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ON THE REGULARITY OF SOLUTIONS TO A NONLINEAR ULTRAPARABOLIC EQUATION ARISING IN MATHEMATICAL FINANCE*

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Abstract. We consider the following nonlinear degenerate parabolic equation which arises in some recent problems of mathematical finance:

$$\partial_{xx}u + u\partial_y u - \partial_t u = f.$$

Using a harmonic analysis technique on Lie groups, we prove that, if the solution u satisfies condition $\partial_x u \neq 0$ in an open set $\Omega \subset \mathbb{R}^3$ and $f \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.

1. Introduction

In this paper we study the interior regularity properties of the solutions to the equation in the variables $z = (x, y, t) \in \mathbb{R}^3$

$$Lu \equiv \partial_{xx}u + u\partial_y u - \partial_t u = f \tag{1.1}$$

satisfying the condition

$$\partial_x u(z) \neq 0 \qquad \forall z \in \Omega. \tag{1.2}$$

This equation arises in mathematical finance, when studying agents' decisions under risk. The problem is the representation of agents' preferences over consumption processes. Epstein and Zin in [9] have proposed a utility functional which is the solution of a backward stochastic differential equation. Recently Antonelli, Barucci and Mancino [1] proposed a more sophisticated utility functional that takes into account some aspects of decision making, such as the agents' habit formation, which is described as a smoothed average of past consumption and expected utility. In that model the couple

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of processes utility and habit is described by a system of backward-forward stochastic differential equations. The solution of such a system as a function of consumption and time satisfies the partial differential equation (1.1).

Several existence and uniqueness results are known for viscosity solutions of the Cauchy problem associated with equation (1.1), under different hypotheses on the initial data ([23], [10], [1]). However no regularity results are known. Here we are concerned with the regularity of the classical solutions of (1.1) (see Section 3 for the precise definition of classical solution). To this end, condition (1.2) is of crucial importance not only because it is suggested by the model, but also because equation (1.1) could have nonregular solutions if it is suppressed. For example any solution u independent of the variable x satisfies the Burgers equation

$$u\partial_y u - \partial_t u = 0,$$

which is of hyperbolic type. Our main result is the following.

Theorem 1.1. *Let Ω be an open subset of $\mathbb{R}^3$ and let $f \in C^\infty(\Omega)$. If u is a classical solution of equation* (1.1) *in Ω and satisfies condition* (1.2), *then $u \in C^\infty(\Omega)$.*

The operator L in (1.1) can be seen as a degenerate parabolic operator, and it can formally be represented as a sum of squares of nonlinear vector fields. Indeed if we set

$$X = \partial_x \quad \text{and} \quad Y_u = u\partial_y - \partial_t, \tag{1.3}$$

then L can be expressed as

$$Lu = X^2 u + Y_u u. \tag{1.4}$$

Condition (1.2) ensures that the vector fields X, Y_u and their commutator $[X, Y_u] = \partial_x u \partial_y$ are linearly independent at every point. This fact suggests a link with the theory of Hörmander's operators. These operators can be written in the form

$$H = \sum_{i=1}^{p} X_i^2 + X_0 \tag{1.5}$$

where X_i, $i = 0, \dots, p$ $(p \le N)$, are linear, smooth vector fields in $\mathbb{R}^N$ whose generated Lie algebra has maximum rank at every point. It is well known that this last condition, called the Hörmander condition, yields that H is hypoelliptic (see [14]). Under this condition there exists a fundamental solution Γ of the equation (1.5) whose properties have been investigated by [14], [20], [21], [17]. In particular in these papers a control distance d associated to the vector fields and their commutators has been introduced. Moreover estimates of Γ and its derivatives are proved in terms of d. Things are particularly easy when the Lie algebra generated by $X_1, \dots, X_p$ is nilpotent and stratified. In this case there exists a nonnegative integer Q, which is called the homogeneous dimension of the space, such that

$$\Gamma(z,\zeta) \le C d(z,\zeta)^{-Q+2} \tag{1.6}$$

for every $z, \zeta \in \Omega$. Hence a theory of the regularity similar to the classical one has been developed for this type of operator. If we denote by $C_d^{k,\alpha}$ the class of functions with derivatives of order k Hölder continuous with respect to the control distance d, then some a priori estimates formally analogous to the classical Schauder ones hold for the solutions of the equation $Hu = f$ (see [13], [11], [20]). Obviously, if $\bar{u}$ is a fixed function satisfying (1.2), then the linear operator

$$L_{\bar{u}} u = X^2 u + Y_{\bar{u}} u \tag{1.7}$$

is formally represented as in (1.5), and the associated classes of Hölder-continuous functions will be denoted by $C_{\bar{u}}^{k,\alpha}$. Then we have the following result (see, for example, [20]):

Theorem RS1. *Let the coefficient $\bar{u}$ of $L_{\bar{u}}$ be of class $C^\infty(\Omega)$, and let $f \in C_{\bar{u}}^{k-2,\alpha}(\Omega)$, $k \in \mathbb{N}$, $0 < \alpha < 1$. If u is a solution of $L_{\bar{u}} u = f$, then u is of class $C_{\bar{u}}^{k,\alpha}(\Omega)$.*

This result is optimal if the coefficient $\bar{u}$ is of class C^∞, and it can be easily extended with the same technique to less-regular vector fields. In this case, the following holds.

Theorem RS2. *Assume that $\bar{u} \in C_{\bar{u}}^{k+1,\alpha}(\Omega)$ and $f \in C_{\bar{u}}^{k-2,\alpha}(\Omega)$. If u is a solution of $L_{\bar{u}} u = f$, then u is of class $C_{\bar{u}}^{k,\alpha}(\Omega)$.*

We stress that this result can not be applied in our nonlinear situation, since the vector fields in (1.3) have only the regularity of the solution; then Theorem RS2 does not provide any gain of regularity. Hence we adapt to this framework a technique introduced by one of the authors in [6] and extended in [7] for studying the regularity of the solutions of another nonlinear operator. Here we are able to prove the following.

Theorem 1.2. *Assume that $\Omega \subseteq \mathbb{R}^3$ is an open set, $\bar{u} \in C_{\bar{u}}^{k-1,\alpha}(\Omega)$ satisfies (1.2), and $f \in C_{\bar{u}}^{k-2,\alpha}(\Omega)$, for any $\alpha \in (0,1)$. If u is a classical solution of $L_{\bar{u}} u = f$ in Ω, then $u \in C_{\bar{u}}^{k,\alpha}(\Omega)$, for any $\alpha \in (0,1)$.*

This result easily yields Theorem 1.1. The proof of Theorem 1.2 is established in two steps: a freezing method and a regularization procedure.

1.1. **Overview of the freezing method.** The freezing method is a well-known technique, classically used to study the regularity of solutions to linear parabolic operators of the form

$$\sum_{i,j=1}^{N} a_{ij}\partial_{x_i}\partial_{x_j} - \partial_t, \tag{1.8}$$

where $z = (x, t)$ denotes the point in $\mathbb{R}^N \times \mathbb{R}$. In this case, the associated frozen operator is simply obtained by evaluating the coefficients at a fixed point z_0:

$$\sum_{i,j=1}^{N} a_{ij}(z_0)\partial_{x_i}\partial_{x_j} - \partial_t.$$

This new operator is, up to a linear change of coordinates, the heat operator, and its fundamental solution can be considered as a parametrix of the fundamental solution of the operator in (1.8). An argument much more complicated was used to prove the existence of a fundamental solution for Hörmander-type operators (1.5). Indeed the properties of the operator rely on the vector fields $X_1, \dots, X_p$ and their commutators. If X_i are represented by

$$X_i = \sum_{j=1}^{N} a_{ij}\partial_{x_j} \qquad i = 1, \dots, p,$$

then the constant-coefficient operators

$$\widetilde{X}_{i,z_0} = \sum_{j=1}^{N} a_{ij}(z_0)\partial_{x_j} \qquad i = 1, \dots, p$$

commute, and the generated Lie algebra is $\mathbb{R}^p$ with $p \leq N$. Hence, in general, the operator $\sum_{i=1}^{p} \widetilde{X}_{i,z_0}^2 - \widetilde{X}_{0,z_0}$ is not hypoelliptic, and it has not a fundamental solution. Folland and Stein first pointed out that the model operators in this case are operators of the form (1.5) such that the Lie algebra generated by $X_1 \cdots X_p$ is nilpotent and stratified. Later on Rothschild and Stein introduced an abstract and very general version of the freezing method. The choice of the frozen vector fields X_{i,z_0} was made in such a way that their generated Lie algebra is nilpotent and stratified, and, at low orders, it has the same structure as $\mathrm{Lie}(X_1 \cdots X_p)$. With this choice of vector fields, the operator $\sum_{i=1}^{p} X_{i,z_0}^2 - X_{0,z_0}$ is hypoelliptic and nilpotent, and its fundamental solution Γ_{z_0} is a parametrix for the fundamental solution of (1.5). After the existence of the fundamental solution of (1.5) was established, a wide class of Hörmander operators has been studied with the same argument as (1.8): the operators of the form $\sum_{i,j=1}^{p} a_{ij}X_iX_j+X_0$, where $a_{i,j}$ are not even continuous but the vector fields X_i are of class C^∞ and satisfy the Hörmander condition for hypoellipticity. Here, for every point z_0, it is convenient to consider as frozen operator $\sum_{i,j=1}^{p} a_{ij}(z_0)X_iX_j + X_0$, that is an operator with C^∞ coefficients. In the same spirit as the results known for elliptic operators, by using the known properties of these operators, many sophisticated results have been obtained under very weak hypotheses on a_{ij} (see, for example, [15], [19], [3], [2]). See also [16] where the regularity properties of this kind of operator have been investigated by a different approach.

The few known results about nonlinear operators refer to operators whose nonlinearity depends on C^∞ vector fields (see, for example, [25], [4], [24], [26]).

Things are different when the vector fields themselves are not smooth, since that operators can not always be considered as simple perturbations of known linear operators. A first regularity result for solutions of a linear equation with continuous vector fields is due to Franchi and Lanconelli [12]. In a more recent study one of the authors introduced a simplification of the freezing method of Rothschild and Stein, for a second-order partial differential operator, based on the notion of "intrinsic" Taylor expansion of the coefficients [6].

In this paper we use a technique similar to the one in [6]. We consider the linearized operator $L_{\bar u}$ in (1.7) defined in an open subset Ω of $\mathbb{R}^3$. Assuming (1.2), we have that $X, Y_{\bar u}, [X, Y_{\bar u}]$ are linearly independent at every point. We observe that the simplest nilpotent Lie algebra with two generators and of dimension 3 is the Heisenberg algebra. Then, for every point $z_0 \in \Omega$, we associate with X and $Y_{\bar u}$ two frozen vector fields X and Y_{z_0} of class C^∞ and whose generated Lie algebra is the Heisenberg one. This choice ensures that the frozen operator $L_{z_0} = X^2 + Y_{z_0}$ is a nilpotent Hörmander-type operator. In particular L_{z_0} has a fundamental solution Γ_{z_0} and an associated control distance d_{z_0}. Unfortunately the distance d_{z_0} is not equivalent to the distance $d_{\bar u}$ associated with $L_{\bar u}$, nor are equivalent two distances d_{z_0} and d_{z_1} associated with different points z_0 and z_1. Since the qualitative behaviour of the solution strictly depends on the control distance, we have to study in detail the properties of these distances. This is done in Section 2 where we also study the properties of the Hölder classes related to these control distances.

1.2. **Overview of the regularization procedure.** In order to introduce our regularity procedure we consider a solution u in Ω of the linearized equation $L_{\bar u}u = f$. Fixing $z_0 \in \Omega$, we represent the function u in terms of the fundamental solution Γ_{z_0} of the frozen operator L_{z_0}:

$$u(z) = \int \Gamma_{z_0}(z,\zeta) L_{z_0} u(\zeta)\, d\zeta$$
$$= \int \Gamma_{z_0}(z,\zeta) f(\zeta)\, d\zeta + \int \Gamma_{z_0}(z,\zeta) K_{z_0}(\zeta)\, d\zeta, \tag{1.9}$$

where K_{z_0} is a kernel with the behaviour $K_{z_0}(\zeta) \sim d^q_{z_0}(z_0,\zeta)$, and the exponent q depends on the regularity of $\bar{u}$. In their classical paper [20], Rothschild and Stein choose $z_0 = z$ in the representation formula (1.9). Therefore the kernel which appears in the second term of (1.9) becomes $\Gamma_z(z,\zeta)K_z(\zeta)$, and it is less singular then Γ_{z_0}. Hence it is possible to perform higher-order derivatives with respect to z and to estimate them. On the other hand, Chiarenza, Frasca and Longo [5] noticed for the first time that, even in the parabolic case, it seems to be convenient, when dealing with nonregular coefficients, to keep z different from z_0 as long as it is possible. Using their technique in our situation, we can differentiate twice u with respect to z, and we obtain

$$D^2 u(z) = \int D^2\Gamma_{z_0}(z,\zeta) f(\zeta)\, d\zeta + \int D^2\Gamma_{z_0}(z,\zeta) K_{z_0}(\zeta)\, d\zeta.$$

Then, we evaluate the second order derivative of u at z_0:

$$D^2 u(z_0) = \int D^2\Gamma_{z_0}(z_0,\zeta) f(\zeta)\, d\zeta + \int D^2\Gamma_{z_0}(z_0,\zeta) K_{z_0}(\zeta)\, d\zeta.$$

In this way we compute the second derivative of u without differentiating the coefficient of Γ_{z_0}. The same idea has been used in [7] also for higher-order derivatives, and here we further extend it. Obviously, we can not repeat the preceding arguments for the third derivatives since, for $z \neq z_0$, the kernel $D^3\Gamma_{z_0}(z,\zeta)K_{z_0}(\zeta)$ is not locally integrable. Nevertheless, a rather delicate argument, based on the use of some high-order difference quotients (see Section 3), yields

$$D^3 u(z_0) = \int D\Gamma_{z_0}(z_0,\zeta) D^2 f(\zeta)\, d\zeta + \int D^3\Gamma_{z_0}(z_0,\zeta) K_{z_0}(\zeta)\, d\zeta.$$

In this way, we obtain some regularity results for the solutions even though the coefficients of the vector fields and of the fundamental solution of the frozen operator are not regular.

2. Freezing method

In this section we describe the freezing method for the linear equation

$$L = X^2 + Y_{\bar{u}},$$

where $X = \partial_x$, $Y_{\bar{u}} = \bar{u}\partial_y - \partial_t$ and $\bar{u}$ is a given function satisfying (1.2). In Subsection 2.1, we study the relation between the distances $d_{\bar{u}}$, associated with the vector fields X and $Y_{\bar{u}}$, and the distance d_{z_0} associated with the vector fields X and Y_{z_0}. In Subsection 2.2, we define classes $C^{k,\alpha}_{\bar{u}}$ of Hölder-continuous functions with respect to the distance $d_{\bar{u}}$, and we prove the existence of a polynomial expansion of Taylor type for functions of class $C^{k,\alpha}_{\bar{u}}$. Finally, in Subsection 2.3, we study the properties of the fundamental solution of the frozen operator.

2.1. **Heisenberg group and distances.** Here we recall some properties of the Heisenberg algebra and we establish some relations between the control distances corresponding to the linear and to the frozen operators.

The Heisenberg algebra is a Lie algebra with two generators, and nilpotent of step two. The simplest representation of the Heisenberg vector fields is

$$X_H = \partial_{\theta_1} - \frac{\theta_2}{2}\partial_{\theta_3} \quad \text{and} \quad Y_H = \partial_{\theta_2} + \frac{\theta_1}{2}\partial_{\theta_3}.$$

Clearly $[X_H, Y_H] = \partial_{\theta_3}$ and all the other commutators vanish. The associated Lie group $\mathbb{H}_1$ is then $\mathbb{R}^3$, endowed with the following composition law:

$$\theta \oplus \theta' = \big(\theta_1 + \theta_1', \theta_2 + \theta_2', \theta_3 + \theta_3' + \frac{1}{2}(\theta_1\theta_2' - \theta_2\theta_1')\big).$$

A natural dilations group on $\mathbb{H}_1$ is defined by

$$\delta^H_\lambda(\theta) = (\lambda\theta_1, \lambda^2\theta_2, \lambda^3\theta_3), \qquad \lambda > 0.$$

Since the Jacobian $J\delta^H_\lambda = \lambda^6$, the homogeneous dimension of $\mathbb{H}_1$ with respect to $(\delta^H_\lambda)_{\lambda>0}$ is the exponent $Q = 6$. A norm homogeneous with respect to this dilations group is given by $\|\theta\|_H = |\theta_1| + |\theta_2|^{\frac{1}{2}} + |\theta_3|^{\frac{1}{3}}$. The associated distance is obviously defined by $d_H(\theta', \theta) = \|\theta^{-1} \oplus \theta'\|_H$. Clearly X_H and Y_H are respectively δ^H_λ-homogeneous of degree one and two; that is,

$$X_H(u \circ \delta^H_\lambda) = \lambda(X_H u) \circ \delta^H_\lambda, \qquad Y_H(u \circ \delta^H_\lambda) = \lambda^2(Y_H u) \circ \delta^H_\lambda.$$

Thus, the second-order differential operator $L_H = X_H^2 + Y_H$ has a fundamental solution Γ_H which is invariant with respect to the left $\oplus$-translations and δ^H_λ-homogeneous of degree $-Q + 2$.

We next introduce the canonical coordinates corresponding to the linear operator $L_{\bar{u}}$. If D is a Lipschitz-continuous vector field, and $[0, 1]$ is contained in the domain of the local solution to the Cauchy problem

$$\gamma'(s) = D(\gamma(s)), \quad \gamma(0) = z,$$

we let $\exp(D)(z) = \gamma(1)$ and call an exponential map the application $D \mapsto \exp(D)(z)$. Since $\gamma(s) = \exp(sD)(z)$, then $\exp(D)(z)$ is defined for D sufficiently small. If D_1, D_2, D_3 are Lipschitz-continuous vector fields, linearly independent at every point, then the map

$$F_z : \theta \mapsto \exp(\theta \cdot D)(z) = \exp(\theta_1 D_1 + \theta_2 D_2 + \theta_3 D_3)(z)$$

is a diffeomorphism of a neighborhood of the origin of $\mathbb{R}^3$ to a neighborhood U_z of z. Its inverse function $\theta_{D,z} = F_z^{-1}$ defines the canonical change of variable associated with the vector field D, and center z. When $D_1, D_2, D_3 \in C^1$, by the properties of the solutions of the Cauchy problem, the Jacobian matrix JF_z of the function F_z depends continuously on z. Then, by the local invertibility theorem, the open set U_z continuously depends on z.

By our assumptions, $X, Y_{\bar{u}}$ and $[X, Y_{\bar{u}}] = \bar{u}_x \partial_y$ are linearly independent at every point, but $\bar{u}_x$ is not Lipschitz continuous, so we cannot define $\exp\left(\theta \cdot (X, Y_{\bar{u}}, [X, Y_{\bar{u}}])\right)$. We instead consider

$$\nabla_{\bar{u}} = (X, Y_{\bar{u}}, \partial_y)$$

and denote by $\theta_{\bar{u},z_0}$ the associated canonical change of coordinates, defined on $U_{z_0} \subseteq \Omega$. This function allows us to introduce a topological structure in a neighborhood of z_0, naturally associated with the vector fields X and $Y_{\bar{u}}$. Indeed, by the continuity of U_z, there exists $r = r(z_0) > 0$ such that the Euclidean ball $B(z_0, r)$ satisfies

$$B(z_0, r) \subseteq U_z, \qquad \forall z \in B(z_0, r). \tag{2.1}$$

Thus, if $z, \zeta \in B(z_0, r)$, then $\zeta \in U_z$, $\theta_{\bar{u},z}(\zeta)$ is defined, and we can set

$$d_{\bar{u}}(z, \zeta) = \|\theta_{\bar{u},z}(\zeta)\|_H. \tag{2.2}$$

More explicitly, we have

$$\theta_{\bar{u},z}(\zeta) = \Big(\xi - x, -(\tau - t), \eta - y + (\tau - t) \int_0^1 \bar{u}(\gamma(s))\, ds\Big), \tag{2.3}$$

and

$$d_{\bar{u}}(z, \zeta) = |\xi - x| + |\tau - t|^{\frac{1}{2}} + \left|\eta - y + (\tau - t) \int_0^1 \bar{u}(\gamma(s))\, ds\right|^{\frac{1}{3}},$$

where $\gamma(s) = \exp\left(s\theta_{\bar{u},z}(\zeta) \cdot \nabla_{\bar{u}}\right)(z)$ with $s \in [0, 1]$.

Remark 2.1. Let $z_0 \in \Omega$ and $z \in U_{z_0}$. An integral curve of $\nabla_{\bar{u}}$, connecting z and z_0, is $\gamma(s) = \exp\left(s\theta_{\bar{u},z_0}(z) \cdot \nabla_{\bar{u}}\right)(z_0)$ with $s \in [0, 1]$. Then, for every $s \in [0, 1]$, we have

$$d_{\bar{u}}(\gamma(s), z_0) = \|s\theta_{\bar{u},z_0}(z)\|_H \leq \|\theta_{\bar{u},z_0}(z)\|_H = d_{\bar{u}}(z_0, z).$$

Since $\theta_{\bar{u},z}(\zeta)$ is only a local diffeomorphism, it does not introduce a group structure on $\mathbb{R}^3$. In order to overcome this problem, we define a new vector field in the following way: for every fixed $z_0 \in \Omega$, we define the frozen operator of $Y_{\bar{u}}$ as follows: $Y_{z_0} = Y_{\bar{u}(z_0)+\bar{u}_x(z_0)(x-x_0)}$, where we have denoted, as usual, $u_x(z_0) = \partial_x u(z_0)$. Obviously X, Y_{z_0} and $[X, Y_{z_0}] = \bar{u}_x(z_0)\partial_y$ are of class C^∞ and all the commutators of higher order are null. Now we set $\nabla_{z_0} = (X, Y_{z_0}, [X, Y_{z_0}])$ and $L_{z_0} = X^2 + Y_{z_0}$. The map $\theta \mapsto \exp(\theta \cdot \nabla_{z_0})(z)$ is a global diffeomorphism. We denote by $\theta_z^{(z_0)}$ the canonical coordinates associated with ∇_{z_0} and of center z. As a consequence of the Campbell-Hausdorff formula we have

$$\begin{aligned} &X\big(u \circ \theta_{z_0}^{(z_0)}\big) = (X_H u) \circ \theta_{z_0}^{(z_0)}, \quad Y_{z_0}\big(u \circ \theta_{z_0}^{(z_0)}\big) = (Y_H u) \circ \theta_{z_0}^{(z_0)}, \\ &\bar{u}_x(z_0)\partial_y\big(u \circ \theta_{z_0}^{(z_0)}\big) = (\partial_{\theta_3} u) \circ \theta_{z_0}^{(z_0)}. \end{aligned} \tag{2.4}$$

Then, as a direct consequence, we have $L_{z_0}\big(u \circ \theta_{z_0}^{(z_0)}\big) = (L_H u) \circ \theta_{z_0}^{(z_0)}$. The diffeomorphism $\theta_{z_0}^{(z_0)}$ naturally induces a Lie group structure with dilations on $\mathbb{R}^3$. Indeed, we define the composition law

$$z \circ \zeta = (\theta_{z_0}^{(z_0)})^{-1}(\theta_{z_0}^{(z_0)}(z) \oplus \theta_{z_0}^{(z_0)}(\zeta)), \tag{2.5}$$

the dilations

$$\delta_\lambda^{(z_0)}(z) = (\theta_{z_0}^{(z_0)})^{-1}(\delta_\lambda^H(\theta_{z_0}^{(z_0)}(z))), \qquad \lambda > 0,$$

and the function d_{z_0} defined by

$$d_{z_0}(z, \zeta) = \big\|\big(\theta_{z_0}^{(z_0)}(z)\big)^{-1} \oplus \theta_{z_0}^{(z_0)}(\zeta)\big\|_H,$$

which is a quasi-distance, in the sense that there exists a positive constant $\widetilde{C} = \widetilde{C}(z_0)$ such that

$$d_{z_0}(z_0, \zeta) \le \widetilde{C}\left(d_{z_0}(z_0, \eta) + d_{z_0}(\eta, \zeta)\right), \qquad z, \eta, \zeta \in \mathbb{R}^3. \tag{2.6}$$

Then $G_{z_0} = (\mathbb{R}^3, \circ)$ is the Lie group associated with the Lie algebra $\mathcal{L}_{z_0} = \mathrm{Lie}(X, Y_{z_0})$, generated by X and Y_{z_0}, and it is isomorphic to $\mathbb{H}_1$. The quasi-distance we have introduced can be represented as

$$d_{z_0}(z, \zeta) = \|\theta_z^{(z_0)}(\zeta)\|_H, \qquad z, \zeta \in \mathbb{R}^3, \tag{2.7}$$

and, more explicitly,

$$\theta_z^{(z_0)}(\zeta) = \tag{2.8}$$

$$\Big(\xi - x, -(\tau - t), \frac{1}{\bar{u}_x(z_0)}\big(\eta - y + (\tau - t)\big(\bar{u}(z_0) + \bar{u}_x(z_0)\frac{\xi + x - 2x_0}{2}\big)\big)\Big),$$

$$d_{z_0}(z, \zeta) = \tag{2.9}$$

$$|\xi - x| + |\tau - t|^{\frac{1}{2}} + \Big|\frac{1}{\bar u_x(z_0)}\big(\eta - y + (\tau - t)\big(\bar u(z_0) + \bar u_x(z_0)\frac{\xi + x - 2x_0}{2}\big)\big)\Big|^{\frac{1}{3}}.$$

We next describe some relations between d_{z_0} and $d_{\bar u}$.

Remark 2.2. In the sequel, we shall also use the distance defined by

$$\widetilde d_{z_0}(z,\zeta) = \|\widetilde\theta_z^{(z_0)}(\zeta)\|_H, \quad \widetilde\theta_z^{(z_0)}(\zeta) = (\xi - x, -(\tau - t), \eta - y + (\tau - t)\bar u(z_0)) \,. \tag{2.10}$$

This distance is equivalent to d_{z_0}, in the sense that there exists a positive constant C, which depends only on $\bar u_x(z_0)$, such that

$$\frac{1}{C}\widetilde d_{z_0}(z_0,\zeta) \le d_{z_0}(z_0,\zeta) \le C\widetilde d_{z_0}(z_0,\zeta), \qquad \forall \zeta \in \mathbb{R}^3.$$

Here and in the sequel, C will denote a constant which will not always be the same. The proof of the above statement relies only on the following elementary inequality,

$$(ab)^{\frac{1}{3}} \le \frac{1}{3}a + \frac{2}{3}b^{\frac{1}{2}}, \qquad \forall a, b > 0, \tag{2.11}$$

and on the explicit expression of d_{z_0} provided in (2.9).

Lemma 2.3. *Let $U_{z_0} \subset\subset \Omega$ be a neighborhood of z_0 such that $(\theta_1, \theta_2, \theta_3) \equiv \theta_{\bar u, z_0}(\zeta)$ is defined for every $\zeta \in U_{z_0}$. If $(\widetilde\theta_1, \widetilde\theta_2, \widetilde\theta_3) \equiv \widetilde\theta_{z_0}^{(z_0)}(\zeta)$ is defined as in* (2.10), *then we have*

$$|\theta_3 - \widetilde\theta_3| \le C|\tau - t_0| d_{\bar u}(z_0,\zeta), \qquad \forall \zeta \in U_{z_0}. \tag{2.12}$$

Proof. Since $\bar u$ is a locally Lipschitz-continuous function (in the Euclidean sense), we get

$$|\bar u(\gamma(s)) - \bar u(z_0)| \le C_1|\gamma(s) - z_0| \le C_2 d_{\bar u}(z_0, \gamma(s)),$$

where $\gamma(s) = \exp(s\theta_{\bar u, z_0}(\zeta)\cdot\nabla_{\bar u})(z_0)$, for $s \in [0,1]$, and C_1, C_2 depend only on U_{z_0}. Thus the assertion follows from expressions (2.3) and (2.10):

$$|\theta_3 - \widetilde\theta_3| \le |\tau - t_0| \int_0^1 |\bar u(\gamma(s)) - \bar u(z_0)|\, ds \le C_2|\tau - t_0| \int_0^1 d_{\bar u}(z_0, \gamma(s))\, ds$$

(by Remark 2.1) $\le C_2|\tau - t_0| d_{\bar u}(z_0,\zeta)$.

Proposition 2.4. *For every $z_1 \in \Omega$, there exists a compact neighborhood $K \subseteq \Omega$ of z_1, and a positive constant $C = C(K)$, such that*

i) $C^{-1} d_{z_0}(z_0,\zeta) \le d_{\bar u}(z_0,\zeta) \le C d_{z_0}(z_0,\zeta)$,
ii) $d_{z_0}(z_0, z) \le C(d_{z_0}(z_0,\zeta) + d_\zeta(\zeta, z))$,
iii) $d_{\bar u}(z_0, z) \le C(d_{\bar u}(z_0,\zeta) + d_{\bar u}(\zeta, z))$, *for every $z, z_0, \zeta \in K$.*

Proof. We first remark that there exists $r = r(z_1) > 0$ such that

$$K \equiv B(z_1, r) \subseteq U_{z_0} \cap U_z \cap U_\zeta.$$

(i) If $\theta_{\bar u, z_0}(\zeta)$ and $\widetilde{\theta}_{z_0}^{(z_0)}(\zeta)$ are the functions defined in (2.3) and (2.10), they have the first two components in common. Then, using Lemma 2.3, we get

$$\begin{aligned}\widetilde{d}_{z_0}(z_0,\zeta) &= \|\widetilde{\theta}_{z_0}^{(z_0)}(\zeta)\|_H \le \|\theta_{\bar u,z_0}(\zeta)\|_H + |\theta_3 - \widetilde{\theta}_3|^{\frac13}\\ &\le d_{\bar u}(z_0,\zeta) + C|\tau - t_0|^{\frac23} d_{\bar u}(z_0,\zeta)^{\frac13} \le C_1 d_{\bar u}(z_0,\zeta).\end{aligned}$$

On the other hand, again from Lemma 2.3 and (2.11), we get

$$|\theta_3 - \widetilde{\theta}_3| \le C|\tau - t_0| d_{\bar u}(z_0,\zeta) \le C\Big(\frac23\Big(\frac{|\tau - t_0|}{\delta}\Big)^{\frac32} + \frac13 (\delta d_{\bar u}(z_0,\zeta))^3\Big)$$

for any positive δ. Therefore

$$\begin{aligned} d_{\bar u}(z_0,\zeta) &\le \widetilde{d}_{z_0}(z_0,\zeta) + |\theta_3 - \widetilde{\theta}_3|^{\frac13}\\ &\le \Big(\delta^{-\frac12}\Big(\frac{2C}{3}\Big)^{\frac13} + 1\Big)\widetilde{d}_{z_0}(z_0,\zeta) + \delta\Big(\frac{C}{3}\Big)^{\frac13} d_{\bar u}(z_0,\zeta).\end{aligned}$$

By choosing a suitably small $\delta > 0$ and by Remark 2.2, we get the thesis.

(ii) We observe that

$$\begin{aligned} |y - y_0 + \bar u(z_0)(t - t_0)|^{\frac13} &\le |y - \eta + \bar u(\zeta)(t-\tau)|^{\frac13}\\ &\quad + |\eta - y_0 + \bar u(z_0)(\tau - t_0)|^{\frac13} + |(t-\tau)(\bar u(z_0) - \bar u(\zeta))|^{\frac13}\\ &\le \widetilde{d}_\zeta(\zeta, z) + \widetilde{d}_{z_0}(z_0,\zeta) + (|t-\tau| d_{\bar u}(z_0,\zeta))^{\frac13},\end{aligned}$$

since $\bar u$ is Lipschitz continuous. The last term can be estimated by (2.11) and (i), as follows:

$$|(t-\tau) d_{\bar u}(z_0,\zeta)|^{\frac13} \le \tfrac23 |t - \tau|^{\frac12} + C_1 d_{\bar u}(z_0,\zeta) \le \tfrac23 \widetilde{d}_\zeta(\zeta, z) + C\widetilde{d}_{z_0}(z_0,\zeta).$$

By the definition of $\widetilde{d}_{z_0}(z_0, z)$, this last inequality and Remark 2.2 yield the assertion.

(iii) It is a direct consequence of (i) and (ii). □

Remark 2.5. Since we are proving a local result, from now on we shall always work in a compact set $K \subseteq \Omega$, satisfying the assumptions of Proposition 2.4.

2.2. **Hölder-continuous functions and Taylor polynomials.**

Definition 2.6. Let $z_0 \in \Omega$, $0 < \alpha < 1$ and let D be a locally Lipschitz continuous vector field on Ω. We say that $u \in C^\alpha_D(z_0)$ if there exists a positive constant C such that

$$|u(\exp(hD)(z_0)) - u(z_0)| \leq C|h|^\alpha, \tag{2.13}$$

for every suitably small h. We say that $u \in C^\alpha_D(\Omega)$, if (2.13) holds uniformly on compact subsets of Ω.

Definition 2.7. Let $z_0 \in \Omega$ and D be a Lipschitz-continuous vector field in Ω. We say that there exists the Lie derivative of u with respect to D in z_0, if the following limit exists:

$$Du(z_0) \equiv \lim_{h\to 0} \frac{u(\exp(hD)(z_0)) - u(z_0)}{h}.$$

We denote by $C^0_{\bar u}(\Omega)$ (or $C_{\bar u}(\Omega)$) the set of continuous functions in Ω. If $u \in C_{\bar u}(\Omega)$, and there exists $Xu \in C_{\bar u}(\Omega)$, we say that $u \in C^1_{\bar u}(\Omega)$. If $k \geq 2$, $Xu \in C^{k-1}_{\bar u}(\Omega)$ and $Y_{\bar u}u \in C^{k-2}_{\bar u}(\Omega)$, then we say that $u \in C^k_{\bar u}(\Omega)$.

Remark 2.8. We remark that if $u \in C^1(\Omega)$ and D can be expressed as $D = d_1\partial_x + d_2\partial_y + d_3\partial_t$, with d_1, d_2, d_3 Lipschitz-continuous functions, then there exists the Lie derivative of u and it can be expressed as

$$Du(z_0) = d_1\partial_x u(z_0) + d_2\partial_y u(z_0) + d_3\partial_t u(z_0), \qquad z_0 \in \Omega.$$

We next define the spaces of Hölder-continuous functions related to the linear operator $L_{\bar u}$.

Definition 2.9. Let $\bar u$ be a C^1 function satisfying (1.2), and let $0 < \alpha < 1$.

We say that $u \in C^\alpha_{\bar u}(\Omega)$ if $u \in C^\alpha_X(\Omega)$ and $u \in C^{\frac{\alpha}{2}}_{Y_{\bar u}}(\Omega)$.

We say that $u \in C^{1,\alpha}_{\bar u}(\Omega)$ if $Xu \in C^\alpha_{\bar u}(\Omega)$ and $u \in C^{\frac{1+\alpha}{2}}_{Y_{\bar u}}(\Omega)$.

We say that $u \in C^{2,\alpha}_{\bar u}(\Omega)$ if $Xu \in C^{1,\alpha}_{\bar u}(\Omega)$ and $Y_{\bar u}u \in C^\alpha_{\bar u}(\Omega)$.

Let $k \geq 3$ and suppose that $\bar u \in C^{k-2,\alpha}_{\bar u}(\Omega)$. We say that $u \in C^{k,\alpha}_{\bar u}(\Omega)$ if $Xu \in C^{k-1,\alpha}_{\bar u}(\Omega)$ and $Y_{\bar u}u \in C^{k-2,\alpha}_{\bar u}(\Omega)$.

For the sake of simplicity, in the sequel, when we write $u \in C^{k,\alpha}_{\bar u}(\Omega)$, for $k \geq 3$, we will assume implicitly that $\bar u \in C^{k-2,\alpha}_{\bar u}(\Omega)$.

Remark 2.10. For every fixed $z \in \Omega$, we agree to work only in a compact neighborhood K of z_1 satisfying the assumptions of Proposition 2.4. Thus $d_{\bar u}(z_0, z)$ is defined for every $z_0, z \in K$. We will see that, as a simple consequence of Theorem 2.16, that the class $C^\alpha_{\bar u}(\Omega)$ is defined in such a way that, for every $u \in C^{\alpha}_{\bar{u}}(\Omega)$,

$$|u(z) - u(z_0)| \le C d_{\bar{u}}(z_0, z)^{\alpha}, \qquad \forall z, z_0 \in K.$$

Remark 2.11. In the sequel we will use the following simple result: if $0 < \alpha < 1$ and $k \ge 1$, then $C^{k,\alpha}_{\bar{u}}(\Omega) \subseteq C^{k-1,\beta}_{\bar{u}}(\Omega)$, $\forall \beta \in (0,1)$.

We next prove some regularity results in the direction ∂_y for a function belonging to the spaces $C^{k,\alpha}_{\bar{u}}(\Omega)$.

Proposition 2.12. *Let $\bar{u}$ be a C^1-function satisfying* (1.2) *and let* $0<\alpha<1$.

i) *If $u \in C^{\alpha}_{\bar{u}}(\Omega)$, then $u \in C^{\frac{\alpha}{3}}_{\partial_y}(\Omega)$;*

ii) *If $u \in C^{1,\alpha}_{\bar{u}}(\Omega)$, then $u \in C^{\frac{1+\alpha}{3}}_{\partial_y}(\Omega)$;*

iii) *If $u \in C^{2,\alpha}_{\bar{u}}(\Omega)$, then $u \in C^{\frac{2+\alpha}{3}}_{\partial_y}(\Omega)$;*

iv) *If $u \in C^{k,\alpha}_{\bar{u}}(\Omega)$, with $k \ge 3$, then there exists $\partial_y u$ and it belongs to $C^{k-3,\alpha}_{\bar{u}}(\Omega)$.*

Remark 2.13. We explicitly note that $C^{3k,\alpha}_{\bar{u}}(\Omega) \subseteq C^{k,\frac{\alpha}{3}}(\Omega) \subseteq C^{k,\frac{\alpha}{3}}_{\bar{u}}(\Omega)$, for every $0 < \alpha < 1$ and $k \ge 1$, where $C^{k,\beta}(\Omega)$ is the space of functions with derivatives up to order k that are β-Hölder-continuous functions in the Euclidean sense. Thus, if $u \in C^{k,\alpha}_{\bar{u}}(\Omega)$ for every $k \in \mathbb{N}$, then $u \in C^{\infty}(\Omega)$.

Proof of Proposition 2.12. We start with a preliminary remark. If $\bar{u} \in C^{\infty}(\Omega)$, the proof is a consequence of the Campbell-Hausdorff formula. The assertion could be deduced from the general theory also if $\bar{u} \in C^{4,\alpha}_{\bar{u}}(\Omega)$. Since $\bar{u}$ is only of class C^1, we proceed by direct computation. Let z_0 be a point in Ω and, for a suitably small $\delta > 0$, let $z_1 = \exp(\delta X)(z_0)$, $z_2 = \exp\left(\delta^2 Y_{\bar{u}}\right)(z_1)$, $z_3 = \exp(-\delta X)(z_2)$, $z_4 = \exp\left(-\delta^2 Y_{\bar{u}}\right)(z_3)$. We claim that

$$z_4 = \exp\left((\delta^3 + o(\delta^3))\bar{u}_x \partial_y\right)(z_0), \qquad \text{as } \delta \to 0. \tag{2.14}$$

We denote by $(x_j, y_j, t_j) = z_j$, for $j = 0, 1, \dots, 4$. Let $\gamma_j : [0,1] \to \Omega$ be the integral path connecting the point z_{j-1} to z_j, $j = 1, 2, 3, 4$. It is easy to see that

$$\begin{aligned}
&\gamma_1(s) = (x_0 + \delta s, y_0, t_0), \qquad \text{then } z_1 = (x_0 + \delta, y_0, t_0),\\
&\gamma_2(s) = \left(x_0 + \delta, y_0 + \delta^2 \int_0^s \bar{u}(\gamma_2(\tau))\, d\tau, t_0 - \delta^2 s\right),\\
&\gamma_3(s) = \left(x_0 + \delta(1-s), y_0 + \delta^2 \int_0^1 \bar{u}(\gamma_2(\tau))\, d\tau, t_0 - \delta^2\right),
\end{aligned} \tag{2.15}$$

$$\gamma_4(s) = \Big(x_0, y_0 + \delta^2\Big(\int_0^1 \bar u(\gamma_2(\tau))\,d\tau - \int_0^s \bar u(\gamma_4(\tau))\,d\tau\Big), t_0 + \delta^2(s-1)\Big);$$

then it is clear that $x_4 = x_0$ and $t_4 = t_0$. In order to estimate $y_4 - y_0$, we observe that

$$\begin{aligned}\bar u(\gamma_2(\tau)) - \bar u(\gamma_4(\tau)) &= \nabla \bar u(\widetilde z(\tau)) \cdot (\gamma_2(\tau) - \gamma_4(\tau)) \\ &= \delta \bar u_x(\widetilde z(\tau)) + \delta^2 \bar u_y(\widetilde z(\tau))\Big(-\int_\tau^1 \bar u(\gamma_2(\sigma))\,d\sigma + \int_0^\tau \bar u(\gamma_4(\sigma))\,d\sigma\Big) \\ &\quad + (1-2\tau)\delta^2 \bar u_t(\widetilde z(\tau)) = \delta \bar u_x(z_0)\,(1+o(1))\,, \qquad \text{as } \delta \to 0,\end{aligned} \tag{2.16}$$

since $\bar u_x$ is a continuous function. As a consequence of (2.15), we get

$$y_4 - y_0 = \delta^2 \int_0^1 (\bar u(\gamma_2(\tau)) - \bar u(\gamma_4(\tau)))\,d\tau = \delta^3 \bar u_x(z_0)(1+o(1)); \tag{2.17}$$

as $\delta \to 0$, then (2.14) is proved. Moreover, since y_4 depends continuously on δ and $u_x(z_0) \neq 0$, the function $\delta \mapsto y_4$ is surjective in a neighborhood of y_0. Hence, for every β sufficiently small, there exists a $\delta = \delta(\beta)$ such that the point $(x_0, y_0 + \beta, t_0)$ can be written as z_4 in (2.15) (with $\delta = \delta(\beta)$). We stress that (2.17) also yields

$$\frac{\delta(\beta)^3}{\beta} \longrightarrow \frac{1}{\bar u_x(z_0)} \qquad \text{as } \beta \to 0. \tag{2.18}$$

After these preliminary considerations, we conclude the proof as follows.

(i) Let β be chosen as above. Since $u \in C^\alpha_{\bar u}(\Omega)$, we have

$$\begin{aligned} &|u(z_0) - u(z_1)| \le C\delta^\alpha, \qquad |u(z_1) - u(z_2)| \le C\delta^\alpha, \\ &|u(z_2) - u(z_3)| \le C\delta^\alpha, \qquad |u(z_3) - u(z_4)| \le C\delta^\alpha; \end{aligned} \tag{2.19}$$

then, since $\bar u_x(z_0) \neq 0$,

$$|u(z_0) - u(z_4)| \le 4C\delta^\alpha = 4C\Big(\frac{\beta}{\bar u_x(z_0)(1+o(1))}\Big)^{\frac{\alpha}{3}} \qquad \text{as } \beta \to 0,$$

and this proves (i).

(ii) We consider the functions $\gamma_j : [0,1] \to \mathbb{R}$, defined in (2.15) for $j = 1, \dots, 4$ and we apply the Taylor expansion of first order to $u \circ \gamma_j$. Since, by hypothesis, u is of class $C^{1,a}$ as a function of the first variable x, we have

$$\begin{aligned} u(z_0) - u(z_1) &= \delta u_x(z_1) + O(\delta^{\alpha+1}), \\ u(z_2) - u(z_3) &= -\delta u_x(z_2) + O(\delta^{\alpha+1}). \end{aligned} \tag{2.20}$$

Since $u \in C^{\frac{1+\alpha}{2}}_{Y_{\bar{u}}}(\Omega)$,

$$|u(z_1) - u(z_2)| \le C\delta^{\alpha+1}, \qquad |u(z_3) - u(z_4)| \le C\delta^{\alpha+1}.$$

Then, by using (2.20), (2.18) and the fact that $u_x \in C^{\alpha}_{\bar{u}}(\Omega)$, we conclude that $u(z_0) - u(z_4) = O(\delta^{\alpha+1}) = O(\beta^{\frac{\alpha+1}{3}})$ as $\beta \to 0$.

(iii) In this case we use Taylor polynomials of order 2. As δ tends to zero, we have

$$\begin{aligned}
u(z_0) - u(z_1) &= \delta u_x(z_1) + \frac{\delta^2}{2} u_{xx}(z_1) + O(\delta^{2+\alpha}),\\
u(z_2) - u(z_3) &= -\delta u_x(z_2) - \frac{\delta^2}{2} u_{xx}(z_2) + O\left(\delta^{2+\alpha}\right),\\
u(z_1) - u(z_2) &= \delta^2 Y_{\bar{u}} u(z_2) + O(\delta^{\alpha+2}),\\
u(z_3) - u(z_4) &= -\delta^2 Y_{\bar{u}} u(z_3) + O(\delta^{\alpha+2}).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
u(z_0) - u(z_4) =& \delta \left(u_x(z_1) - u_x(z_2)\right) + \frac{\delta^2}{2}\left(u_{xx}(z_1) - u_{xx}(z_2)\right)\\
&+ \delta^2 \left(Y_{\bar{u}} u(z_2) - Y_{\bar{u}} u(z_3)\right) + O\left(\delta^{2+\alpha}\right)
\end{aligned}$$

(since $u_x \in C^{1,\alpha}_{\bar{u}}(\Omega)$ and $u_{xx}, Y_{\bar{u}} \in C^{\alpha}_{\bar{u}}(\Omega)) = O\left(\delta^{2+\alpha}\right)$, as $\delta \to 0$, and this yields (iii).

(iv) We first consider the problem for $k = 3$. By using a Taylor polynomial of higher order, we get, as δ tends to zero,

$$\begin{aligned}
u(z_0) - u(z_1) &= \delta u_x(z_1) + \frac{\delta^2}{2} u_{xx}(z_1) + \frac{\delta^3}{6} u_{xxx}(z_1) + O\left(\delta^{3+\alpha}\right),\\
u(z_1) - u(z_2) &= -\delta^2 Y_{\bar{u}} u(z_2) + O\left(\delta^{3+\alpha}\right),\\
u(z_2) - u(z_3) &= -\delta u_x(z_2) - \frac{\delta^2}{2} u_{xx}(z_2) - \frac{\delta^3}{6} u_{xxx}(z_2) + O\left(\delta^{3+\alpha}\right),\\
u(z_3) - u(z_4) &= \delta^2 Y_{\bar{u}} u(z_3) + O\left(\delta^{3+\alpha}\right).
\end{aligned} \tag{2.21}$$

Then

$$\begin{aligned}
u(z_0) - u(z_4) &= \delta \left(u_x(z_1) - u_x(z_2)\right) + \frac{\delta^2}{2}\left(u_{xx}(z_1) - u_{xx}(z_2)\right)\\
&\quad + \delta^2 \left(Y_{\bar{u}} u(z_2) - Y_{\bar{u}} u(z_3)\right) + \frac{\delta^3}{6}\left(u_{xxx}(z_1) - u_{xxx}(z_2)\right) + O\left(\delta^{3+\alpha}\right).
\end{aligned} \tag{2.22}$$

Since $u_x \in C^{2,\alpha}_{\bar{u}}(\Omega)$, $Y_{\bar{u}} u_x \in C^{\alpha}_{\bar{u}}(\Omega)$ and $d_{\bar{u}}(z_0, z_2) \le \delta$, we have

$$u_x(z_1) - u_x(z_2) = -\delta^2 Y_{\bar{u}} u_x(z_2) + O\left(\delta^{2+\alpha}\right) = -\delta^2 Y_{\bar{u}} u_x(z_0) + O\left(\delta^{2+\alpha}\right).$$

Since $u_{xx} \in C_{\bar{u}}^{\frac{1+\alpha}{2}}(\Omega)$ and $u_{xxx} \in C_{\bar{u}}^{\alpha}(\Omega)$, we have

$$u_{xx}(z_1) - u_{xx}(z_2) = O\left(\delta^{1+\alpha}\right), \quad u_{xxx}(z_1) - u_{xxx}(z_2) = O\left(\delta^{\alpha}\right).$$

Moreover, by the fact that $Y_{\bar{u}}u \in C_{\bar{u}}^{1,\alpha}(\Omega)$, $\partial_x Y_{\bar{u}}u \in C_{\bar{u}}^{\alpha}(\Omega)$ and $d_{\bar{u}}(z_0, z_2) \leq \delta$, we get

$$Y_{\bar{u}}u(z_2) - Y_{\bar{u}}u(z_3) = \delta\partial_x Y_{\bar{u}}u(z_2) + O\left(\delta^{1+\alpha}\right) = \delta\partial_x Y_{\bar{u}}u(z_0) + O\left(\delta^{1+\alpha}\right).$$

Inserting in (2.22), we finally obtain

$$u(z_0) - u(z_4) = \delta^3\left(\partial_x Y_{\bar{u}}u(z_0) - Y_{\bar{u}}u_x(z_0)\right) + O(\delta^{3+\alpha})$$

or, in other words,

$$\begin{aligned} \frac{u(x_0, y_0 + \beta, t_0) - u(x_0, y_0, t_0)}{\beta} &= \frac{\delta(\beta)^3}{\beta}\big[X, Y_{\bar{u}}\big]u(z_0) + O\left(\delta(\beta)^{\alpha}\right) \\ \longrightarrow \frac{1}{\bar{u}_x(z_0)}\big[X, Y_{\bar{u}}\big]u(z_0), & \qquad \text{as } \beta \to 0. \end{aligned} \tag{2.23}$$

This proves the existence of $\partial_y u$. The regularity follows from the fact that

$$u_y(z) = \frac{1}{\bar{u}_x(z_0)}\left(\partial_x Y_{\bar{u}}u(z) - Y_{\bar{u}}u_x(z)\right), \qquad \forall z \in \Omega.$$

The proof in the case $k > 3$ is immediate: the existence of $\partial_y u$ has been proved, while its regularity directly follows from the above identity. $\square$

We introduce the "Taylor polynomials" related to the spaces $C_{\bar{u}}^{k,\alpha}(\Omega)$ above considered.

Definition 2.14. Let $z_0 = (x_0, y_0, t_0) \in \Omega, k \in \mathbb{N}$ and let $\bar{u}$ be a C^1 function such that (1.2) holds. We denote by $P_{z_0}^k$ any function of the form

$$P_{z_0}^k(x, y, t) = \sum_{i+2j+3m \leq k} c_{i,j,m}(x - x_0)^i(t - t_0)^j(y - y_0 + (t - t_0)\bar{u}(z_0))^m \tag{2.24}$$

where $i, j, m \in \mathbb{N} \cup \{0\}$ and $c_{i,j,m}$ are real constants. We say that $P_{z_0}^k$ is a polynomial of initial point z_0 and $\delta_\lambda^{(z_0)}$-degree k.

Remark 2.15. The functions $x - x_0, t - t_0$ and $y - y_0 + (t - t_0)\bar{u}(z_0)$ are $\delta_\lambda^{(z_0)}$-homogeneous of degree $1, 2$ and 3, respectively, since they are $\theta_1, -\theta_2$ and $\bar{u}_x(z_0)(\theta_3 + \theta_1\theta_2)$, where $(\theta_1, \theta_2, \theta_3) \equiv \theta_{z_0}^{(z_0)}(z)$ defined in (2.8).

We next state the main result of this subsection.

Theorem 2.16. *Let $z_0 \in \Omega$, $0 < \alpha < 1$, $k \in \mathbb{N} \cup \{0\}$ and assume that $u \in C^{k,\alpha}_{\bar u}(\Omega)$. Then there exists a polynomial function $P^k_{z_0} u$, of $\delta^{(z_0)}_\lambda$-degree k, such that*

$$u(z) = P^k_{z_0} u(z) + O(d_{\bar u}(z_0, z)^{k+\alpha}) \qquad \text{as } z \to z_0. \tag{2.25}$$

Proof. We prove our result by a classical argument that relies on Proposition 2.12. We observe that the regularity assumption is given in terms of the geometry corresponding to $\theta_{\bar u, z_0}$ while we obtain polynomial functions that are homogeneous with respect to $\delta^{(z_0)}_\lambda$.

The assertion is obvious if $k = 0$. We first prove (2.25) for $k = 1, 2$. We proceed essentially as in the proof of Proposition 2.12. We consider a point $z_0 = (x_0, y_0, t_0) \in \Omega$ and a compact neighborhood $K \subset \Omega$ of z_0 such that every $z = (x, y, t) \in K$ can be connected to z_0 as follows. Let

$$\begin{aligned}
&\gamma_1(s) = \exp(s(x - x_0)X)(z_0) && \text{and} && z_1 = \gamma_1(1) = (x, y_0, t_0)\\
&\gamma_2(s) = \exp(-s(t - t_0)Y_{\bar u})(z_1) && \text{and} && z_2 = \gamma_2(1) = (x, y_2, t)\\
&\gamma_3(s) = \exp(s(y - y_2)\partial_y)(z_2).
\end{aligned}$$

Clearly

$$|x - x_0| \le d_{\bar u}(z_0, z), \qquad |t - t_0| \le d_{\bar u}(z_0, z)^2. \tag{2.26}$$

Let us now estimate $|y - y_2|$. We have

$$\begin{aligned}
|y - y_2| &= d_{\bar u}(z, z_2)^3 \le \widetilde{C}^2 \left(d_{\bar u}(z, z_0) + d_{\bar u}(z_0, z_1) + d_{\bar u}(z_1, z_2)\right)^3\\
&\le \widetilde{C}^2 \left(d_{\bar u}(z, z_0) + |x - x_0| + |t - t_0|^{\frac{1}{2}}\right)^3 \le C d_{\bar u}(z_0, z)^3.
\end{aligned} \tag{2.27}$$

Let us give a detailed proof of (2.25) for $k = 1$. The case $k = 2$ can be treated analogously.

$$u(z) - u(z_0) = u(z_1) - u(z_0) + u(z_2) - u(z_1) + u(z) - u(z_2) =$$

(since $u \in C^{1,\alpha}(\Omega)$ as a function of its first variable, $u \in C^{\frac{1+\alpha}{2}}_{Y_{\bar u}}(\Omega)$ and $u \in C^{\frac{2+\alpha}{3}}_{\partial_y}(\Omega)$)

$$= u(z_0) + (x - x_0) u_x(z_0) + O(d_{\bar u}(z_0, z)^{1+\alpha}) + O(|t - t_0|^{\frac{1+\alpha}{2}}) + O(|y - y_2|^{\frac{2+\alpha}{3}}) =$$

(by (2.26) and (2.27))

$$= u(z_0) + (x - x_0) u_x(z_0) + O(d_{\bar u}(z_0, z)^{1+\alpha}), \qquad \text{as } z \to z_0.$$

For $k \ge 3$ we simply argue by induction. We recall that, by our convention, we assume $\bar u \in C^{k-2,\alpha}_{\bar u}(\Omega)$. If $\widetilde{\gamma}$ is the Euclidean segment connecting z and

z_0, we have

$$\begin{aligned} u(z)-u(z_0) = (x-x_0)\int_0^1 \partial_x u(\widetilde{\gamma}(s))\,ds - (t-t_0)\int_0^1 Y_{\bar u} u(\widetilde{\gamma}(s))\,ds \\ + (y-y_0+\bar u(z_0)(t-t_0))\int_0^1 \partial_y u(\widetilde{\gamma}(s))\,ds \\ + (t-t_0)\int_0^1 (\bar u(\widetilde{\gamma}(s))-\bar u(z_0))\,\partial_y u(\widetilde{\gamma}(s))\,ds. \end{aligned} \tag{2.28}$$

By the inductive hypothesis, u_x has a Taylor expansion of the form (2.24)–(2.25) of $\delta_\lambda^{(z_0)}$-degree $k-1$. Thus we have

$$\begin{aligned} &(x-x_0)\int_0^1 \partial_x u(\widetilde{\gamma}(s))\,ds = (x-x_0)\int_0^1 \sum_{i+2j+3m\le k-1} c_{i,j,m}(x-x_0)^i(t-t_0)^j \\ &\times (y-y_0+(t-t_0)\bar u(z_0))^m s^{i+j+m}\,ds + (x-x_0)\int_0^1 O(d_{\bar u}(z_0,\widetilde{\gamma}(s))^{k-1+\alpha})ds \\ &= \sum_{i+2j+3m\le k-1} \frac{c_{i,j,m}}{i+j+m+1}(x-x_0)^{i+1}(t-t_0)^j\,(y-y_0+(t-t_0)\bar u(z_0))^m \\ &+ O\left(d_{\bar u}\,(z_0,z)^{k+\alpha}\right), \qquad \text{as } z\to z_0. \end{aligned}$$

In the same way we can handle the second and the third term in the right-hand side of (2.28), since, by our assumption, $Y_{\bar u}u \in C_{\bar u}^{k-2,\alpha}(\Omega)$ and $\partial_y u \in C_{\bar u}^{k-3,\alpha}(\Omega)$. The last term can be estimated as follows. Let $\bar c_{i,j,m}$ (respectively $\hat c_{i,j,m}$) denote the coefficients of $P_{z_0}^{k-2}\bar u$ (respectively $P_{z_0}^{k-3}u_y$). Then we have

$$\begin{aligned} &(t-t_0)\int_0^1 (\bar u(\widetilde{\gamma}(s))-\bar u(z_0))\,\partial_y u(\widetilde{\gamma}(s))\,ds \\ &= (t-t_0)\int_0^1 \Big(P_{z_0}^{k-2}\bar u(\widetilde{\gamma}(s)) - \bar u(z_0) + O(d_{\bar u}(z_0,\widetilde{\gamma}(s))^{k-2+\alpha})\Big)\cdot \\ &\cdot \Big(P_{z_0}^{k-3}u_y(\widetilde{\gamma}(s)) + O(d_{\bar u}(z_0,\widetilde{\gamma}(s))^{k-3+\alpha})\Big)\,ds \end{aligned}$$

(since $P_{z_0}^{k-2}\bar u\,(\widetilde{\gamma}(s)) - \bar u(z_0) = O\,(d_{\bar u}(z_0,z))$, as $z\to z_0$)

$$\begin{aligned} &= \int_0^1 \sum_{\substack{0<i+2j+3m\le k-2,\\ i_1+j_1+m_1\le k-3}} \bar c_{i,j,m}\hat c_{i_1,j_1,m_1}(x-x_0)^{i+i_1}(t-t_0)^{j+j_1+1}\cdot \\ &\cdot (y-y_0+(t-t_0)\bar u(z_0))^{m+m_1}\, s^{i+i_1+j+j_1+m+m_1}ds + O(d_{\bar u}(z_0,z)^{k+\alpha}). \quad \square \end{aligned}$$

2.3. **Parametrix.** In this subsection, we provide some results about the fundamental solution of the frozen operator L_{z_0}. As we previously noticed, the second-order differential operator

$$L_H = X_H^2 + Y_H$$

has a fundamental solution Γ_H, which is invariant with respect to the left $\oplus$-translations and δ_λ^H-homogeneous of degree $-Q+2$. Hence a fundamental solution of L_{z_0} is given by

$$\Gamma_{z_0}(z,\zeta) = \frac{1}{\bar{u}_x(z_0)}\Gamma_H\left(\left(-\theta_{z_0}^{(z_0)}(\zeta)\right) \oplus \theta_{z_0}^{(z_0)}(z)\right) \tag{2.29}$$

and it is $\delta_\lambda^{(z_0)}$-homogeneous of degree $-Q+2$. We remark that in [22], Kolmogorov wrote explicitly the fundamental solution of the operator

$$\partial_{xx} + x\partial_y - \partial_t,$$

which is, up to a canonical change of coordinates, the fundamental solution of L_H or L_{z_0}. However here we don't make use of that explicit formula, but we use only its local behavior.

For the sake of convenience, here and in the sequel we systematically use the following notation:

$$\begin{aligned} D_1 &= X, \qquad D_1^{(z_0)} = X, \qquad D_1^H = X_H,\\ D_2 &= Y_{\bar{u}}, \qquad D_2^{(z_0)} = Y_{z_0}, \qquad D_2^H = Y_H. \end{aligned} \tag{2.30}$$

Besides we denote $D_3 = \partial_y$, $D_3^{(z_0)} = \bar{u}_x(z_0)\partial_y$, $D_3^H = \partial_{\theta_3}$. We also denote the identity by $D_0 = D_0^{(z_0)} = D_0^H$. For every multi-index $\sigma = (\sigma_1, \dots, \sigma_m)$, with $\sigma_r \in \{0,1,2,3\}$, $1 \le r \le m \in \mathbb{N}$, we set

$$D_\sigma = D_{\sigma_1}\cdots D_{\sigma_m}, \quad D_\sigma^{(z_0)} = D_{\sigma_1}^{(z_0)}\cdots D_{\sigma_m}^{(z_0)}, \quad D_\sigma^H = D_{\sigma_1}^H\cdots D_{\sigma_m}^H. \tag{2.31}$$

We call height of σ the natural number

$$|\sigma| = \sum_{r=1}^m \sigma_r. \tag{2.32}$$

We remark that $D_\sigma^{(z_0)}$ (resp. D_σ^H) is a $\delta_\lambda^{(z_0)}$(respectively δ_λ^H)-homogeneous operator of degree $|\sigma|$. Since Γ_{z_0} depends on many variables, the notation $D(z_1)\Gamma_{z_0}(\cdot,\zeta)$ shall denote the D-derivative of $\Gamma_{z_0}(z,\zeta)$ with respect to the variable z, evaluated at the point z_1.

If $\varphi \in C_0^\infty$, a simple relation holds between the derivatives $D_\sigma\varphi$ and $D_\sigma^{(z_0)}\varphi$.

Lemma 2.17. *If $\bar{u} \in C^{k,\alpha}_{\bar{u}}(\Omega)$, and $\sigma \in \{0,1,2\}^{k+1}$ with $|\sigma| \le k+1$, then for every function $\varphi \in C^\infty_0(\Omega)$ the derivative $D_\sigma\varphi$ can be represented as*

$$D_\sigma\varphi(z) = \sum_{\varrho\in I_\sigma} C_\varrho \frac{(\bar{u} - P^1_{z_0}\bar{u})^{k_\varrho}(z)}{(\bar{u}_x(z_0))^{h_\varrho}} \prod_{\mu\in J_\varrho} D_\mu\bar{u}(z) D^{(z_0)}_\varrho\varphi(z),$$

where I_σ and J_ϱ are suitable subsets of $\{0,1,2,3\}^{k+1}$, C_ϱ are nonnegative constants, and h_ϱ and k_ϱ are nonnegative integers such that

$$|\varrho| \le |\sigma| + k_\varrho, \qquad k_\varrho \le h_\varrho, \qquad |\mu| \le k_\varrho \le 3\,|\sigma|\,, \qquad \forall\mu\in I_\varrho, \forall\varrho\in I_\sigma.$$

If J_ϱ is empty, we set $\prod_{\mu\in J_\varrho} D_\sigma\bar{u} = 1$.

Proof. Since the function φ is of class $C^\infty_0(\Omega)$, by Remark 2.8 its Lie derivatives can be represented in terms of the standard partial derivatives, and it is not necessary to use the exponential function. If $|\sigma| \le 2$, the assertion follows directly from the definition. Indeed, if $\sigma \in \{(1),(1,1)\}$, then $D_\sigma = D^{(z_0)}_\sigma$, while, if $\sigma = (2)$, then

$$D_\sigma = D^{(z_0)}_\sigma + \frac{(\bar{u} - P^1_{z_0}\bar{u})}{\bar{u}_x(z_0)} D^{(z_0)}_3.$$

The general assertion follows by induction on $|\sigma|$. □

Analogously it is not difficult to prove the following.

Lemma 2.18. *If $\bar{u} \in C^{1,\alpha}_{\bar{u}}(\Omega)$, $\sigma \in \{0,1,2\}^{k+1}$ with $|\sigma| \le k+1$, and z_0, $z_1 \in \Omega$, then for every function $\varphi \in C^\infty_0(\Omega)$*

$$D^{(z_1)}_\sigma\varphi(z) =$$
$$D^{(z_0)}_\sigma\varphi(z) + \sum_{\varrho\in J_\sigma} C_\varrho (\bar{u} - P^1_{z_1}\bar{u})^{k_\varrho}(z_0)(x - x_0)^{h_\varrho} \left(\bar{u}_x(z_0) - \bar{u}_x(z_1)\right)^{j_\varrho} D^{(z_0)}_\varrho\varphi(z),$$

where J_σ is a suitable family of subsets of $\{0,1,2,3\}^{k+1}$, C_ϱ are nonnegative constants, and j_ϱ and k_ϱ are nonnegative integers such that

$$|\varrho| \le |\sigma| + k_\varrho + h_\varrho \qquad h_\varrho \le j_\varrho, \forall\varrho\in I_\sigma.$$

Proof. If $|\sigma| \in \{(1),(1,1)\}$ the assertion is obvious. If $\sigma = (2)$, then

$$\begin{aligned} D^{(z_1)}_\sigma\varphi(z) - D^{(z_0)}_\sigma\varphi(z) &= \left(P^1_{z_1}\bar{u}(z) - P^1_{z_0}\bar{u}(z)\right)\partial_y\varphi(z) \\ &= -\left(\bar{u}(z_0) - P^1_{z_1}\bar{u}(z_0) - (x - x_0)\left(\bar{u}_x(z_1) - \bar{u}_x(z_0)\right)\right)\partial_y\varphi(z). \end{aligned}$$

Now, let us suppose that the assertion is true for every multi-index of height less than or equal to $k-1$. We choose $\sigma = (\sigma_1, \sigma')$, of height $|\sigma| = k$. We assume for simplicity that $\sigma_1 = 1$, since the proof is similar if $\sigma_1 = 2$. Then

$$
\begin{aligned}
&D^{(z_1)}_{\sigma}\varphi = \\
&D^{(z_0)}_1\Big(D^{(z_0)}_{\sigma'}\varphi + \sum_{\varrho\in J_{\sigma'}} C_\varrho(\bar u - P^1_{z_1}\bar u)^{k_\varrho}(z_0)(x-x_0)^{h_\varrho}\left(\bar u_x(z_0) - \bar u_x(z_1)\right)^{j_\varrho} D^{(z_0)}_{\varrho}\varphi\Big)\\
&= D^{(z_0)}_{\sigma}\varphi + \sum_{\varrho\in J_{\sigma'}} h_\varrho C_\varrho(\bar u - P^1_{z_1}\bar u)^{k_\varrho}(z_0)(x-x_0)^{h_\varrho-1}\left(\bar u_x(z_0) - \bar u_x(z_1)\right)^{j_\varrho} D^{(z_0)}_{\varrho}\varphi\\
&+ \sum_{\varrho\in J_{\sigma'}} C_\varrho(\bar u - P^1_{z_1}\bar u)^{k_\varrho}(z_0)(x-x_0)^{h_\varrho}\left(\bar u_x(z_0) - \bar u_x(z_1)\right)^{j_\varrho} D^{(z_0)}_1 D^{(z_0)}_{\varrho}\varphi.
\end{aligned}
$$

Remark 2.19. It is well known that for every compact set $K \subset \Omega$ and for every multi-index σ, there exists a positive constant C such that

$$|D^{(z_0)}_{\sigma}(z)\Gamma_{z_0}(\,\cdot\,,\zeta)| \le C d_{z_0}(z,\zeta)^{-Q+2-|\sigma|}, \qquad \forall z, z_0, \zeta \in K,\ z \neq \zeta.$$

By Lemma 2.17, we also have

$$|D_{\sigma}(z)\Gamma_{z_0}(\,\cdot\,,\zeta)| \le C d_{z_0}(z,\zeta)^{-Q+2-|\sigma|}.$$

Fixing an open subset Ω of $\mathbb{R}^3$, a positive constant M and two points $z_0, z \in \Omega$, we set

$$\Omega_M = \{\zeta \in \Omega : d_{z_0}(z_0,\zeta) \ge M d_{z_0}(z_0,z)\}. \tag{2.33}$$

This set is defined in such a way that the function $z \mapsto \Gamma_{z_0}(z,\zeta)$ is smooth, if ζ belongs to Ω_M. Indeed, if M is sufficiently large, then, by (2.6),

$$d_{z_0}(z,\zeta) \ge \frac{1}{\widetilde C} d_{z_0}(z_0,\zeta) - d_{z_0}(z_0,z) \ge \left(\frac{1}{\widetilde C} - \frac{1}{M}\right) d_{z_0}(z_0,\zeta), \tag{2.34}$$

for every $\zeta \in \Omega_M$.

Proposition 2.20. *Let $k \in \mathbb{N}$. Let $\bar u \in C^{k-1,\alpha}_{\bar u}(\Omega)$ and let K be a compact subset of Ω. There is a positive constant C, such that, for every multi-index σ, $|\sigma| = k$, we have*

$$
\begin{aligned}
&|(D_\sigma(z) - D_\sigma(z_0))\,\Gamma_{z_0}(\cdot,\zeta)|\\
&\le C\left(d_{z_0}(z_0,z) d_{z_0}(z_0,\zeta)^{-Q+1-k} + d_{z_0}(z_0,z)^{\alpha} d_{z_0}(z_0,\zeta)^{-Q+2-k}\right),
\end{aligned}
\tag{2.35}
$$

for every $z, z_0 \in K$, $\zeta \in \Omega_M$ defined in (2.33) *for suitable $M > 0$.*

Proof. We apply Lemma 2.17, and denote $\widetilde{I}_\sigma = \{\varrho : k_\varrho = 0\}$. Then, we have

$$
\begin{aligned}
&(D_\sigma(z) - D_\sigma(z_0))\Gamma_{z_0}(\cdot,\zeta)\\
&= \sum_{\varrho\in I_\sigma\setminus\widetilde{I}_\sigma} C_\varrho \frac{(\bar u - P^1_{z_0}\bar u)^{k_\varrho}(z)}{(\bar u_x(z_0))^{h_\varrho}} \prod_{\mu\in J_\varrho} D_\mu \bar u(z) D^{(z_0)}_\varrho(z)\Gamma_{z_0}(\cdot,\zeta)\\
&+ \sum_{\varrho\in\widetilde{I}_\sigma} C_\varrho \Big(\prod_{\mu\in J_\varrho} D_\mu\bar u(z) - \prod_{\mu\in J_\varrho} D_\mu \bar u(z_0)\Big) D^{(z_0)}_\varrho(z_0)\Gamma_{z_0}(\cdot,\zeta)\\
&+ \sum_{\varrho\in\widetilde{I}_\sigma} C_\varrho \prod_{\mu\in J_\varrho} D_\mu\bar u(z)\left(D^{(z_0)}_\varrho(z) - D^{(z_0)}_\varrho(z_0)\right)\Gamma_{z_0}(\cdot,\zeta).
\end{aligned}
$$

Now we estimate each term separately. By simplicity let us call them S_1, S_2, S_3 respectively. We first note that if with $M > \widetilde{C}$, $\zeta\in\Omega_M$ and $\bar z\in\Omega$ is such that $d_{z_0}(z_0,\bar z)\le d_{z_0}(z_0,z)$, then we have

$$d_{z_0}(\bar z,\zeta) \ge \frac{1}{\widetilde{C}} d_{z_0}(z_0,\zeta) - d_{z_0}(z_0,\bar z) \ge \frac{1}{\widetilde{C}} d_{z_0}(z_0,\zeta) - d_{z_0}(z_0,z)$$

(by definition of Ω_M)

$$\ge \Big(\frac{1}{\widetilde{C}} - \frac{1}{M}\Big) d_{z_0}(z_0,\zeta). \tag{2.36}$$

We first consider S_1

$$|S_1| \le C \sum_{|\varrho|\le|\sigma|+k_\varrho} d_{z_0}(z_0,z)^{k_\varrho(1+\alpha)} d_{z_0}(z,\zeta)^{-Q+2-|\varrho|} \le$$

(using (2.33), (2.34), the fact that $|\varrho|\le|\sigma|+k_\varrho$ and that K is bounded)

$$\le C d_{z_0}(z_0,z)^\alpha d_{z_0}(z_0,\zeta)^{-Q+2-|\sigma|}.$$

Analogously we can estimate S_2. Indeed, since $D_\mu\bar u\in C^\alpha$ for every μ such that $|\mu|\le k$ and $|\varrho|\le|\sigma|$, we get

$$|S_2| \le C d_{z_0}(z_0,z)^\alpha d_{z_0}(z_0,\zeta)^{-Q+2-|\sigma|}.$$

Finally

$$|S_3| \le \sum_{|\varrho|\le|\sigma|} C_\varrho \prod_{\mu\in I_\varrho} D_\mu\bar u(z)\left(D^{(z_0)}_\varrho(z) - D^{(z_0)}_\varrho(z_0)\right)\Gamma_{z_0}(\cdot,\zeta) =$$

(by the mean value theorem, for some $\bar z$ such that $d_{z_0}(z_0,\bar z)\le d_{z_0}(z_0,z)$)

$$= \sum_{|\varrho|\le|\sigma|} C_\varrho \langle \theta^{(z_0)}_{z_0}(z), (\nabla_{z_0} D^{(z_0)}_\varrho)(\bar z)\Gamma_{z_0}(\cdot,\zeta)\rangle$$

$$\leq \sum_{|\varrho|\leq|\sigma|} C_\varrho \sum_{r=1}^{3} d_{z_0}(z_0,z)^r d_{z_0}(\bar{z},\zeta)^{-Q-r+2-|\varrho|}$$

(by (2.36), and the definition of Ω_M)

$$\leq C d_{z_0}(z_0,z) d_{z_0}(z_0,\zeta)^{-Q+1-|\sigma|}.$$

The main results of this section are contained in the following statement.

Proposition 2.21. *Let $k \in \mathbb{N}$, $\bar{u} \in C_{\bar{u}}^{k-1,\alpha}(\Omega)$ and K be a compact subset of Ω. There exist two positive constants C and M, such that*

$$\begin{aligned} &|D_\sigma(z)\Gamma_z(\cdot,\zeta) - D_\sigma(z_0)\Gamma_{z_0}(\cdot,\zeta)| \\ &\leq C\left(d_{z_0}(z_0,z)d_{z_0}(z_0,\zeta)^{-Q+1-|\sigma|} + d_{z_0}(z_0,z)^\alpha d_{z_0}(z_0,\zeta)^{-Q+2-|\sigma|}\right), \end{aligned} \tag{2.37}$$

for every multi-index σ, $|\sigma| = k$, and for every $z, z_0 \in K$, $\zeta \in \Omega_M$ defined in (2.33).

The proof of the above statement relies on the following.

Lemma 2.22. *Let Ω be a bounded open set, $\bar{u} \in C_{\bar{u}}^{1,\alpha}(\Omega)$, K a compact subset of Ω and $M > 0$. Then there exists a positive constant M_0 such that*

$$|(\theta_{z_0}^{(z_0)}(\zeta) \oplus (-\theta_z^{(z)}(\zeta)))_3| \leq M_0(d_{z_0}(z_0,\zeta)^3 d_{z_0}(z_0,z)^\alpha + d_{z_0}(z_0,\zeta)^2 d_{z_0}(z_0,z)), \tag{2.38}$$

for every $z, z_0 \in K$ and for every $\zeta \in \Omega_M$ (the notation $(\cdot)_3$ in the left-hand side denotes the third component of the considered vector). Moreover, if M is sufficiently big, there exist two constants $M_1, M_2 > 1$ such that, for every $z, z_0 \in K$ and for every $\zeta \in \Omega_M$, we have

$$M_1\|\theta_{z_0}^{(z_0)}(\zeta) \oplus (-\theta_z^{(z)}(\zeta))\|_H \leq \|\theta_{z_0}^{(z_0)}(\zeta)\|_H \tag{2.39}$$

and, for every $\theta \in \mathbb{R}^3$ such that $d_H(\theta_{z_0}^{(z_0)}(\zeta),\theta) \leq d_H(\theta_z^{(z)}(\zeta),\theta_{z_0}^{(z_0)}(\zeta))$, we have

$$d_{z_0}(z_0,\zeta) \leq M_2\|\theta\|_H. \tag{2.40}$$

Proof. A straightforward computation gives

$$\begin{aligned} \theta_{z_0}^{(z_0)}(\zeta) \oplus (-\theta_z^{(z)}(\zeta)) = \big(&x - x_0, t_0 - t, \frac{1}{2}(x-x_0)(2\tau - t - t_0) \\ &+ \frac{1}{\bar{u}_x(z_0)}(\eta - y_0 + \bar{u}(z_0)(\tau - t_0)) - \frac{1}{\bar{u}_x(z)}(\eta - y + \bar{u}(z)(\tau - t))\big). \end{aligned}$$

If we denote by θ_3 the last component of $\theta_{z_0}^{(z_0)}(\zeta) \oplus (-\theta_z^{(z)}(\zeta))$, we have

$$|\theta_3| \leq \frac{1}{2}|x - x_0|(|\tau - t_0| + |\tau - t|)$$

$$+\big|\big(\frac{1}{\bar u_x(z)}-\frac{1}{\bar u_x(z_0)}\big)(\eta-y_0+\bar u(z_0)(\tau-t_0))\big|$$
$$+\big|\frac{1}{\bar u_x(z)}(y-y_0+\bar u(z_0)(\tau-t_0)-\bar u(z)(\tau-t))\big|$$

$$\begin{aligned}
&\le C_1\big(|x-x_0|(|\tau-t_0|+|\tau-t|)\\
&+\big|\frac{\bar u_x(z)-\bar u_x(z_0)}{\bar u_x(z)\bar u_x(z_0)}\big||\eta-y_0+\bar u(z_0)(\tau-t_0)|\\
&+|y-y_0+\bar u(z_0)(t-t_0)|+|t-\tau||\bar u(z)-\bar u(z_0)-\bar u_x(z_0)(x-x_0)|\\
&+|\bar u_x(z_0)||t-\tau||x-x_0|\big)\\
&\le C_2\big(d_{z_0}(z_0,z)(d_{z_0}(z_0,\zeta)^2+d_{z_0}(z,\zeta)^2)+d_{z_0}(z_0,z)^\alpha d_{z_0}(z_0,\zeta)^3\\
&+d_{z_0}(z_0,z)^3+d_{z_0}(z,\zeta)^2d_{z_0}(z_0,z)^{1+\alpha}+d_{z_0}(z,\zeta)^2d_{z_0}(z_0,z)\big),
\end{aligned}$$

since $\bar u_x\in C^\alpha_{\bar u}(\Omega)$. By using the inequality

$$d_{z_0}(z,\zeta)\le\widetilde C\,(d_{z_0}(z_0,z)+d_{z_0}(z_0,\zeta))$$

we then obtain

$$\begin{aligned}
|\theta_3|\le &C_3\big(d_{z_0}(z_0,\zeta)^2d_{z_0}(z_0,z))+d_{z_0}(z_0,\zeta)^3d_{z_0}(z_0,z)^\alpha+d_{z_0}(z_0,z)^3\\
&+d_{z_0}(z_0,\zeta)^{3+\alpha}+d_{z_0}(z_0,\zeta)^2d_{z_0}(z_0,z)^{1+\alpha}\big).
\end{aligned}$$

Since $z_0,z\in K$ and $\zeta\in\Omega_M$, from the above inequality we immediately deduce (2.38). Moreover, using the fact that Ω is bounded and that $\zeta\in\Omega_M$, we get

$$\begin{aligned}
\|\theta^{(z_0)}_{z_0}(\zeta)\oplus(-\theta^{(z)}_z(\zeta))\|_H&\le d_{z_0}(z_0,z)+|\theta_3|^{\frac13}\\
&\le\frac{d_{z_0}(z_0,\zeta)}{M}+M_0^{\frac13}\big(\frac{d_{z_0}(z_0,\zeta)^3}{M^\alpha}+\frac{d_{z_0}(z_0,\zeta)^3}{M}\big)^{\frac13};
\end{aligned}$$

then, by choosing M sufficiently great, we also obtain (2.39). Finally, if $d_H(\theta^{(z_0)}_{z_0}(\zeta),\theta)\le d_H(\theta^{(z)}_z(\zeta),\theta^{(z_0)}_{z_0}(\zeta))$, from (2.39) we obtain

$$\begin{aligned}
\|\theta\|_H&\ge d_H(\theta^{(z_0)}_{z_0}(\zeta),0)-d_H(\theta^{(z_0)}_{z_0}(\zeta),\theta)\\
&\ge d_H(\theta^{(z_0)}_{z_0}(\zeta),0)-d_H(\theta^{(z)}_z(\zeta),\theta^{(z_0)}_{z_0}(\zeta))\\
&\ge\big(1-\frac{1}{M_1}\big)d_H(\theta^{(z_0)}_{z_0}(\zeta),0)\ge\frac{1}{M_2}d_{z_0}(z_0,\zeta).
\end{aligned}$$

This proves (2.40) and concludes the proof of the lemma. □

Proof of Proposition 2.21. Let us first prove that, for every multi-index σ, $|\sigma| \geq 0$, we have

$$\begin{aligned} &\left|D^H_\sigma \Gamma_H(-\theta^{(z)}_z(\zeta)) - D^H_\sigma \Gamma_H(-\theta^{(z_0)}_{z_0}(\zeta))\right| \\ &\leq C\left(d_{z_0}(z_0,z)^\alpha d_{z_0}(z_0,\zeta)^{-Q+2-|\sigma|} + d_{z_0}(z_0,z) d_{z_0}(z_0,\zeta)^{-Q+1-|\sigma|}\right), \end{aligned} \tag{2.41}$$

for some positive constant C. Indeed, we have

$$|D^H_\sigma \Gamma_H(-\theta^{(z)}_z(\zeta)) - D^H_\sigma \Gamma_H(-\theta^{(z_0)}_{z_0}(\zeta))| =$$

(by the mean value theorem and denoting $\nabla_H = (X_H, Y_H, \partial_{\theta_3})$)

$$= |\langle \theta^{(z_0)}_{z_0}(\zeta) \oplus (-\theta^{(z)}_z(\zeta)), \nabla_H D^H_\sigma \Gamma_H(\theta)\rangle| \leq$$

(where $d_H(\theta^{(z_0)}_{z_0}(\zeta), \theta) \leq d_H(\theta^{(z)}_z(\zeta), \theta^{(z_0)}_{z_0}(\zeta))$)

$$\begin{aligned} &\leq |x-x_0||X_H D^H_\sigma \Gamma_H(\theta)| + |t-t_0||Y_H D^H_\sigma \Gamma_H(\theta)| \\ &+ |(\theta^{(z_0)}_{z_0}(\zeta) \oplus (-\theta^{(z)}_z(\zeta)))_3||\partial_{\theta_3} D^H_\sigma \Gamma_H(\theta)| \leq \end{aligned}$$

(by (2.40) and (2.38))

$$\leq C(d_{z_0}(z_0,z) d_{z_0}(z_0,\zeta)^{-Q+1-|\sigma|} + d_{z_0}(z_0,z)^\alpha d_{z_0}(z_0,\zeta)^{-Q+2-|\sigma|}).$$

Assertion (2.41) is then proved. Now we can apply Lemma 2.17, again denoting $\widetilde{I}_\sigma = \{\varrho : k_\varrho = 0\}$. We have

$$\begin{aligned} &|D_\sigma(z_0)\Gamma_{z_0}(\cdot,\zeta) - D_\sigma(z)\Gamma_z(\cdot,\zeta)| \\ &= \Big|\sum_{\varrho\in\widetilde{I}_\sigma}\Big(\frac{C_\varrho}{\bar u_x(z_0)^{h_\varrho}}\prod_{\mu\in I_\varrho} D_\mu \bar u(z_0) D^{(z_0)}_\varrho(z_0)\Gamma_{z_0}(\cdot,\zeta) \\ &- \frac{C_\varrho}{\bar u_x(z)^{h_\varrho}}\prod_{\mu\in I_\varrho} D_\mu \bar u(z) D^{(z)}_\varrho(z)\Gamma_{z_0}(\cdot,\zeta)\Big)\Big| \\ &\leq \sum_{\varrho\in\widetilde{I}_\sigma} C_\varrho \Big|\frac{1}{\bar u_x(z_0)^{h_\varrho}}\prod_{\mu\in I_\varrho} D_\mu \bar u(z_0) - \frac{1}{\bar u_x(z)^{h_\varrho}}\prod_{\mu\in I_\varrho} D_\mu \bar u(z)\Big| |D^{(z_0)}_\varrho(z_0)\Gamma_{z_0}(\cdot,\zeta)| \\ &+ \sum_{\varrho\in\widetilde{I}_\sigma} C_\varrho \Big|\prod_{\mu\in I_\varrho} D_\mu \bar u(z)\Big| |D^{(z)}_\varrho(z)\Gamma_z(\cdot,\zeta) - D^{(z_0)}_\varrho(z_0)\Gamma_{z_0}(\cdot,\zeta)| \end{aligned}$$

(by (2.29))

$$= \sum_{\varrho\in\widetilde{I}_\sigma} C_\varrho \Big|\frac{1}{\bar u_x(z_0)^{h_\varrho}}\prod_{\mu\in I_\varrho} D_\mu \bar u(z_0) - \frac{1}{\bar u_x(z)^{h_\varrho}}\prod_{\mu\in I_\varrho} D_\mu \bar u(z)\Big| |D^{(z_0)}_\varrho(z_0)\Gamma_{z_0}(\cdot,\zeta)|$$

$$+\sum_{\varrho\in\widetilde{I}_\sigma} C_\varrho \Big|\prod_{\mu\in I_\varrho} D_\mu \bar{u}(z)\Big| \Big| \frac{1}{\bar{u}_x(z_0)}(D_\varrho^H\Gamma_H)(-\theta_{z_0}^{(z_0)}(\zeta)) - \frac{1}{\bar{u}_x(z)}(D_\varrho^H\Gamma_H)(-\theta_z^{(z)}(\zeta))\Big|$$

(by 2.41)

$$\leq C\sum_{\varrho\in\widetilde{I}_\sigma}\left(d_{z_0}(z_0,z)d_{z_0}(z_0,\zeta)^{-Q-|\varrho|+1} + d_{z_0}(z_0,z)^\alpha d_{z_0}(z_0,\zeta)^{-Q-|\varrho|+2}\right)$$

(since $|\varrho|\leq|\sigma|$, $\forall\varrho\in I_\varrho$)

$$\leq C\left(d_{z_0}(z_0,z)d_{z_0}(z_0,\zeta)^{-Q-|\sigma|+1} + d_{z_0}(z_0,z)^\alpha d_{z_0}(z_0,\zeta)^{-Q-|\sigma|+2}\right).$$

Proposition 2.23. *Let $\varphi\in C^\infty(\Omega)$. Then*

$$D_\sigma^{(z_0)}(z)\left(\varphi(\cdot\circ\zeta^{-1})\right) = \sum_{\varrho\in J_\sigma} p_\varrho(\zeta)(D_\varrho^{(z_0)}\varphi)(z\circ\zeta^{-1})$$

where $\circ$ denotes the law group (2.5) in G_{z_0} and p_ϱ is a polynomial of δ_λ^H-degree k_ϱ in the first two components of ζ such that $|\varrho|\leq k_\varrho+|\sigma|$, $\forall\varrho\in J_\sigma$.

Proof. If we prove the assertion on the Heisenberg group, it will be proved in any group $(G_{z_0},\circ)$, by the canonical change of variables. Let us start with $|\sigma|=1$.

$$X_H(\theta)\varphi(\cdot\oplus\widetilde{\theta}^{-1}) = (\partial_{\theta_1} - \frac{\theta_2}{2}\partial_{\theta_3})\varphi\big(\theta_1-\widetilde{\theta}_1, \theta_2-\widetilde{\theta}_2, \theta_3-\widetilde{\theta}_3-\tfrac{1}{2}(\theta_1\widetilde{\theta}_2-\theta_2\widetilde{\theta}_1)\big)$$

$$= \big(\partial_1\varphi - \frac{\widetilde{\theta}_2}{2}\partial_3\varphi - \frac{\theta_2}{2}\partial_3\varphi\big)(\theta\circ\widetilde{\theta}^{-1}) = \big(\partial_1\varphi - \frac{\theta_2-\widetilde{\theta}_2}{2}\partial_3\varphi - \widetilde{\theta}_2\partial_3\varphi\big)(\theta\circ\widetilde{\theta}^{-1})$$

$$= (X_H\varphi)(\theta\circ\widetilde{\theta}^{-1}) - \widetilde{\theta}_2\partial_3\varphi(\theta\circ\widetilde{\theta}^{-1}).$$

This proves the claim since the δ_λ^H-degree of $\widetilde{\theta}_2$ is two. An analogous direct computation shows that

$$Y_H(\theta)(\varphi(\cdot\circ\widetilde{\theta}^{-1})) = (Y_H\varphi)(\theta\circ\widetilde{\theta}^{-1}) + \widetilde{\theta}_1\partial_3\varphi(\theta\circ\widetilde{\theta}^{-1}).$$

The general assertion follows by iterating the previous arguments. □

In the sequel we shall need the following results.

Remark 2.24. If $u\in C_{\bar{u}}^{k,\alpha}(\Omega)$, the coefficients $c_{i,j,m}$ in $P_{z_0}^k u$ depend on the derivatives of u of order less than or equal to k. Hence $c_{i,j,m}\in C_{\bar{u}}^\alpha(\Omega)$. If K is a compact subset of Ω and σ is a multi-index, then there exists $C>0$ such that

$$|D_\sigma P_z^k u(\zeta) - D_\sigma P_{z_0}^k u(\zeta)| \leq C d_{z_0}(z_0,z)^\alpha,$$

and

$$|(P_{z_0}^k u(\zeta) - P_{z_0}^1 u(\zeta)) - (P_z^k u(\zeta) - P_z^1 u(\zeta))| \leq C d_{z_0}(z_0,z)^\alpha d_{z_0}(z_0,\zeta)^2,$$

for every $z, z_0, \zeta \in K$.

Remark 2.25. Let $u \in C^{k,\alpha}_{\bar{u}}(\Omega)$, with $0 \le k \le 5$. Then

$$|P^k_{z_0}u(\zeta) - P^k_z u(\zeta)| \le C d_{z_0}(z_0, z)^\alpha d_{z_0}(z_0, \zeta)^k,$$

for every $z, z_0 \in \Omega$ and $\zeta \in \Omega_M$, where Ω_M is defined in (2.33).

3. Regularization results

In this section we prove Theorems 1.1 and 1.2. We consider the linear equation in $\Omega \subseteq \mathbb{R}^3$,

$$L_{\bar{u}}u = u_{xx} + \bar{u}u_y - u_t = f. \tag{3.1}$$

We say that a function $u \in C^1(\Omega)$ is a *classical* solution of (3.1), if there exists $\partial_{xx}u \in C(\Omega)$ and equation (1.1) is satisfied at every point of Ω. In order to study the regularity of u, we first represent it in terms of the fundamental solution Γ_{z_0}:

$$(u\varphi)(z) = \int \Gamma_{z_0}(z, \zeta) L_{z_0}(u\varphi)(\zeta)\, d\zeta \tag{3.2}$$

for any $C^\infty_0(\Omega)$ function φ. Then we set, as usual,

$$U_\varepsilon(z, z_0) = \int \Gamma_{z_0}(z, \zeta) \chi_{z_0,\varepsilon}(z, \zeta) L_{z_0}(u\varphi)(\zeta)\, d\zeta \tag{3.3}$$

where $\chi_{z_0,\varepsilon}(z, \cdot)$ is a cut-off function, vanishing in a neighborhood of the pole of $\Gamma_{z_0}(z, \cdot)$.

As we pointed out in the introduction we can not use the standard theory, based on uniform convergence of $U_\varepsilon(z, z_0)$ and its derivatives to u and its derivatives. Instead we use a different technique, introduced in [6] and [7] and based on a weak definition of local uniform convergence and on the representation of higher-order derivatives as limits of suitable different quotients.

We represent the functions u and U_ε in (3.2) and (3.3) as the sum of two terms

$$u = I_1(\cdot, z_0) + I_2(\cdot, z_0) \qquad U_\varepsilon = I_{1,\varepsilon}(\cdot, z_0) + I_{2,\varepsilon}(\cdot, z_0),$$

where $I_1(\cdot, z_0)$ is C^∞, $I_{1,\varepsilon}$ uniformly converges to I_1, while $I_{2,\varepsilon}$ converges to I_2 in the sense of the following definition.

Definition 3.1. Let (F_ε) be a family of continuous functions on $\Omega \times \Omega$, let $f : \Omega \to \mathbb{R}$, let $\alpha \in (0,1)$ and $k \in \mathbb{N}$. We say that $F_\varepsilon(z, z_0) \longrightarrow f(z_0)$, as $\varepsilon \to 0$, locally uniformly of order $k + \alpha$ if for every compact set $K \subset \Omega$ there exists $C > 0$ such that

$$|F_\varepsilon(z, z_0) - f(z_0)| \le C\varepsilon^{k+\alpha}, \qquad \forall z, z_0 \in K,\ d_{z_0}(z, z_0) \le \varepsilon.$$

We next state an existence result for the derivatives $D_\sigma u$ (introduced in formula (2.31)).

Lemma 3.2. *Let $|\sigma| \geq 1$ and let $u \in C_{\bar{u}}^{|\sigma|-1,\alpha}(\Omega)$, a function which can be represented as $u(z) = I_1(z, z_0) + I_2(z, z_0)$, where I_1 is smooth as a function of z in Ω, and the function $z \to D_\sigma(z) I_1(\cdot, z_0)$ is continuous in z uniformly in z_0. Assume that there exists a family $I_{2,\varepsilon}$ of smooth functions and a continuous function I_2^σ such that $I_{2,\varepsilon}(z, z_0) \longrightarrow I_2(z_0, z_0)$, as $\varepsilon \to 0$, locally uniformly of order $|\sigma| + \alpha$, and $D_\sigma I_{2,\varepsilon}(z, z_0) \longrightarrow I_2^\sigma(z_0)$, as $\varepsilon \to 0$, locally uniformly of order α. Then $D_\sigma u(z_0)$ exists and, for every z_0 in Ω,*

$$D_\sigma u(z_0) = D_\sigma(z_0) I_1(\cdot, z_0) + I_2^\sigma(z_0).$$

The proof is postponed to Subsection 3.1. In Subsection 3.2, we prove Theorem 1.2 by using Lemma 3.2. Finally, in Subsection 3.3, we conclude the proof of Theorem 1.1.

3.1. **Derivatives and difference quotients.** The main ideas of the proof of Lemma 3.2 are already contained in [6] and [7], but the lemma is not stated explicitly; hence we give here the proof. It is based on the following definition:

Definition 3.3. If $g : \Omega \longrightarrow \mathbb{R}$, for every $z \in \Omega$ and $h \in \mathbb{R}$ sufficiently small, we define

$$\triangle_{(i)}(z) g(h) = \frac{g(\exp(h^i D_{(i)})(z)) - g(z)}{h^i}, \qquad i = 1, 2.$$

For every multi-index $\sigma = (\sigma_1, \ldots, \sigma_m) \in \{1, 2\}^m$, we define by recurrence

$$\triangle_\sigma(z) g(h) = \triangle_{(\sigma_1)}(z)(\triangle_{(\sigma_2,\ldots,\sigma_m)} g(h))(h).$$

Remark 3.4. If $g \in C_{\bar{u}}^{|\sigma|}(\Omega)$ then, by the mean value theorem, we have

$$\triangle_\sigma(z) g(h) = D_\sigma g(z_h),$$

for a suitable z_h such that $d_{\bar{u}}(z_h, z) \leq h$. Hence there exists

$$\lim_{h \to 0} \triangle_\sigma(z) g(h) = D_\sigma g(z),$$

uniformly on the compact sets.

As in [7], Remark 4.2, the following result holds.

Lemma 3.5. *Let $|\sigma| \geq 1$ and let $g \in C_{\bar{u}}^{|\sigma|-1}(\Omega)$. If there exists*

$$\lim_{h \to 0} \triangle_\sigma g(h) = w$$

uniformly on the compact subsets of Ω, then there exists $D_\sigma g = w$.

Proof of Lemma 3.2 Since $z \mapsto I_1(z, z_0)$ is smooth, by Remark 3.4, we have

$$\Delta_\sigma(h) I_1(\cdot, z_0)(z_0) \longrightarrow D_\sigma(z_0) I_1(\cdot, z_0),$$

as $h \to 0$, locally uniformly on compact sets. On the other side

$$\begin{aligned}&|\Delta_\sigma(z_0) I_2(\cdot, z_0)(h) - I_2^\sigma(z_0)|\\ \le &|\Delta_\sigma(z_0) I_2(\cdot, z_0)(h) - \Delta_\sigma(z_0) I_{2,\varepsilon}(\cdot, z_0)(h)| + |\Delta_\sigma(z_0) I_{2,\varepsilon}(\cdot, z_0)(h) - I_2^\sigma(z_0)|\end{aligned}$$

(by the hypotheses on the local uniform convergence of order $k+\alpha$ and Remark 3.4)

$$\le C_1 \varepsilon^\alpha + |D_\sigma(z_h) I_{2,\varepsilon}(\cdot, z_0) - I_2^\sigma(z_0)| \le C\varepsilon^\alpha.$$

Then

$$\Delta_\sigma(z_0) u(h) \longrightarrow D_\sigma(z_0) I_1(\cdot, z_0) + I_2^\sigma(z_0),$$

as $h \to 0$, uniformly on the compact sets. By Lemma 3.5, we infer that

$$D_\sigma(z_0) u = D_\sigma(z_0) I_1(\cdot, z_0) + I_2^\sigma(z_0).$$

3.2. **Linear operators with $C^{k,\alpha}_{\bar{u}}$ coefficients.** The aim of this subsection is the proof of Theorem 1.2. Let K be fixed according to Remark 2.5 and let K_1 be any compact set $K_1 \subset\subset \mathrm{int}(K)$. We study the regularity of u in K_1. We fix a function $\varphi \in C_0^\infty(\mathrm{int}(K))$ such that $\varphi \equiv 1$ in a neighborhood of K_1. It is nonrestrictive to assume that, if $z, z_0 \in K_1$, then

$$M d_{z_0}(z_0, z) \le d_{z_0}(K_1, \mathrm{supp}(\nabla\varphi)), \quad M d_z(z, z_0) \le d_z(K_1, \mathrm{supp}(\nabla\varphi)), \quad (3.4)$$

where M is the constant of Lemma 2.22.

Remark 3.6. With this choice of function φ and compact set K_1, we have

$$C_1 d_z(z,\zeta) \le d_{z_0}(z_0,\zeta) \le C_2 d_z(z,\zeta), \qquad \forall \zeta \in \mathrm{supp}(\nabla\varphi), z, z_0 \in K_1 \quad (3.5)$$

where C_1, C_2 are positive constants depending only on $\bar{u}$ and K. In particular,

$$d_{z_0}(z_0, z) \le C d_{z_0}(z, \zeta), \qquad \forall \zeta \in \mathrm{supp}(\nabla\varphi),\ z, z_0 \in K. \qquad (3.6)$$

Proof. By (3.4) and Proposition 2.4–(ii), we have

$$d_{z_0}(z_0,\zeta) \le C(d_{z_0}(z_0,z) + d_z(z,\zeta)) \le C(\frac{1}{M} d_{z_0}(z_0,\zeta) + d_z(z,\zeta)),$$

$\forall \zeta \in \mathrm{supp}(\nabla\varphi)$; thus, if M is sufficiently large, we get

$$d_{z_0}(z_0,\zeta) \le C d_z(z,\zeta), \qquad (3.7)$$

for every $\zeta \in \mathrm{supp}(\nabla\varphi)$. Exchanging the role of z_0 and z in (3.7), we get (3.5).

Proposition 3.7. *Let us assume that the coefficient $\bar{u}$ in equation* (3.1) *is of class $C^{k-1,\alpha}_{\bar{u}}(\Omega)$, $2 \le k \le 6$, and that f is of class $C^{k-2,\alpha}_{\bar{u}}(\Omega)$. Let $u \in C^{k-1,\alpha}_{\bar{u}}(\Omega)$ be a classical solution of* (3.1). *Then, for every $z, z_0 \in K_1$, $u = u\varphi$ can be represented as*

$$u(z) = u\varphi(z) = \int_\Omega \Gamma_{z_0}(z,\zeta) N_1(\zeta, z_0)\, d\zeta + \int_\Omega \Gamma_{z_0}(z,\zeta) N_{2,k}(\zeta, z_0)\, d\zeta + \int_\Omega \Gamma_{z_0}(z,\zeta) N_{3,k}(\zeta, z_0)\, d\zeta. \tag{3.8}$$

where $N_{i,k}(\cdot, z_0)$ is supported in the support of φ, and
(i) $\mathrm{supp}(N_1(\cdot, z_0)) \subseteq \mathrm{supp}(\nabla\varphi)$ *and*

$$|N_1(\zeta, z_0) - N_1(\zeta, z)| \le C d_{z_0}(z_0, z)^\alpha d_{z_0}(z_0, \zeta); \tag{3.9}$$

(ii) $N_{2,k}(\cdot, z_0)$ *is of class C^∞ and for every multi-index σ*

$$|D_\sigma(\zeta) N_{2,k}(\cdot, z_0) - D_\sigma(\zeta) N_{2,k}(\cdot, z)| \le C d_{z_0}(z_0, z)^\alpha, \qquad \forall \zeta \in K;$$

(iii) *there exists a constant C, dependent only on the choice of φ and K_1, such that for every $\zeta \in K$ and $z, z_0 \in K_1$*

$$|N_{3,k}(\zeta, z_0)| \le C d^{k-2+\alpha}_{z_0}(z_0, \zeta), \tag{3.10}$$

$$|N_{3,k}(\zeta, z_0) - N_{3,k}(\zeta, z)| \le C d_{z_0}(z_0, z)^\alpha d_{z_0}(z_0, \zeta)^{k-2}. \tag{3.11}$$

Proof. By definition of the fundamental solution, we have

$$\begin{aligned} u\varphi(z) &= \int_\Omega \Gamma_{z_0}(z,\zeta) L_{z_0}(u\varphi)(\zeta)\, d\zeta = \int_\Omega \Gamma_{z_0}(z,\zeta) \left(u L_{z_0}\varphi + 2XuX\varphi\right)\, d\zeta \\ &+ \int_\Omega \Gamma_{z_0}(z,\zeta) L_{\bar{u}} u(\zeta) \varphi(\zeta)\, d\zeta + \int_\Omega \Gamma_{z_0}(z,\zeta)(L_{z_0} - L_{\bar{u}}) u(\zeta)\varphi(\zeta)\, d\zeta \\ &= \int_\Omega \Gamma_{z_0}(z,\zeta) \left(u L_{z_0}\varphi + 2XuX\varphi\right)\, d\zeta \\ &+ \int_\Omega \Gamma_{z_0}(z,\zeta) f(\zeta)\varphi(\zeta)\, d\zeta - \int_\Omega \Gamma_{z_0}(z,\zeta)(\bar{u} - P^1_{z_0}\bar{u})(\zeta)\partial_\eta u(\zeta)\varphi(\zeta)\, d\zeta. \end{aligned} \tag{3.12}$$

In order to use a uniform notation, in the sequel we will set $P^k_{z_0} u = 0$ for k a negative integer. Then, for every $k \ge 2$, we have

$$\begin{aligned} (\bar{u} - P^1_{z_0}\bar{u})(\zeta)\partial_\eta u(\zeta) &= (P^{k-1}_{z_0}\bar{u}(\zeta) - P^1_{z_0}\bar{u}(\zeta))(u_\eta(\zeta) - P^{k-4}_{z_0} u_\eta(\zeta)) \\ &+ (P^{k-1}_{z_0}\bar{u}(\zeta) - P^1_{z_0}\bar{u}(\zeta)) P^{k-4}_{z_0} u_\eta(\zeta) + (\bar{u} - P^{k-1}_{z_0}\bar{u})(\zeta)\partial_\eta u(\zeta). \end{aligned}$$

Then (3.8) is satisfied by inserting these expressions in formula (3.12) and by choosing the kernels N_1, $N_{2,k}$, and $N_{3,k}$ as follows:

$$\begin{aligned}
N_1(\zeta, z_0) &= u(\zeta) L_{z_0}\varphi(\zeta) + 2Xu(\zeta)X\varphi(\zeta),\\
N_{2,k}(\zeta, z_0) &= P^{k-2}_{z_0} f(\zeta)\varphi(\zeta) + (P^{k-1}_{z_0}\bar{u}(\zeta) - P^{1}_{z_0}\bar{u}(\zeta))P^{k-4}_{z_0} u_\eta(\zeta)\varphi(\zeta),\\
N_{3,k}(\zeta, z_0) &= (f(\zeta) - P^{k-2}_{z_0} f(\zeta))\varphi(\zeta) + (\bar{u}(\zeta) - P^{k-1}_{z_0}\bar{u}(\zeta))u_\eta(\zeta)\varphi(\zeta)\\
&\quad + (P^{k-1}_{z_0}\bar{u}(\zeta) - P^{1}_{z_0}\bar{u}(\zeta))(u_\eta(\zeta) - P^{k-4}_{z_0} u_\eta(\zeta))\varphi(\zeta).
\end{aligned}$$

Let us prove (i). The support of $N_1(\cdot, z_0)$ is clearly a subset of $\mathrm{supp}(\nabla\varphi)$. Formula (3.9) can be proved as follows:

$$\begin{aligned}
|N_1(\zeta, z_0) - N_1(\zeta, z)| &= |u(\zeta)\left(L_{z_0}\varphi(\zeta) - L_z\varphi(\zeta)\right)|\\
&= |u(\zeta)|\left|P^1_{z_0}\bar{u}(\zeta) - P^1_z\bar{u}(\zeta)\right||\partial_\eta\varphi(\zeta)| \le
\end{aligned}$$

(by Remark 2.25)

$$\le C\left(d_{z_0}(z_0, z)^{1+\alpha} + d_{z_0}(z_0, z)^\alpha d_{z_0}(z_0, \zeta)\right) \le C d_{z_0}(z_0, z)^\alpha d_{z_0}(z_0, \zeta).$$

Condition (ii) easily follows from the definition of $N_{2,k}$ and Remark 2.24.

Let us prove (3.10) for $k = 2$:

$$N_{3,2}(\zeta, z_0) = (f(\zeta) - f(z_0))\varphi(\zeta) + \left(\bar{u}(\zeta) - P^1_{z_0}\bar{u}(\zeta)\right) u_\eta(\zeta)\varphi(\zeta) \le C d_{z_0}(z_0, \zeta)^\alpha.$$

We observe that, for every $k \ge 3$,

$$P^{k-1}_{z_0}\bar{u}(\zeta) - P^1_{z_0}\bar{u}(\zeta) = O\left(d_{z_0}(z_0, \zeta)^2\right), \qquad \text{as } \zeta \to z_0.$$

Hence, we get

$$\begin{aligned}
N_{3,3}(\zeta, z_0) &= \left(f(\zeta) - P^1_{z_0} f(\zeta)\right)\varphi(\zeta) + \left(\bar{u}(\zeta) - P^2_{z_0}\bar{u}(\zeta)\right) u_\eta(\zeta)\varphi(\zeta)\\
&\quad + O\left(d_{z_0}(z_0, \zeta)^2\right) = O\left(d_{z_0}(z_0, \zeta)^{1+\alpha}\right), \qquad \text{as } \zeta \to z_0.
\end{aligned}$$

This proves (3.10) for $k = 3$. We can proceed analogously for $k \ge 4$.

Let us prove (3.11):

$$\begin{aligned}
&|N_{3,k}(\zeta, z_0) - N_{3,k}(\zeta, z)| \le |P^{k-2}_{z_0} f(\zeta) - P^{k-2}_z f(\zeta)||\varphi(\zeta)|\\
&+ |P^{k-1}_{z_0}\bar{u}(\zeta) - P^1_{z_0}\bar{u}(\zeta)||P^{k-4}_{z_0} u_\eta(\zeta) - P^{k-4}_z u_\eta(\zeta)||\varphi(\zeta)|\\
&+ |(P^{k-1}_{z_0}\bar{u}(\zeta) - P^1_{z_0}\bar{u}(\zeta)) - (P^{k-1}_z\bar{u}(\zeta) - P^1_z\bar{u}(\zeta))||u_\eta(\zeta) - P^{k-4}_{z_0} u_\eta(\zeta)||\varphi(\zeta)|\\
&+ |P^{k-1}_{z_0}\bar{u}(\zeta) - P^{k-1}_z\bar{u}(\zeta)||u_\eta(\zeta)||\varphi(\zeta)| \le
\end{aligned}$$

(by Remark 2.25 and Remark 2.24)

$$\le C d_{z_0}(z_0, z)^\alpha d_{z_0}(z_0, \zeta)^{k-2}.$$

Remark 3.8. As stated in formula (3.3), we introduce a cut-off function $\chi_{z_0,\varepsilon}(z,\zeta)$ with the following properties:

(i) $\chi_{z_0,\varepsilon}(\cdot,\zeta) \in C^\infty(\Omega,[0,1])$;
(ii) $\chi_{z_0,\varepsilon}(z,\zeta) = 0$ if $d_{z_0}(z,\zeta) \le 2\widetilde{C}\varepsilon$;
(iii) $\chi_{z_0,\varepsilon}(z,\zeta) = 1$ if $d_{z_0}(z,\zeta) \ge 4\widetilde{C}\varepsilon$;
(iv) $|D_\sigma(z)\chi_{z_0,\varepsilon}(\cdot,\zeta)| \le \frac{C}{\varepsilon^{|\sigma|}}$, for every multi-index σ;

for every $z_0,\zeta \in \Omega$, $\varepsilon > 0$, where $\widetilde{C}$ is the constant in (2.6).

Proof. We consider a smooth function g_{z_0} defined in terms of the composition law (2.5) of G_{z_0}. We assume that g_{z_0} is $\delta_\lambda^{(z_0)}$-homogeneous of degree one and that

$$\frac{3}{4}d_{z_0}(z,\zeta) \le g_{z_0}(\zeta^{-1}\circ z) \le \frac{5}{4}d_{z_0}(z,\zeta), \qquad \forall z,\zeta \in \Omega.$$

We next denote by χ a $C^\infty([0,+\infty),[0,1])$ function such that

$$\chi(s) = 0, \quad \text{for } s \le \frac{5}{4}, \qquad \chi(s) = 1, \quad \text{for } s \ge \frac{3}{2},$$

and we define $\chi_{z_0}(z,\zeta) = \chi(\frac{g_{z_0}(\zeta^{-1}\circ z)}{2\widetilde{C}\varepsilon})$.

Proposition 3.9. *Assume that u is of class $C_{\bar{u}}^{k-1,\alpha}(\Omega)$, with $2 \le k \le 6$, and that it can be represented as in* (3.8)*, for every $z, z_0 \in K$, with the kernels $N_1, N_{2,k}$ and $N_{3,k}$ satisfying* (i), (ii) *and* (iii)*. Then u is of class $C_{\bar{u}}^{k}(\Omega)$.*

Proof. Since u is represented as in (3.8), we apply Lemma 3.2 to it. If we set

$$I_1(z,z_0) = \int_\Omega \Gamma_{z_0}(z,\zeta)N_1(\zeta,z_0)\,d\zeta \ \text{ and } \ I_{2,k}(z,z_0) = \int_\Omega \Gamma_{z_0}(z,\zeta)N_{2,k}(\zeta,z_0)\,d\zeta,$$

then $z \mapsto I_1(z,z_0)$ is C^∞, since $N_1(\zeta,z_0)$ is null in a neighborhood of the pole of $\Gamma_{z_0}(z,\zeta)$, by (i) and (3.6). Also $I_{2,k}(\cdot,z_0)$ is C^∞ since, by the change of variable $\zeta^{-1}\circ z = \omega$, it can be represented as

$$I_{2,k}(z,z_0) = \int \Gamma_{z_0}(\omega,0)N_{2,k}(z\circ\omega^{-1},z_0)\,d\omega, \tag{3.13}$$

where $N_{2,k}(\cdot,z_0)$ is of class $C_0^\infty(\Omega)$. Next, we set

$$I_{3,k}(z,z_0) = \int_\Omega \Gamma_{z_0}(z,\zeta)N_{3,k}(\zeta,z_0)\,d\zeta$$

and, for every multi-index σ of height $|\sigma| = k$,

$$I_{3,k}^\sigma(z_0) = \int_\Omega D_\sigma(z_0)\Gamma_{z_0}(\cdot,\zeta)N_{3,k}(\zeta,z_0)\,d\zeta. \tag{3.14}$$

We remark that $I_{3,k}^{\sigma}$ is well defined and continuous by (iii). Let us define

$$I_{3,k,\varepsilon}(z,z_0)=\int_{\Omega}\Gamma_{z_0}(z,\zeta)\chi_{z_0,\varepsilon}(z,\zeta)N_{3,k}(\zeta,z_0)\,d\zeta.$$

Clearly $I_{3,k,\varepsilon}(\cdot,z_0)$ is smooth.

In order to apply Lemma 3.2, we have only to prove that

$$\sup_{d_{z_0}(z_0,z)\le\varepsilon}|I_{3,k}(z,z_0)-I_{3,k,\varepsilon}(z,z_0)|\le C\varepsilon^{k+\alpha}, \tag{3.15}$$

$$\sup_{d_{z_0}(z_0,z)\le\varepsilon}|D_\sigma(z)I_{3,k,\varepsilon}(\cdot,z_0)-I_{3,k}^{\sigma}(z_0)|\le C\varepsilon^{\alpha}, \tag{3.16}$$

where C is a positive constant which depends only on K_1. Then we will deduce that

$$D_\sigma u(z_0)=D_\sigma(z_0)I_1(\cdot,z_0)+D_\sigma(z_0)I_{2,k}(\cdot,z_0)+I_{3,k}^{\sigma}(z_0). \tag{3.17}$$

Indeed, we have, by (iii),

$$|I_{3,k}(z,z_0)-I_{3,k,\varepsilon}(z,z_0)|\le C_1\int_{d_{z_0}(z,\zeta)\le 2\varepsilon} d_{z_0}(z,\zeta)^{-Q+2}d_{z_0}(z_0,\zeta)^{k-2+\alpha}d\zeta$$

$$\le C_2\int_{d_{z_0}(z,\zeta)\le 2\varepsilon} d_{z_0}(z,\zeta)^{-Q+2}(d_{z_0}(z_0,z)^{k-2+\alpha}+d_{z_0}(z,\zeta)^{k-2+\alpha})d\zeta\le C\varepsilon^{k+\alpha},$$

since $d_{z_0}(z_0,z)\le\varepsilon$ and (3.15) holds. We remark that

$$D_\sigma(fg)=\sum_{|\sigma'|+|\sigma''|=|\sigma|}c(\sigma,\sigma',\sigma'')D_{\sigma'}fD_{\sigma''}g,$$

for some constants $c(\sigma,\sigma',\sigma'')$. Thus, we have

$$|D_\sigma(z)I_{3,k,\varepsilon}(\cdot,z_0)-I_{3,k}^{\sigma}(z_0)|\le|B_1(\varepsilon)|+|B_2(\varepsilon)|+|B_3(\varepsilon)|,$$

where

$$B_1(\varepsilon)=\int_\Omega(D_\sigma(z)-D_\sigma(z_0))\Gamma_{z_0}(\cdot,\zeta)\chi_{z_0,\varepsilon}(z,\zeta)N_{3,k}(\zeta,z_0)\,d\zeta,$$

$$B_2(\varepsilon)=\int_\Omega D_\sigma(z_0)\Gamma_{z_0}(\cdot,\zeta)\left(1-\chi_{z_0,\varepsilon}(z,\zeta)\right)N_{3,k}(\zeta,z_0)\,d\zeta,$$

$$B_3(\varepsilon)=\int_\Omega\sum_{\substack{|\sigma'|+|\sigma''|=|\sigma|,\\ |\sigma''|\neq 0}}c(\sigma,\sigma',\sigma'')D_{\sigma'}(z)\Gamma_{z_0}(\cdot,\zeta)D_{\sigma''}(z)\chi_{z_0,\varepsilon}(z,\zeta)N_{3,k}(\zeta,z_0)d\zeta.$$

Let us estimate each term separately. We observe that, for every $\zeta \in \operatorname{supp}(\chi_{z_0,\varepsilon}(z,\zeta))$, we have $d_{z_0}(z,\zeta) \geq 2\widetilde{C}\varepsilon$, where $\widetilde{C}$ is the constant in (2.6); then

$$d_{z_0}(z_0,\zeta) \geq \frac{1}{\widetilde{C}} d_{z_0}(z,\zeta) - d_{z_0}(z_0,z) \geq \varepsilon. \tag{3.18}$$

Thus, by Proposition 2.20, we get

$$|B_1(\varepsilon)| \leq$$

$$C_1 \int_\Omega \left(d_{z_0}(z_0,z) d_{z_0}(z_0,\zeta)^{-Q-1+\alpha} + d_{z_0}(z_0,z)^\alpha d_{z_0}(z_0,\zeta)^{-Q+\alpha} \right) \chi_{z_0,\varepsilon}(z,\zeta)\, d\zeta$$

(by (3.18))

$$\leq C_2 \int_{d_{z_0}(z_0,\zeta)\geq\varepsilon} \left(\varepsilon d_{z_0}(z_0,\zeta)^{-Q-1+\alpha} + \varepsilon^\alpha d_{z_0}(z_0,\zeta)^{-Q+\alpha} \right) |\varphi(\zeta)| d\zeta \leq C\varepsilon^\alpha.$$

By Remark 2.19 and since $d_{z_0}(z_0,\zeta) \leq \widetilde{C}(d_{z_0}(z_0,z) + d_{z_0}(z,\zeta))$, we obtain

$$|B_2(\varepsilon)| \leq C_1 \int_{d_{z_0}(z_0,\zeta)\leq \widetilde{C}\left(1+4\widetilde{C}\right)\varepsilon} d_{z_0}(z_0,\zeta)^{-Q+\alpha} d\zeta \leq C\varepsilon^\alpha.$$

Finally, by using again Remark 2.19 and property (iv) of $\chi_{z_0,\varepsilon}$, we obtain

$$|B_3(\varepsilon)| \leq \sum_{\substack{|\sigma'|+|\sigma''|=|\sigma|,\\ |\sigma''|\neq 0}} \frac{|c(\sigma,\sigma',\sigma'')|}{\varepsilon^{|\sigma''|}} \int\limits_{\varepsilon\leq d_{z_0}(z_0,\zeta)\leq \widetilde{C}\left(1+4\widetilde{C}\right)\varepsilon} d_{z_0}(z,\zeta)^{-Q+2-|\sigma'|} d_{z_0}(z_0,\zeta)^{|\sigma|-2+\alpha} d\zeta$$
$$\leq C\varepsilon^\alpha.$$

This concludes the proof of (3.16).

Lemma 3.10. *Assume that u is of class $C^{k-1,\alpha}_{\bar{u}}(\Omega)$, with $2 \leq k \leq 6$, $\alpha > \frac{1}{2}$, and that it can be represented as in* (3.8), *for every $z, z_0 \in K$, with the kernels $N_1, N_{2,k}$ and $N_{3,k}$ satisfying* (i), (ii) *and* (iii). *Then*
(i) *$D_\sigma u \in C^{\alpha'}_{\bar{u}}(\Omega)$, for every multi-index σ of height k;*
(ii) *$D_{\sigma'} u \in C^{\alpha'}_{Y_{\bar{u}}}(\Omega)$, for every multi-index σ' of height $k-1$,*
for every $\alpha' \in (0,\alpha)$.

Proof. Let us prove (i). In formula (3.17), we gave an explicit expression of $D_\sigma u$ as a sum of three terms. Since these terms have similar behaviour and $D_\sigma I_1$ is the simplest one, we study only $I^\sigma_{3,k}$ and $D_\sigma I_{2,k}$.

Let us start with $I^\sigma_{3,k}$. Let $M = M(K)$ be as in Lemma 2.22 and let Ω_M be the set defined in (2.33). Then, we have

$$I^\sigma_{3,k}(z) - I^\sigma_{3,k}(z_0) = A_1(z,z_0) + A_2(z,z_0) + A_3(z,z_0) + A_4(z,z_0), \tag{3.19}$$

where

$$A_1(z,z_0) = \int_{\Omega_M} (D_\sigma(z)\Gamma_z(\cdot,\zeta) - D_\sigma(z_0)\Gamma_{z_0}(\cdot,\zeta))N_{3,k}(\zeta,z_0)\,d\zeta,$$
$$A_2(z,z_0) = \int_{\Omega_M} D_\sigma(z)\Gamma_z(\cdot,\zeta)\left(N_{3,k}(\zeta,z) - N_{3,k}(\zeta,z_0)\right)\,d\zeta,$$
$$A_3(z,z_0) = \int_{\Omega\setminus\Omega_M} D_\sigma(z)\Gamma_z(\cdot,\zeta)N_{3,k}(\zeta,z)\,d\zeta,$$
$$A_4(z,z_0) = -\int_{\Omega\setminus\Omega_M} D_\sigma(z_0)\Gamma_{z_0}(\cdot,\zeta)N_{3,k}(\zeta,z_0)\,d\zeta.$$

We get immediately, by Remark 2.19,

$$|A_3(z,z_0)|,\ |A_4(z,z_0)| \le C d_{z_0}(z_0,z)^\alpha.$$

Moreover, by Proposition 2.21, we have

$$\begin{aligned}|A_1(z,z_0)| &\le C_1 \int_{\Omega_M} (d_{z_0}(z_0,z)d_{z_0}(z_0,\zeta)^{-Q+\alpha-1} + d_{z_0}(z_0,z)^\alpha d_{z_0}(z_0,\zeta)^{-Q+\alpha})\,d\zeta \\ &\le C d_{z_0}(z_0,z)^\alpha.\end{aligned}$$

By Remark 2.19 and condition (iii) of Proposition 3.7, we have

$$|A_2(z,z_0)| \le C_1 \int_{\Omega_M} d_z(z,\zeta)^{-Q} d_{z_0}(z_0,z)^\alpha |\varphi(\zeta)|\,d\zeta \le$$

(by (2.34) which holds for every $\zeta \in \Omega_M$, for every $\alpha' < \alpha$)

$$\le C_2 d_{z_0}(z_0,z)^{\alpha'} \int_{\Omega_M} d_{z_0}(z_0,\zeta)^{-Q+\alpha-\alpha'}|\varphi(\zeta)|\,d\zeta \le C d_{z_0}(z_0,z)^{\alpha'}. \tag{3.20}$$

This concludes the proof of the Hölder continuity of $I^\sigma_{3,k}$ of order α', for every $\alpha' < \alpha$. Let us consider $I_{2,k}$. By (3.13),

$$I_{2,k}(z,z_0) = \int \Gamma_{z_0}(\omega)N_{2,k}(z\circ\omega^{-1},z_0)\,d\omega,$$

where $N_{2,k}(\zeta,z_0)$ is introduced in Proposition 3.7 and $\Gamma_{z_0}(\omega) \equiv \Gamma_{z_0}(\omega,0)$. We have to show that $D_\sigma(z_0)I_{2,k}(\cdot,z_0) \in C^{\alpha'}_{\bar{u}}(\Omega)$. By differentiating the above integral, we obtain

$$D_\sigma(z_0)I_{2,k}(\cdot,z_0) = \int \Gamma_{z_0}(\omega)D_\sigma(z_0)N_{2,k}(\cdot\circ\omega^{-1},z_0)\,d\omega =$$

(by Lemma 2.17)

$$= \sum_{\varrho \in I_\sigma} \frac{C_\varrho}{(\bar{u}_x(z_0))^{h_\varrho}} \prod_{\mu \in J_\varrho} D_\mu \bar{u}(z_0) \int \Gamma_{z_0}(\omega) D_\varrho^{(z_0)}(z_0) N_{2,k}(\cdot \circ \omega^{-1}, z_0)\, d\omega =$$

(by Proposition 2.23)

$$= \sum_{\varrho \in I_\sigma} \frac{C_\varrho}{(\bar{u}_x(z_0))^{h_\varrho}} \prod_{\mu \in J_\varrho} D_\mu \bar{u}(z_0) \sum_{\nu \in J'} \int \Gamma_{z_0}(\omega) p_\nu(\omega) D_\nu^{(z_0)}(z_0) N_{2,k}(\cdot \circ \omega^{-1}, z_0)\, d\omega.$$

Each term of this sum is of the form $\int \Gamma_{z_0}(z_0, \zeta)\psi(\zeta, z_0) d\zeta$, where $\psi(\cdot, z_0) \in C_0^\infty(\Omega)$ and, by property (ii) of $N_{2,k}$,

$$|\psi(\zeta, z_0) - \psi(\zeta, z)| \le C d_{z_0}(z_0, z)^\alpha.$$

Thus each term can be treated separately as $I_{3,k}^\sigma$.

The proof of (ii) is analogous. $\square$

Proof of Theorem 1.2 As a consequence of Proposition 3.9 and Lemma 3.10, the assertion is true for $2 \le k \le 6$. Now we assume $k \ge 7$ and we proceed by induction. Let us assume that $u, \bar{u} \in C_{\bar{u}}^{k-1,\alpha}(\Omega)$ and $f = L_{\bar{u}} u \in C_{\bar{u}}^{k-2,\alpha}(\Omega)$. We prove that $u \in C_{\bar{u}}^{k,\alpha}(\Omega)$.

By assumption $u_y \in C_{\bar{u}}^{k-4,\alpha}(\Omega)$. Thus, differentiating the equation with respect to the variable y, we get $L_{\bar{u}}(u_y) = f_y - \bar{u}_y u_y \in C_{\bar{u}}^{k-5,\alpha}(\Omega)$. By induction, $u_y \in C_{\bar{u}}^{k-3,\alpha'}(\Omega)$ for $\alpha' \in (0, \alpha)$.

On the other hand, $u_x \in C_{\bar{u}}^{k-2,\alpha}(\Omega)$ and, differentiating the equation with respect to x, we have $L_{\bar{u}}(u_x) = f_x - \bar{u}_x u_y \in C_{\bar{u}}^{k-3,\alpha'}(\Omega)$, for $\alpha' \in (0, \alpha)$. Therefore, by induction, $u_x \in C_{\bar{u}}^{k-1,\alpha'}(\Omega)$ for $\alpha' \in (0, \alpha)$.

Finally, $Y_{\bar{u}} u \in C_{\bar{u}}^{k-3,\alpha}(\Omega)$ and, differentiating the equation with respect to $Y_{\bar{u}}$, we have $L_{\bar{u}}(Y_{\bar{u}} u) = Y_{\bar{u}} f + \bar{u}_{xx} u_y + 2\bar{u}_x u_{xy} \in C_{\bar{u}}^{k-4,\alpha'}(\Omega)$, for every $\alpha' < \alpha$. Thus, by induction, $Y_{\bar{u}} u \in C_{\bar{u}}^{k-2,\alpha'}(\Omega)$ for $\alpha' \in (0, \alpha)$. This proves that $u \in C_{\bar{u}}^{k,\alpha}(\Omega)$ for any $\alpha \in (0, 1)$.

3.3. **The nonlinear operator.** In this subsection we prove the regularity of the classical solutions of (1.1).

Proof of Theorem 1.1. We first show that $u \in C_u^{1,\alpha}(\Omega)$, for every $\alpha \in (0, 1)$. Then, by Theorem 1.2, it will follow that $u \in C_u^{k,\alpha}(\Omega)$ for every $k \in \mathbb{N}$ and $\alpha \in (0, 1)$. Thus, by Remark 2.13, the thesis will be proved.

Since u is a classical solution of (1.1), we have only to show that $Xu \in C_u^\alpha(K)$ for a fixed compact set K and for every $\alpha \in (0, 1)$. Representing u

by formula (3.8), we infer that

$$Xu(z) = \int_\Omega X(z)\Gamma_{z_0}(\cdot,\zeta)\psi(\zeta)d\zeta,$$

for some suitable $\psi \in C_0(\Omega)$ and $z_0 \in \Omega$. We have

$$|Xu(z) - Xu(z_0)| \leq A_1(z,z_0) + A_2(z,z_0),$$

where, for Ω_M defined in (2.33),

$$A_1(z,z_0) = \int_{\Omega\setminus\Omega_M} |(X(z) - X(z_0))\Gamma_{z_0}(\cdot,\zeta)|\,|\psi(\zeta)|\;d\zeta,$$
$$A_2(z,z_0) = \int_{\Omega_M} |(X(z) - X(z_0))\Gamma_{z_0}(\cdot,\zeta)|\,|\psi(\zeta)|\;d\zeta.$$

By Remark 2.19, we obtain the following estimate of $A_1(z,z_0)$:

$$A_1(z,z_0) \leq C_1 \int_{\Omega\setminus\Omega_M} \left(d_{z_0}(z,\zeta)^{-Q+1} + d_{z_0}(z_0,\zeta)^{-Q+1}\right)\, d\zeta \leq C d_{z_0}(z_0,z).$$

On the other hand, by Proposition 2.20, we have

$$\begin{aligned} A_2(z,z_0) &\leq C \int_{\Omega_M} \left(d_{z_0}(z_0,z) d_{z_0}(z_0,\zeta)^{-Q} + d_{z_0}(z_0,z)^\alpha d_{z_0}(z_0,\zeta)^{-Q+1}\right) d\zeta \\ &\leq C d_{z_0}(z_0,z)^\alpha, \end{aligned}$$

for every $\alpha \in\ (0,1)$. Thus the proof is completed. □

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References

[1] F. Antonelli, E. Barucci, and M.E. Mancino, *Backward-forward stochastic differential utility: Existence, optimal consumption and equilibrium analysis*, to appear.

[2] M. Bramanti and L. Brandolini, *L^p estimates for uniformly hypoelliptic operators with discontinuous coefficients on homogeneous groups*, to appear.

[3] M. Bramanti, Cerutti, and M. Manfredini, *L^p estimates for some ultraparabolic operators with discontinuous coefficients*, J. Math. Anal. Appl., 200 (1996), 332–354.

[4] L. Capogna, D. Danielli, and N. Garofalo, *An embedding theorem and the Harnack inequality for nonlinear subelliptic equations*, Comm. Partial Diff. Eq., 18 (1993), 1765–1794.

[5] F. Chiarenza, M. Frasca, and P. Longo, *Interior $W^{2,p}$ estimates for non-divergence elliptic equations with discontinuous coefficients*, Ric. Mat., 40 (1991), 149–168.

[6] G. Citti, *C^∞ regularity of solutions of a quasilinear equation related to the Levi operator*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., Serie IV, Vol. XXIII (1996), 483–529.

[7] G. Citti and A. Montanari, *C^∞ regularity of solutions of an equation of Levi's type in R^{2n+1}*, Ann. Mat. Pura Appl., 4, to appear.

[8] D. Duffie and L. Epstein, *Stochastic differential utility*, Econometrica, 60 (1992), 353–394.

[9] L. Epstein and S. Zin, *Substitution, risk aversion, and the temporal behavior of consumption and asset returns: a theoretical framework*, Econometrica, 57 (1989), 937–969.

[10] M. Escobedo, J.L. Vazquez, and E. Zuazua, *Entropy solutions for diffusion-convection equations with partial diffusivity*, Trans. Am. Math. Soc., 343 (1994), 829–842.

[11] G.B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat., 13 (1975), 161–207.

[12] B. Franchi and E. Lanconelli, *Hölder regularity theorem for a class of linear non-uniformly elliptic operators with measurable coefficients*, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser., 10 (1983), 523–541.

[13] G.B. Folland and E.M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math., 20 (1974), 429–522.

[14] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math., 119 (1967), 147–171.

[15] G. Lu, *Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications*, Rev. Math Iberoamericana, 8 (1992), 367–439.

[16] A. Lunardi, *Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in* $\mathbb{R}^n$, Ann. Scuola Norm. Sup. Pisa , Serie IV, Vol. XXIV (1997), 133–164.

[17] A. Nagel, E.M. Stein, and S. Wainger, *Balls and metrics defined by vector fields I: basic properties*, Acta Math., 155 (1985), 103–147.

[18] R. Peszek, *PDE models for pricing stocks and options with memory feedback*, Appl. Math. Finance, 2 (1995), 211–223.

[19] S. Polidoro, *On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type*, Le Matematiche, 49 (1994), 53–105.

[20] L.P. Rothschild and E.M. Stein, *Hypoelliptic differential operators on nilpotent groups*, Acta Math., 137 (1977), 247–320.

[21] A. Sánchez-Calle, *Fundamental solutions and geometry of the sum of squares of vector fields*, Invent. Math., 78 (1984), 143–160.

[22] A.N. Shiryayev (ed.), "Selected works of A. N. Kolmogorov," vol. II, Probability theory and mathematical statistics, Kluver Academic Publishers, Dodrecht-Boston-London (1991).

[23] A.I. Vol'pert and S.I. Hudjaev, *Cauchy's problem for degenerate second order quasilinear parabolic equations*, Math. USSR Sb., 7 (1969), 365–387.

[24] S.K. Vodop'yanov and I.G. Markina, *Exceptional sets for solutions to subelliptic equations*, Siberian Math. J., 36 (1995), 694–706.

[25] C. Xu, *Regularity for quasilinear second-order subelliptic equations*, Comm. Pure Appl. Math., 45 (1992), 77–96.

[26] C. Xu, *Dirichlet problems for the quasilinear second order subelliptic equations*, Acta Math. Sinica, New Ser., 12 (1996), 18–32.