mathbb{N}$,

$$p_n(\zeta) := \int_{\Gamma(z;\zeta)<1/l} \left(\Gamma(z;\zeta) - \frac{1}{l}\right)\varphi(z)\psi_n(|z-\zeta|)\,\mathrm{d}z$$

is a continuous function in $\mathbb{R}^{N+1}$. We now verify that p_n is uniformly convergent to

$$p(\zeta) := \int_{\Gamma(z;\zeta)>1/l} \left(\Gamma(z;\zeta) - \frac{1}{l}\right)\varphi(z)\,\mathrm{d}z. \tag{4.5}$$

We first point out that, by Lemma 4.3,

$$\int_{\mathbb{R}^{N+1}} \Gamma(z;\zeta)(1-\psi_n(|z-w|))\,\mathrm{d}z \leqslant \frac{1}{n}, \quad \forall \zeta, w \in \mathbb{R}^{N+1}. \tag{4.6}$$

Then, since $\varphi \geqslant 0$,

$$
\begin{aligned}
0 &\leqslant p(\zeta) - p_n(\zeta) \\
&= \int_{\Gamma(z;\zeta)>1/l} \left(\Gamma(z;\zeta) - \frac{1}{l}\right) \varphi(z)(1 - \psi_n(|z - w|))\, \mathrm{d}z \\
&\leqslant \|\varphi\|_\infty \int_{\mathbb{R}^{N+1}} \Gamma(z;\zeta)(1 - \psi_n(|z - w|))\, \mathrm{d}z \leqslant \frac{\|\varphi\|_\infty}{n}, \quad \forall \zeta \in \mathbb{R}^{N+1}.
\end{aligned}
$$

The last inequality follows from (4.6) for $w = \zeta$. This proves that p_n converges uniformly to p. From the continuity of p, the continuity of G follows straightforwardly.

Defining

$$
\mu_n = -Lu_n, \quad n \in N,
$$

by a change of the order of integration, we have

$$
\int_{\mathbb{R}^{N+1}} \varphi(z) \Phi_r u_n(z)\, \mathrm{d}z = \int_{\mathbb{R}^{N+1}} G(\zeta)\, \mathrm{d}\mu_n(\zeta), \quad n \in \mathbb{N}.
$$

Now we consider a sequence $(g_n)_{n\in\mathbb{N}}$ of smooth functions which converges uniformly to G in $\mathbb{R}^{N+1}$ and such that

$$
\operatorname{supp}(g_n), \operatorname{supp}(G) \subseteq K \subset\subset \mathbb{R}^{N+1}, \quad \forall n \in \mathbb{N}.
$$

Besides, let $\chi \in C_0^\infty(\mathbb{R}^{N+1})$ such that

$$
0 \leqslant \chi_K \leqslant \chi \leqslant 1,
$$

where χ_K denotes the characteristic function of K. Then, we have

$$
\begin{aligned}
&\left| \int G(z)\, \mathrm{d}\mu(z) - \int G(z)\, \mathrm{d}\mu_n(z) \right| \\
&\quad \leqslant \left| \int (G(z) - g_j(z))\, \mathrm{d}\mu_n(z) \right| \\
&\qquad + \left| \int (G(z) - g_j(z))\, \mathrm{d}\mu(z) \right| \\
&\qquad + \left| \int g_j(z)(\mathrm{d}\mu(z) - \mathrm{d}\mu_n(z)) \right| \\
&\quad := I_1^{(n,j)} + I_1^{(n,j)} + I_3^{(n,j)}, \quad n, j \in \mathbb{N}.
\end{aligned}
$$

For every $\varepsilon > 0$, there exists $j = j_\varepsilon \in \mathbb{N}$, independent of n, such that $I_1^{(n,j)}$, $I_2^{(n,j)} \leqslant \varepsilon$. Indeed

$$\begin{aligned} I_1^{(n,j)} &\leqslant \|G - g_j\|_\infty \int \chi(z)\,\mathrm{d}\mu_n(z) \\ &= \|G - g_j\|_\infty \int L^*\chi(z) u_n(z)\,\mathrm{d}z \\ &\leqslant \|G - g_j\|_\infty \int L^*\chi(z) u(z)\,\mathrm{d}z \end{aligned}$$

and

$$I_2^{(n,j)} \leqslant \|G - g_j\|_\infty \int \chi(z)\,\mathrm{d}\mu(z).$$

On the other hand, for ε and j fixed as above, there exists $\bar{n} = \bar{n}(\varepsilon, j) \in \mathbb{N}$ such that for every $n \geqslant \bar{n}$

$$I_3^{(n,j)} = \left| \int L^* g_j(z)(u(z) - u_n(z))\,\mathrm{d}z \right| \leqslant \|L^* g_j\|_\infty \int_K (u(z) - u_n(z))\,\mathrm{d}z \leqslant \varepsilon$$

in virtue of Beppo–Levi theorem. The proof of (4.4) is thus completed. □

Proof of Theorem 1.6. We fix $z_0 \in \Omega$. As before, we suppose that μ is compactly supported and $u = \Gamma_\mu$. We distinguish three cases.

(1) *there exists an open neighborhood V of z_0 such that $u|_V \in \mathcal{H}^L(V)$.*

In this case, u and $\Phi_r u$ are continuous functions in a suitable neighborhood of z_0, since $\operatorname{supp}(\mu) \subseteq \mathbb{R}^{N+1} \backslash V$. We now

claim: u_r is continuous in a neighborhood z_0.

Let us take the claim for granted and use it to prove (1.3). We consider a sequence $(\varphi_j)_{j\in\mathbb{N}}$ of continuous nonnegative functions with $\operatorname{supp}(\varphi_j) \subseteq B(z_0, \varepsilon) \subseteq V$, for suitable positive ε, for every $j \in \mathbb{N}$, such that

$$\lim_{j\to\infty} \varphi_j = \delta_{z_0},$$

in the sense that, for every $\psi \in C(\mathbb{R}^{N+1})$,

$$\lim_{j\to\infty} \int_{\mathbb{R}^{N+1}} \psi(z)\varphi_j(z)\,\mathrm{d}z = \psi(z_0).$$

Substituting φ and φ_j in (4.4) and letting j go to infinity, (1.3) follows.

We are thus left with the proof of the claim. Let φ be a smooth cut-off function such that $\varphi \geqslant 0$, $\operatorname{supp}(\varphi) \subseteq V$ and $\varphi \equiv 1$ in $B(z_0, \varepsilon)$. We have

$$u_r = (u\varphi)_r + (u(1-\varphi))_r.$$

It is not difficult to verify that $(u(1-\varphi))_r$ is continuous in a neighborhood z_0.

On the other hand, since $u\varphi \in C_0^\infty(\mathbb{R}^{N+1})$, by the representation formula of Theorem 1.5, we have

$$(u\varphi)_r = u\varphi + \Phi_r(u\varphi). \tag{4.7}$$

From this identity, it follows that $(u\varphi)_r$ is continuous near z_0. Indeed $u\varphi$ is a C^∞ function in V and $\Phi_r(u\varphi)$ is continuous in a suitable neighborhood of z_0, because

$$L(u\varphi) = Lu = 0 \quad \text{on} \quad B(z_0, \varepsilon).$$

(2) $\mu(\{z_0\}) = 0$.

In this case, let $(V_j)_{j\in\mathbb{N}}$ be a decreasing sequence of open neighborhoods of z_0, such that

$$\bigcap_{j\in\mathbb{N}} V_j = \{z_0\}.$$

We set

$$\mu_j = \mu(1 - \chi_{V_j}), \qquad j \in \mathbb{N}.$$

Then, for every nonnegative $\varphi \in C(\mathbb{R}^{N+1})$,

$$\int_{\mathbb{R}^{N+1}} \varphi(z)\,\mathrm{d}\mu_j(z) \uparrow \int_{\mathbb{R}^{N+1}} \varphi(z)\,\mathrm{d}\mu(z) \quad \text{as} \quad j \to \infty.$$

Moreover

$$L\Gamma_{\mu_j} = 0 \quad \text{on} \quad V_j,\ j \in \mathbb{N}.$$

Thus, we can apply the result proved in the case (1) to the function Γ_{μ_j}, obtaining

$$\Gamma_{\mu_j}(z_0) = (\Gamma_{\mu_j})_r(z_0) - \Phi_r\Gamma_{\mu_j}(z_0). \tag{4.8}$$

By the monotone convergence theorem

$$\lim_{j\to\infty} \Gamma_{\mu_j}(z_0) = u(z_0)$$

and

$$\lim_{j\to\infty}(\Gamma_{\mu_j})_r(z_0) = u_r(z_0).$$

In order to prove that

$$\lim_{j\to\infty} \Phi_r\Gamma_{\mu_j}(z_0) = \Phi_r u(z_0), \tag{4.9}$$

we first show that

$$\int_{\mathbb{R}^{N+1}} \psi(z)\,\mathrm{d}\mu_j(z) \uparrow \int_{\mathbb{R}^{N-1}} \psi(z)\,\mathrm{d}\mu(z) \quad \text{as} \ \ j \to \infty, \tag{4.10}$$

for every nonnegative lower semicontinuous function ψ. Indeed, since

$$\int \psi(z)\,\mathrm{d}\mu_j(z) \leqslant \int \psi(z)\,\mathrm{d}\mu(z), \quad \forall j \in \mathbb{N},$$

we have

$$\limsup_{j\to\infty} \int \psi(z)\,\mathrm{d}\mu_j(z) \leqslant \int \psi(z)\,\mathrm{d}\mu(z).$$

On the other hand, if $\varphi \in C(\mathbb{R}^{N+1})$ and $0 \leqslant \varphi \leqslant \psi$, then

$$\liminf_{j\to\infty} \int \psi(z)\,\mathrm{d}\mu_j(z) \geqslant \liminf_{j\to\infty} \int \varphi(z)\,\mathrm{d}\mu_j(z) = \int \varphi(z)\,\mathrm{d}\mu(z). \tag{4.11}$$

From (4.11), by taking the upper bound with respect to φ, we obtain

$$\liminf_{j\to\infty} \int \psi(z)\,\mathrm{d}\mu_j(z) \geqslant \int \psi(z)\,\mathrm{d}\mu(z).$$

Thus, since for every positive l

$$\psi_l := \chi_{\Omega_l(z_0)}\left(\Gamma(z_0;\cdot) - \frac{1}{l}\right)$$

is a lower semicontinuous function, from (4.10) we get

$$\lim_{j\to\infty} \int_{\mathbb{R}^N} \psi_l(z)\,\mathrm{d}\mu_j(z) = \int_{\mathbb{R}^N} \psi_l(z)\,\mathrm{d}\mu(z).$$

The monotone convergence theorem yields (4.9).

(3) $\mu(\{z_0\}) \neq 0$.

In this case

$$\mu = \lambda + c\delta_{z_0},$$

where λ is a non-negative measure such that $\lambda(\{z_0\}) = 0$ and c is a positive constant. Therefore, for every $z \in \mathbb{R}^{N+1}$, it holds

$$u(z) = \Gamma_\lambda(z) + c\Gamma(z; z_0).$$

In particular $u(z_0) = \Gamma_\lambda(z_0)$. Moreover, since $\Gamma(\cdot; z_0) \in S(\mathbb{R}^{N+1})$, we have

$$0 \leqslant (\Gamma(\cdot; z_0))_r(z_0) \leqslant \Gamma(z_0; z_0) = 0$$

and then

$$u_r(z_0) = (\Gamma_\lambda)_r(z_0).$$

Finally, observing that, for every $l > 0$

$$\Omega_l(z_0) \cap \operatorname{supp}(\delta_{z_0}) = \emptyset,$$

we have

$$\int_0^r \int_{\Omega_l(z_0)} \left(\Gamma(z_0; z) - \frac{1}{l}\right) \mathrm{d}(\delta_{z_0}(z))\, \mathrm{d}l = 0.$$

Hence, (1.3) follows from step (2) and this concludes the proof of the theorem. □

Proof of Corollary 1.7. Proceeding as in [4], Theorem 1.6, from (1.3) we get

$$\frac{\mathrm{d}}{\mathrm{d}r} u_r(z_0) = -\frac{1}{r^2} \int_{\Omega_r(z_0)} \lg(r\Gamma(z_0; \cdot))\, \mathrm{d}\mu. \tag{4.12}$$

We remark that the kernel appearing in (4.12) is strictly positive on the domain of integration $\Omega_r(z_0)$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}r} u_r(z_0) \leqslant 0,$$

since $\mu = -Lu$ is non-negative. This proves (i).

Moreover, by the lower semicontinuity of u, for every positive ε, there exists $\rho = \rho_\varepsilon > 0$ such that

$$u(z_0) - u(z) \leqslant \varepsilon, \qquad \forall z \in B(z_0, \rho).$$

Thus, recalling Remark 3.2,

$$
\begin{aligned}
0 \leqslant u(z_0) - u_r(z_0) &= \int_{\Omega_r(z_0)} (u(z_0) - u(z)) E_r(z_0; z)\,\mathrm{d}z \\
&\leqslant \varepsilon \int_{\Omega_r(z_0)} E_r(z_0; z)\,\mathrm{d}z = \varepsilon
\end{aligned}
$$

if r is sufficiently small, so that $\Omega_r(z_0) \subseteq B(z_0, \rho)$. □

5. Appendix

In this paragraph we briefly recall some of the results of [9] which are preliminary to this paper. We begin by giving a simple maximum principle on cylindrical domains. Given a cylinder $Q = O\times]a, b[$, where O is an open subset of $\mathbb{R}^N$ and $a < b$, we define the parabolic boundary of Q by

$$\partial_r Q = (O \times \{a\}) \cup (\partial O \times [a, b]). \tag{5.1}$$

PROPOSITION 5.1. *Let $u \in C^2(Q)$ and $Lu \geqslant 0$ in Q. If $\limsup_{z\to\zeta} u(z) \leqslant 0$ for every $\zeta \in \partial_r Q$ then $u \leqslant 0$ in Q.*

The following theorem states the existence of a fundamental solution of L.

THEOREM 5.2. *There exists a fundamental solution Γ of L having the following properties:*

(i) *Γ is a nonnegative function which is smooth away from the diagonal of $\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$;*
(ii) *for every fixed $z \in \mathbb{R}^{N+1}$, $\Gamma(\cdot; z)$ and $\Gamma(z; \cdot)$ are locally integrable;*
(iii) *for every nonnegative measure μ with compact support and $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$, the following identities hold:*

$$L \int_{\mathbb{R}^{N+1}} \Gamma(\cdot; \zeta)\,\mathrm{d}\mu(\zeta) = -\mu,$$

$$\int_{\mathbb{R}^{N+1}} \Gamma(\cdot; \zeta) L\varphi(\zeta)\,\mathrm{d}\zeta = -\varphi;$$

(iv) $\Gamma(x, t; \xi, \tau) = 0$ *if $t \leqslant \tau$;*
(v) *for every $\zeta \in \mathbb{R}^{N+1}$, $L\Gamma(\cdot; \zeta) = -\delta_\zeta$, where δ_ζ denotes the Dirac measure supported in $\{\zeta\}$;*
(vi) *if we define*

$$\Gamma^*(z; \zeta) = \Gamma(\zeta; z), \qquad \forall z, \zeta \in \mathbb{R}^{N+1},$$

then Γ^* is a fundamental solution of L^*, the formal adjoint of L, satisfying the dual statements of (iii)–(v).

By means of hypothesis (H.3) and by suitably modifying some classical results about caloric functions, we prove the following asymptotic behavior of Γ at infinity.

THEOREM 5.3. *For every* $\zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$ *and for every* $\varepsilon > 0$ *there exists a compact set* $F \subseteq \mathbb{R}^{N+1}$ *and a positive constant* C *such that*

$$\Gamma(z; \zeta) \leqslant CK(z; \zeta(\varepsilon)), \quad \forall z \in \mathbb{R}^{N+1} \backslash F,$$

where $\zeta(\varepsilon) = (\xi, \tau - \varepsilon)$ *and* K *denotes the fundamental solution of the heat operator* H *in* $\mathbb{R}^{N+1}$.

The last section of [9] is devoted to the proof of some further classical properties of the fundamental solution. In particular, we have

THEOREM 5.4. *For every* $\zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$,

$$\limsup_{z \to \zeta} \Gamma(z; \zeta) = \infty. \tag{5.2}$$

Moreover, if $t > \tau$, *we have*

$$\int_{\mathbb{R}^N} \Gamma(x, t; \zeta)\, \mathrm{d}x = 1. \tag{5.3}$$

Acknowledgments

The main results of this paper were announced in [12].

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