HÖLDER REGULARITY FOR SOLUTIONS OF
ULTRAPARABOLIC EQUATIONS IN DIVERGENCE FORM

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ABSTRACT. We consider the second order differential equation
\[
\sum_{i,j=1}^{m_0} \partial_{x_i} \left( a_{i,j} (x,t) \partial_{x_j} u \right) + \sum_{j=1}^{N} b_{i,j} x_i \partial_{x_j} u - \partial_t u = \sum_{j=1}^{m_0} \partial_{x_j} F_j (x,t)
\]
where \((x,t) \in \mathbb{R}^{N+1}, 0 < m_0 \leq N\), the coefficients \(a_{i,j}\) belong to a suitable space of vanishing mean oscillation functions VMO, and \(B = (b_{i,j})\) is a constant real matrix. The aim of this paper is to study interior regularity for weak solutions to the above equation assuming that \(F_j\) belong to a function space of Morrey type.

1. Introduction

The purpose of the present note is to study interior regularity for weak solutions of the ultraparabolic equation
\[
Lu \equiv \sum_{i,j=1}^{m_0} \partial_{x_i} \left( a_{i,j}(z) \partial_{x_j} u \right) + \sum_{j=1}^{N} b_{i,j} x_i \partial_{x_j} u - \partial_t u = \sum_{j=1}^{m_0} \partial_{x_j} F_j (z) \tag{1.1}
\]
where \(z = (x,t) \in \mathbb{R}^{N+1}, 0 < m_0 \leq N\) and \(F_j\) are in some \(L^p\) spaces. We can equivalently rewrite equation (1.1) as follows
\[
\text{div} \ (A(z)Du) + Yu = \text{div} \ (F)
\]
where \(F = (F_1, F_2, \ldots, F_{m_0}, 0, \ldots, 0), Yu = \langle x, BDu \rangle - \partial_t u\) and \(A(z)\) is the \(N \times N\) matrix
\[
A(z) = \begin{pmatrix} A_0(z) & 0 \\ 0 & 0 \end{pmatrix}
\]
We will say that a function \(u\) is a “weak solution” of (1.1) in the open set \(\Omega \subset \mathbb{R}^{N+1}\) if \(u, \partial_{x_1} u, \partial_{x_2} u, \ldots, \partial_{x_{m_0}} u, Yu\) belong to \(L^q_{\text{loc}}(\Omega)\) for some \(q > 1\) and
\[
\int_{\Omega} \langle A(z)Du(z), D\varphi(z) \rangle dz - \int_{\Omega} Y u(z) \varphi(z) dz = \int_{\Omega} \langle F(z), D\varphi(z) \rangle dz \quad \forall \varphi \in C_0^\infty(\Omega).
\]
We also make the following structure assumption

HYPOTHESIS H (i) The matrix \(A_0(z) = (a_{i,j}(z))_{i,j=1,\ldots,m_0}\) is symmetric and there exists \(\Lambda > 0\) such that
\[
\Lambda^{-1} |\xi|^2 \leq \langle A_0(z)\xi, \xi \rangle \leq \Lambda |\xi|^2 \quad \forall z \in \mathbb{R}^{N+1}, \forall \xi \in \mathbb{R}^{m_0}.
\]

Key words and phrases. Hypoelliptic equations, Morrey Spaces.
(ii) The matrix $B = (b_{i,j})_{i,j=1,\ldots,N}$ has constant entries and takes the following form

$$B = \begin{pmatrix}
0 & B_1 & 0 & \cdots & 0 \\
0 & 0 & B_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & B_r \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},$$

where each $B_j$ is a $m_{j-1} \times m_j$ block matrix of rank $m_j$, with $j = 1, 2, \ldots, r$, and $m_0 \geq m_1 \geq \cdots \geq m_r \geq 1$ and $m_0 + m_1 + \cdots + m_r = N$.

This kind of operators arises in the stochastic theory (see Shiryayev [21]) and in the theory of diffusion processes (see Chandrasekhar [2], Nguyen Dong An [14]). We also recall that (1.1) is a linearized version of the Landau equation that in turns is a simplified model for the Boltzmann equation (see [9]).

We are interested in the interior regularity of weak solutions to (1.1). When $A$ is a constant matrix and $F_j$ are $C^\infty$ functions then $u$ is $C^\infty$, since in that case Hypothesis H is equivalent to the Hörmander hypoellipticity condition. For a review of results about the operator in (1.1) with constant coefficients $a_{ij}$ see the paper [8], by Lanconelli and by one of the authors, and the references therein.

Operators (1.1) satisfying Hypothesis H, with Hölder continuous coefficients $a_{ij}$, were studied in [15, 16, 17]; Manfredini in [12] proved interior Schauder estimates. Some related regularity results were proved by Lunardi in [11] in the setting of the semigroup theory (see also Da Prato and Lunardi [4]).

More recently Bramanti, Cerutti and Manfredini in [1] and the authors of the present note in [18] studied the regularity of strong solutions to the nondivergence form of the equation (1.1), while operators in divergence form were considered in [13]. In both cases the most relevant hypothesis is that the coefficients $a_{ij}$ belong to the Sarason class $VMO_L$, that contains the $BMO_L$ functions whose integral oscillation over balls converges uniformly to zero as the radius goes to zero.

Boundary value problems for nonlinear equations like (1.1) have been considered by Lanconell and Lasiecka in [7], and by Lasiecka and Morbidelli in [10]. In both papers the interior regularity results proved in the linear case play a crucial role and apparently some better regularity results may be suitable. In that order, the aim of this note is to improve the results contained in [13] assuming that the right hand term $F = (F_1, F_2, \ldots, F_m, 0, \ldots, 0)$ is such that every $F_j$ belongs to a suitable Morrey space $L^{p,\lambda}(L, \Omega)$ (see Section 2 for its definition). Regularity results for elliptic equations with known term in $L^{p,\lambda}$ are obtained in [5] for equations in nondivergence form and in [19] in the case of divergence form.

In this paper the authors use a method inspired by a technique, introduced by Ciarletza, Frasca and Longo in [3], that is based on explicit representation formulas for the weak solutions of (1.1) and their first derivatives and on the $L^{p,\lambda}$ estimates of some singular integral operators.

We stress that the natural geometry for operators like (1.1) is not euclidean, as was shown in [8], but is given by the following group structures.

**Definition 1.1.** For every $(x, t), (\xi, \tau) \in \mathbb{R}^{N+1}$ we set

$$(x, t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau), \quad E(t) = \exp(-tB^2)$$
and
\[ D(\lambda) = \text{diag}(\lambda I_{m_0}, \lambda^3 I_{m_1}, \ldots, \lambda^{2r+1} I_{m_r}), \]
where \( I_{m_j} \) is the \( m_j \times m_j \) identity matrix. We say that \((\mathbb{R}^{N+1}, \circ)\) is the “translation group associated to \( L \)” and that \((D(\lambda), \lambda^2)_{\lambda > 0} \) is the “dilation group associated to \( L \)”.

In the following, we will call “homogeneous dimension” of \( \mathbb{R}^{N+1} \) the integer \( Q + 2 \), where
\[ Q = m_0 + 3m_1 + \ldots + (2r + 1)m_r. \]

We observe that the zero of the group \((\mathbb{R}^{N+1}, \circ)\) is \((0,0)\) and that \((x,t)^{-1} = (-E(-t)x, -t). \) Moreover \( \theta_{x,j} \) \((j = 1, \ldots, m_0) \) is a first order operator with respect to \((D(\lambda), \lambda^2)_{\lambda > 0} \), while \( Y \) is a second order operator. We next introduce a natural homogeneous norm and a corresponding quasi-distance.

**Definition 1.2.** Let \( \alpha_1, \ldots, \alpha_N \) be the positive integers such that
\[ D(\lambda) = \text{diag}(\lambda^\alpha_1, \ldots, \lambda^\alpha_N). \]
If \( z = 0 \) we set \( \|z\| = 0 \) while, if \( z \in \mathbb{R}^{N+1} \setminus \{0\} \) we define \( \|z\| = \varrho \) where \( \varrho \) is the unique positive solution to the equation
\[ \frac{x_1^2}{\varrho^{2\alpha_1}} + \frac{x_2^2}{\varrho^{2\alpha_2}} + \ldots + \frac{x_N^2}{\varrho^{2\alpha_N}} + \frac{\ell^2}{\varrho^2} = 1. \]

With this norm we associate the “quasidistance” \( d(z, w) = \|w^{-1} \circ z\| \) and we denote by \( B_r(z) \) the \( d \)-sphere centered at \( z \) and with radius \( r \).

The statement of our main results is contained in the following two theorems. In Section 2 the function spaces \( L^{p,\lambda}(L, \Omega) \) and \( VMO_L \) are defined in terms of the group structures of Definition 1.1. Section 2 also contains some preliminary results, while the proof of the main theorems of this note is given in Section 3.

**Theorem 1.3.** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^{N+1} \) and \( u \) a weak solution in \( \Omega \) of equation (1.1):
\[ \text{div} \left( A(x,t) Du \right) + Yu = \text{div}(F). \]

Suppose that the operator \( L \) satisfies hypothesis (H), \( a_{ij} \in VMO_L, \ i,j = 1, \ldots, m_0, \ u \in L^p(\Omega) \) and \( F_j \in L^{p,\lambda}(\Omega) \) \( \forall j = 1, \ldots, m_0, 0 \leq \lambda < Q + 2 \) and \( 1 < p < \infty \).

Then, for any compact set \( K \subset \Omega \) we have that \( \|\theta_{x,j} u \|_{L^{p,\lambda}(K)} \), \( \forall j = 1, \ldots, m_0, \)
for every \( 1 < p < \infty, 0 \leq \lambda < Q + 2. \) Moreover there exists a positive constant \( c \) depending on \( p, \lambda, K, \Omega \) and \( L \) such that
\[ \|\theta_{x,j} u \|_{L^{p,\lambda}(L,K)} \leq \left( \sum_{k=1}^{m_0} \|F_k\|_{L^{p,\lambda}(L,\Omega)} + \|u\|_{L^p(\Omega)} \right) \quad \forall j = 1, \ldots, m_0. \]

**Theorem 1.4.** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^{N+1} \) and \( u \) a weak solution in \( \Omega \) of equation (1.1). Suppose that the operator \( L \) satisfies hypothesis (H), \( a_{ij} \in VMO_L, \ i,j = 1, \ldots, m_0, \ u \in L^p(\Omega) \) and \( F_j \in L^{p,\lambda}(\Omega) \) \( \forall j = 1, \ldots, m_0, 0 \leq \lambda < Q + 2 \) and \( p > Q + 2 - \lambda. \)

Then, for any compact \( K \subset \Omega \) there exists a positive constant \( c \) depending on \( p, \lambda, K, \Omega \) and \( L \) such that
\[ \frac{|u(z) - u(\zeta)|}{\|z^{-1} \circ z\|^2 \cdot x_1 + \ldots + x_N + \frac{\ell^2}{x_n} + \frac{\ell^2}{x_n}} \leq \left( \sum_{k=1}^{m_0} \|F_k\|_{L^{p,\lambda}(L,\Omega)} + \|u\|_{L^p(\Omega)} \right), \quad \forall z, \zeta \in K, \ z \neq \zeta. \]
2. Preliminaries

We first define the function spaces $\text{BMO}_L$ and $\text{VMO}_L$ naturally associated with the group’s structures of Definition 1.1. These spaces extend respectively the John-Nirenberg class $\text{BMO}_L$ of bounded mean oscillation functions introduced in [6] and its subset $\text{VMO}_L$ first defined by Sarason in [20]. Next we define the Morrey spaces $L^{p,\lambda}(L, \Omega)$. Let $u \in L^{1}_{\text{loc}}(\mathbb{R}^{N+1})$ and

$$
\int_{B} u = \frac{1}{\text{meas}(B)} \int_{B} u; \quad u_B = \int_{B} u.
$$

**Definition 2.1.** For every function $u \in L^{1}_{\text{loc}}(\mathbb{R}^{N+1})$ we set

$$
\|u\|_{\ast} = \sup_{B} \int_{B} \left| u(z) - u_B \right| dz; \quad \eta_u(r) = \sup_{v \leq r} \int_{B_v} \left| u(z) - u_{B_v} \right| dz.
$$

Then we define

$$
\text{BMO}_L = \left\{ u \in L^{1}_{\text{loc}}(\mathbb{R}^{N+1}) : \|u\|_{\ast} < +\infty \right\},
$$

$$
\text{VMO}_L = \left\{ u \in \text{BMO}_L : \lim_{r \to 0} \eta_u(r) = 0 \right\}.
$$

**Definition 2.2.** Let $\Omega$ be an open subset of $\mathbb{R}^{N+1}$, $p \in [1, \infty]$ and $\lambda \in [0, Q + 2]$. We set

$$
L^{p,\lambda}(L, \Omega) = \left\{ f \in L^{p}_{\text{loc}}(\Omega) : \|f\|_{L^{p,\lambda}(L, \Omega)} < \infty \right\},
$$

where we set

$$
\|f\|_{L^{p,\lambda}(L, \Omega)} = \left( \sup_{r \in (0, \infty)} \frac{1}{|\Omega \cap B(x, r)|} \int_{|\Omega \cap B(x, r)|} |f(w)|^p \, dw \right)^{\frac{1}{p}}.
$$

As an immediate consequence of the Hölder inequality we get the following

**Remark 2.3.** Let $\Omega \subset \mathbb{R}^{N+1}$ then $L^{p,\nu}(L, \Omega)$ is continuously imbedded in every $L^{p,\lambda}(L, \Omega)$, with $p \leq q$ and $\lambda \leq \frac{Q}{q} + (Q + 2) \left(1 - \frac{Q}{q} \right)$.

We now recall some results that we will need in the following.

**Theorem 2.4.** (see [18], Theorem 3.3) Let $p \in [1, \infty]$, $\lambda \in [0, Q + 2]$, $a \in \text{BMO}_L$ and $g \in L^{p,\lambda}(L, \mathbb{R}^{N+1})$. For every $i, j = 1, \ldots, m_{0}$ we put

$$
T_{ij}g(z) = \lim_{\epsilon \to 0} \int_{|\xi_{r^{2}} - \xi^{2}| \geq \epsilon} \Gamma_{ij}(z; \cdot \xi - \xi_{r^{2}})g(\xi) \, d\zeta,
$$

$$
C_{ij}(a, g)(z) = \lim_{\epsilon \to 0} \int_{|\xi_{r^{2}} - \xi^{2}| \geq \epsilon} \Gamma_{ij}(z; \cdot \xi - \xi_{r^{2}}) \left[a(\xi) - a(z)\right]g(\xi) \, d\zeta.
$$

Then $T_{ij}g, C_{ij}(a, g) \in L^{p,\lambda}(L, \mathbb{R}^{N+1})$ and there exists a positive constant $c$ depending only on $p$, $\lambda$ and on the operator, such that

$$
\|T_{ij}g\|_{L^{p,\lambda}(L, \mathbb{R}^{N+1})} \leq c \|g\|_{L^{p,\lambda}(L, \mathbb{R}^{N+1})},
$$

$$
\|C_{ij}(a, g)\|_{L^{p,\lambda}(L, \mathbb{R}^{N+1})} \leq c \|a\|_{\text{BMO}_L} \|g\|_{L^{p,\lambda}(L, \mathbb{R}^{N+1})}.
$$

We now introduce some notations which are useful in the sequel. Let $r, s \in \mathbb{R}$, with $0 < s < r$ and let $\varphi \in C^{\infty}(\Omega)$ a function such that $\varphi(y) = 1$ for $0 \leq y \leq s$ and $\varphi(y) = 0$ for $y \geq r$. For every $\xi \in \Omega$ and $r > 0$ such that $B_{r} = B_{r}(\xi) \subset \Omega$ we set

$$
\eta(z) = \varphi(|\xi_{r^{2}} - \xi^{2}|).
$$
If \( u \) is a solution of (1.1), we have

\[
L(\eta u) = \text{div} (G) + g
\]

with

\[
G = \eta F + u A D\eta, \quad g = \langle A D\eta , D\eta \rangle - \langle F , D\eta \rangle + u Y^*\eta,
\]

where \( Y^* \) denotes the adjoint of the operator \( Y \).

The next result is contained in [13].

**Theorem 2.5.** Let \( u \) be a weak solution of (1.1) and \( \eta, G_1, G_2, \ldots, G_{m_0}, g \) are the functions defined above. Then, for every \( j = 1, \ldots, m_0 \)

\[
\partial_{x_j}(\eta u)(z) = \sum_{h, k=1}^{m_0} \lim_{h_k \to 0} \int_{\|\zeta^{-1} - \zeta\| \geq \varepsilon} \Gamma_{jk}(z, \zeta^{-1} \circ z) \cdot \\
\{ (a_{hk}(z) - a_{hk}(\zeta)) \partial_{x_k}(\eta u)(z) - G_k(\zeta) \} d\zeta + \\
\int_{\mathbb{R}^{N+1}} \Gamma_j(z, \zeta^{-1} \circ z) g(\zeta) d\zeta + \sum_{k=1}^{m_0} c_{jk}(z) G_k(z), \quad \text{for a.e. } z \in \mathbb{R}^{N+1} \tag{2.3}
\]

where \( c_{jk} = \int_{\|\zeta\|=1} \Gamma_j(z; \zeta) \nu_k(\zeta) d\sigma \) and \((\nu_1, \ldots, \nu_{N+1})\) is the outer normal at the set \( \Sigma_{N+1} \). \( \square \)

**Theorem 2.6.** (see [18], Theorem 2.1.) Let \( \Omega \) be an open subset of \( \mathbb{R}^{N+1}, \alpha \in [0, Q+2], \rho \in [1, \infty[ \) and \( K \in C(\mathbb{R}^{N+1} \setminus \{0\}) \) be a homogeneous function of degree \(-\alpha\) with respect to the group \((D(\lambda), \lambda^2)_{\lambda > 0}\). If \( g \in L^{p,\rho}(L, \Omega) \) then

\[
Tg(z) = \int_{\Omega} K(\zeta^{-1} \circ z) g(\zeta) d\zeta
\]

is a.e. defined. Moreover:

i) If \( \nu + \alpha \rho < Q + 2 \), then \( Tg \in L^{p,\rho}(L, \Omega) \), where \( q \) and \( \mu \) are given by the following equations

\[
\frac{1}{p} + \frac{\alpha}{Q+2} = \frac{1}{q} + 1, \quad \mu = \frac{\nu q}{p},
\]

and there exists \( C = C(p, \nu) \) such that

\[
\|Tg\|_{L^{p,\rho}(L, \Omega)} \leq C \|g\|_{L^{p,\rho}(L, \Omega)}.
\]

ii) If \( \nu + \alpha \rho > Q + 2 \) and \( \Omega = \mathbb{R}^{N+1} \), then \( Tg \in C(\mathbb{R}^{N+1}) \) and there exists \( C = C(p, \nu) \) such that

\[
\|Tg(z) - Tg(\zeta)\| \leq C \|z^{-1} \circ \zeta\|_{\frac{p+\rho+(Q+2)}{p}} \|g\|_{L^{p,\rho}(L, \mathbb{R}^{N+1})}, \quad \forall z, \zeta \in \mathbb{R}^{N+1}. \square
\]

3. MAIN RESULTS

We start with a simple observation. Assume that \( F_1, \ldots, F_{m_0}, u \in L^{p,\nu}(L, \Omega) \) and \( \partial_{x_1} u, \ldots, \partial_{x_{m_0}} u \in L^{p,\nu}(L, \Omega) \), with \( 1 < p < \infty, 0 \leq \nu \leq \lambda < Q + 2 \) and let \( G \) and \( g \) as in (2.2). Then \( G_1, \ldots, G_{m_0} \in L^{p,\nu}(L, \mathcal{B}_r), \ g \in L^{p,\nu}(L, \mathcal{B}_r) \) and there
exists \( c > 0 \) such that
\[
\sum_{j=1}^{m_0} \|G_j\|_{L^p,\lambda(L, B_r)} \leq c \left( \|u\|_{L^p,\lambda(L, B_r)} + \sum_{j=1}^{m_0} \|F_j\|_{L^p,\lambda(L, B_r)} \right),
\]
\[
\|g\|_{L^{p,\nu}(L, B_r)} \leq c \left( \|u\|_{L^{p,\nu}(L, B_r)} + \sum_{j=1}^{m_0} (\|F_j\|_{L^{p,\nu}(L, B_r)} + \|\partial_{x_j} u\|_{L^{p,\nu}(L, B_r)}) \right). \tag{3.1}
\]

In order to simplify the notations, in the following we shall denote by
\[
D_0 u = (\partial_{x_1} u, \partial_{x_2} u, \ldots, \partial_{x_m} u),
\]
\[
\|D_0 u\|_{L^p,\lambda(L, \Omega)} = \sum_{j=1}^{m_0} \|\partial_{x_j} u\|_{L^p,\lambda(L, \Omega)},
\]
\[
\|F\|_{L^p,\lambda(L, \Omega)} = \sum_{j=1}^{m_0} \|F_j\|_{L^p,\lambda(L, \Omega)}.
\]

Lemma 3.1. Let \( B_r \subset \subset \Omega \), and suppose that \( u, F \in L^{p,\lambda}(L, B_r) \) and that \( D_0 u \in L^{p,\nu}(L, B_r) \), with \( 1 < p < \infty \), \( 0 \leq \lambda < Q + 2 \).

Then there exists a positive constant \( r_0 \) depending on the operator \( L \) such that, \( \partial_{x_j} u \in L^{p,\lambda}(B_{r_0}) \), for \( j = 1, \ldots, m_0 \), \( r_0 \) is fixed, with \( \mu = \min(\lambda, \nu + p) \) and
\[
\|D_0 u\|_{L^{p,\nu}(L, B_r)} \leq c \left( \|F\|_{L^{p,\lambda}(L, B_r)} + \|u\|_{L^{p,\lambda}(L, B_r)} + \|D_0 u\|_{L^{p,\nu}(L, B_r)} \right) \tag{3.2}
\]
for every \( s \) such that \( 0 < s < r \). \( \Box \)

Proof. We start to prove (3.2), assuming first that \( \partial_{x_j} u \in L^{p,\mu}(L, B_r) \).

Let \( v(z) = \eta(z)u(z) \), where \( \eta \) is defined above in the ball \( B_r \) as in (2.1). Using the representation formula (2.3) we have
\[
\partial_{x_j} v(z) = \sum_{h,k=1}^{m_0} c_{j,k} a_{h,k} \eta v_k(z) - \sum_{h=1}^{m_0} T_{j,h} G_h(z) + T_{j} g(z) + \sum_{h=1}^{m_0} c_{j,h} G_h(z),
\]
(here \( T_j g(z) = \int \Gamma_j(z, \zeta^{-1} \circ \zeta) g(\zeta) d\zeta \); recall that \( |\Gamma_j(z, \zeta^{-1} \circ \zeta)| \leq c |\zeta^{-1} \circ \zeta|^{-Q+1} \), as shown in [15], Corollary 2.2). Then, since \( c_{j,k} \) are bounded functions, by Theorem 2.4, Theorem 2.6 and Remark 2.3, we have
\[
\|\partial_{x_j} v\|_{L^{p,\nu}(L, B_r)} \leq c \left( \sum_{h,k=1}^{m_0} \|a_{h,k}\| \|\partial_{x_k} v\|_{L^{p,\lambda}(L, B_r)} + \|G\|_{L^{p,\lambda}(L, B_r)} + \|g\|_{L^{p,\nu}(L, B_r)} \right).
\]

Using the VMO\(_L\) hypothesis on the coefficients \( a_{h,k} \) (and since \( \mu \leq \lambda \)) we obtain
\[
\|\partial_{x_j} v\|_{L^{p,\nu}(L, B_r)} \leq c \left( \|G\|_{L^{p,\lambda}(L, B_r)} + \|g\|_{L^{p,\nu}(L, B_r)} \right), \ \forall r \leq r_0.
\]

From the above inequalities (3.1) and since \( \eta \) is a bounded function with support contained in the ball \( B_r \), we obtain the requested formula.

We now remove the assumption that \( \partial_{x_j} u \in L^{p,\mu}(B_r) \). First of all we note that for every \( B \) contained in \( B_r \), the map
\[
\overline{S}: [L^{p,\nu}(L, B)]^{m_0} \rightarrow [L^{p,\nu}(L, B)]^{m_0}
\]
defined for every $\bar{u} \in [L^{p,\nu}(L, B)]^{m_0}$ as
\[
(S\bar{u})_j(z) = \sum_{h,k=1}^{m_0} C_{j,k}[a_{h,k}; \bar{u}](z) - \sum_{h=1}^{m_0} T_{j,h}(G_h)(z) + Tg(z) + \sum_{h=1}^{m_0} c_{j,k} G_h(z),
\]
(j = 1, \ldots, m_0) is a contraction in $[L^{p,\nu}(L, B)]^{m_0}$ and it follows that $\partial_d \bar{v}$ is the unique fixed point of $S$ in $[L^{q,\nu}(L, B)]^{m_0}$. Moreover $S$ is also a contraction in $[L^{q,\nu}(L, B)]^{m_0}$, then $S$ has a unique fixed point $\bar{u} \in [L^{p,\nu}(L, B)]^{m_0}$. Since $[L^{p,\nu}(L, B)]^{m_0} \subset [L^{p,\nu}(L, B)]^{m_0}$ the function $\partial_d \bar{v}$ must coincide with $\bar{u} \in [L^{p,\nu}(L, B)]^{m_0}$.

\[\square\]

**Proof of Theorem 1.3.** We first claim that, if $u, D_0u, F \in L^{p,\nu}(\Omega)$, for some $\nu \in [0, Q + 2]$ and $\mu \in [0, Q + 2]$ is such that $\mu \leq p + \nu$, then $u \in L^{p,\nu}(\Omega)$. We obtain this result as in the proof of Theorem 1.2 of [13]: denote by $\Gamma^0$ the fundamental solution of the operator $L_0 = \Delta_{m_0} + Y$ and let $\eta$ be the function defined in (2.2). Then we represent the function $\eta\bar{u}$ as
\[
(\eta\bar{u})(z) = \int_{\Omega} \Gamma^0(\zeta^{-1} \circ z) \eta^0(\zeta) d\zeta + \sum_{j=1}^{m_0} \int_{\Omega} \Gamma^0_j(\zeta^{-1} \circ z) \eta(\zeta) G_j^0(\zeta) d\zeta, \tag{3.3}
\]
where
\[
G_j^0 = F_j + \partial_d u - \sum_{k=1}^{m_0} a_{j,k} \partial_d u, \quad g^0 = \sum_{j=1}^{m_0} (G_j^0 - 2 \partial_d u) \partial_d \eta - uL\eta,
\]
and we apply Theorem 2.6 (i). We also obtain the estimate
\[
\|\bar{u}\|_{L^{p,\nu}(B_r)} \leq c \left( \|\bar{u}\|_{L^{p,\nu}(B_s)} + \|F\|_{L^{p,\nu}(L, B_s)} + \|D_0u\|_{L^{p,\nu}(L, B_s)} \right). \tag{3.4}
\]
In order to prove Theorem 1.3, we observe that it is sufficient to prove the claim when $\Omega$ is a ball $B_r$ and $K$ is the closure of $B_s$, $0 < s < r$. Let choose $s_1, s_2, s_3 \in \mathbb{R}$ such that $s < s_3 < s_2 < s_1 < r$ and recall that in the paper [13] (Theorem 1.1) the $L^p$-result is contained, thus $D_0u \in L^{p,\nu}(\Omega)$ and
\[
\|D_0u\|_{L^{p,\nu}(B_s)} \leq c \left( \|F\|_{L^{p,\nu}(L, B_s)} + \|u\|_{L^{p,\nu}(L, B_s)} \right). \tag{3.5}
\]
Combining this inequality with (3.4), (letting $\nu = 0$ and $\mu = \min\{p, \lambda\}$) we get
\[
\|u\|_{L^{p,\nu}(L, B_s)} \leq c \left( \|F\|_{L^{p,\nu}(L, B_s)} + \|u\|_{L^{p,\nu}(L, B_s)} \right). \tag{3.6}
\]
Then Lemma 3.1 (with $\nu = 0$), formulas (3.5) and (3.6) give
\[
\|D_0u\|_{L^{p,\nu}(L, B_s)} \leq c \left( \|F\|_{L^{p,\nu}(L, B_s)} + \|u\|_{L^{p,\nu}(L, B_s)} \right).
\]
If $\nu < p$ we get the conclusion, otherwise we proceed as above, choosing $s_4, s_5, s_6 \in \mathbb{R}$ such that $s < s_6 < s_5 < s_4 < s_3$ and using formula (3.4) and Lemma 3.1 with $\nu = p$. We find
\[
\|D_0u\|_{L^{p,\nu}(L, B_s)} \leq c \left( \|F\|_{L^{p,\nu}(L, B_s)} + \|u\|_{L^{p,\nu}(L, B_s)} \right),
\]
where $\mu_1 = \min\{\lambda, 2p\}$ If $\lambda < 2p$ we get to the conclusion, if $\lambda > 2p$ we apply this technique again. We observe that the method is finite because we have used (3.2) with $\nu = 0, \nu = p, \nu = 2p, \ldots$ and since the improvement $p$ is constant we surely arrive to $\nu = \lambda$. \[\square\]

**Proof of Theorem 1.4.** It is sufficient to use the representation formula (3.3) together with Theorems 1.3 and 2.6 (ii). \[\square\]
REFERENCES


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