A GLOBAL LOWER BOUND FOR THE FUNDAMENTAL SOLUTION OF KOLMOGOROV-FOKKER-PLANCK EQUATIONS

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Abstract

The main result of this paper is a global lower bound for the fundamental solution $\Gamma$ of the ultraparabolic differential operator

$$Lu = \sum_{i,j=1}^{p_0} \partial_{x_i}(a_{i,j}(x,t)\partial_{x_j}u) + \sum_{i,j=1}^{N} b_{i,j}x_i\partial_{x_j}u - \partial_t u,$$

where the $a_{i,j}$’s and their first derivatives are Hölder continuous functions and $0 < p_0 < N$. The bound will follow from a local estimate of $\Gamma$ and a Harnack inequality for non-negative solutions of $Lu = 0$, by exploiting the invariance of the Harnack inequality with respect to suitable translation and dilation groups. For non-degenerate parabolic operators, our methods and results generalize those of Aronson and Serrin [1].

1. Introduction and main results

We consider in $\mathbb{R}^{N+1}$ the second order differential operator

$$Lu = \text{div}(A(z) Du) + \langle x, BDu \rangle - \partial_t u, \quad (1.1)$$

where $z = (x,t) \in \mathbb{R}^{N+1}$, $\text{div}(\cdot)$, $D = (\partial_{x_1}, \ldots, \partial_{x_N})$ and $\langle \cdot, \cdot \rangle$ denote, respectively, the divergence, the gradient and the inner product in $\mathbb{R}^N$. In (1.1), $B = (b_{i,j})$ denotes an $N \times N$ matrix with constant real entries and $A(z) = (a_{i,j}(z))_{i,j=1,\ldots,N}$ is a non-negative symmetric matrix for any $z \in \mathbb{R}^{N+1}$.

Equations like (1.1) arise in the stochastic theory of diffusion processes. For example

$$L = \sum_{j=1}^{n} \partial_{x_j}^2 + \sum_{j=1}^{n} x_j \partial_{x_{n+j}} - \partial_t$$

is the prototype of the Kolmogorov operator which, under suitable conditions, describes the probability density of a physical system with $2n$ degree of freedom (see [8], page 167. For further physical meanings, see also Chandrasekhar [2]).

In our treatment we shall always assume that
Hypothesis H1. For some basis in $\mathbb{R}^N$, $A(z)$ and $B$ can be written as

$$A(z) = \begin{pmatrix} A_0(z) & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & B_1 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & B_r \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where each $B_j$ is a $p_j-1 \times p_j$ block matrix of rank $p_j$, $j = 1, 2, \ldots, r$, with $p_0 \geq p_1 \geq \ldots \geq p_r \geq 1$ and $p_0 + p_1 + \ldots + p_r = N$. Moreover, there exists $\Lambda > 0$ such that

$$\frac{1}{\Lambda} |\xi|^2 \leq \langle A_0(z)\xi, \xi \rangle \leq \Lambda |\xi|^2$$

for every $\xi \in \mathbb{R}^{p_0}$ and for every $z \in \mathbb{R}^{N+1}$.

An important consequence of hypothesis H1 is that it ensures that the “frozen” operator

$$L_\zeta u = \text{div} (A(\zeta)Du) + \langle x, BDu \rangle - \partial_t u,$$

(1.2)
is hypoelliptic for every $\zeta \in \mathbb{R}^{N+1}$. This aspect will be fully clarified in Section 2, Theorem A.

In order to state the regularity hypotheses on the coefficients $a_{i,j}$, we introduce the following definitions.

Definition 1.1. For every $(x, t), (\xi, \tau) \in \mathbb{R}^{N+1}$ we define

$$(x, t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau), \quad E(t) = \exp(-tB^T).$$

(1.3)

and

$$D(\lambda) = \text{diag}(\lambda I_{p_0}, \lambda^3 I_{p_1}, \ldots, \lambda^{2r+1} I_{p_r}),$$

(1.4)

where $I_{p_j}$ denotes the $p_j \times p_j$ identity matrix.

We shall say that $(\mathbb{R}^{N+1}, \circ)$ is the translation group associated to $L$ and that $G = (D(\lambda), \lambda^2)_{\lambda > 0}$ is the dilation group associated to $L$.

Definition 1.2. Let $\{q_j\}_{j=1}^N$ be such that $D(\lambda) = \text{diag}(\lambda^{q_1}, \ldots, \lambda^{q_N}, \lambda^2)$. For every $z = (x, t) \in \mathbb{R}^{N+1}$, we put

$$|x|_B = \sum_{j=1}^N |x_j|^{1/q_j}, \quad \|z\|_B = |t|^{1/2} + |x|_B.$$

Hypothesis H2. There exist $\alpha \in (0, 1]$ and $M > 0$ such that

$$|a_{i,j}(z) - a_{i,j}(\zeta)| \leq M \|\zeta^{-1} \circ z\|_B^\alpha,$$

$$|\partial_x a_{i,j}(z) - \partial_x a_{i,j}(\zeta)| \leq M \|\zeta^{-1} \circ z\|_B^\alpha,$$

$$|\partial_{x^2} a_{i,j}(z) - \partial_{x^2} a_{i,j}(\zeta)| \leq M \|\zeta^{-1} \circ z\|_B^\alpha.$$
for all $z, \zeta \in \mathbb{R}^{N+1}$ and for any $i, j = 1, \ldots, p_0$.

In a recent work [7] we constructed the fundamental solution for operators of the form (1.1), verifying hypotheses H1 and H2, by means of an adaptation of Levi’s parametrix method.

The parametrix method provides an upper bound for the fundamental solution $\Gamma$ of $L$. It was shown in [7, Corollary 2.5] that there exists a positive constant $\lambda$ such that, if $\Gamma^+$ denotes the fundamental solution of

$$L^+ = \lambda \Delta_{p_0} + \langle x, BD \rangle - \partial_t,$$

then for every $T > 0$ there exists a positive constant $c^+$ with the property that

$$\Gamma(z; \zeta) \leq c^+ \Gamma^+(z; \zeta)$$

(1.5)

for every $z = (x, t), \zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$, with $0 < t - \tau < T$. Here $\Delta_{p_0} = \sum_{j=1}^{p_0} \partial^2_{x_j}$.

The aim of this paper is to provide a lower bound analogous to (1.5), as stated in the following

**Main Theorem (Global lower bound).** Let $L$ be as in (1.1), verifying hypotheses H1 and H2. Then there exists a positive constant $\lambda$ such that, if $\Gamma^-$ denotes the fundamental solution of

$$L^- = \lambda^{-1} \Delta_{p_0} + \langle x, BD \rangle - \partial_t,$$

then for every $T > 0$ there exists a positive constant $c^-$ with the property that

$$c^- \Gamma^-(z; \zeta) \leq \Gamma(z; \zeta)$$

(1.6)

for every $z = (x, t), \zeta = (\xi, \tau) \in \mathbb{R}^{N+1}, 0 \leq t - \tau \leq T$.

Here, we emphasize the fact that the functions $\Gamma^-$ and $\Gamma^+$ appearing in (1.5) and (1.6) can be expressed explicitly, since they are fundamental solutions of the operators (1.1) with $A(z)$ a constant matrix (see Section 2, Theorem A4).

The main theorem relies only on a local estimate of $\Gamma$ and on a Harnack inequality for non-negative solutions of $Lu = 0$, which is invariant with respect to the translation and dilation groups described in Definition 1.1. These results were proved in [7] and are stated in Section 2. In Section 2 we also recall some results from [4] concerning operators (1.1) with constant $A(z)$.

In Section 3 we prove a “Harnack principle” that extends the local Harnack inequality to a global Harnack inequality, where the Harnack constant is explicitly expressed in terms of the location of $z$ and $\zeta$. We obtain this result by using a rather complicated technique which is inspired by the method of Aronson and Serrin [1] for classical parabolic operators.

The proof of the main theorem will be given in Section 4, using the Harnack principle and the local estimate of $\Gamma$.

2. Some known results

We first recall some results from [4] for operators (1.1) with constant matrix $A(z)$.
**Theorem A.** Let $A$ and $B$ be two $N \times N$ constant matrices, with $A$ symmetric and non-negative, and let

$$L_0 = \text{div}(AD) + \langle x, BD \rangle - \partial_t.$$  \hspace{1cm} (2.1)

Then

A1. The operator $L_0$ is invariant with respect to left translations of the group $(\mathbb{R}^{N+1}, \circ)$, defined in (1.3).

A2 [4, Propositions A.1 and 2.1]. The following conditions are equivalent:

i) $L_0$ is hypoelliptic;

ii) For some basis of $\mathbb{R}^N$ the matrices $A$ and $B$ take the form

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} * & B_1 & 0 & \cdots & 0 \\ * & * & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & B_r \end{pmatrix},$$  \hspace{1cm} (2.2)

where $A_0$ is a $p_0 \times p_0$ non-singular matrix, each $B_j$ is a $p_j-1 \times p_j$ block matrix of rank $p_j$, $j = 1, 2, \ldots, r$, with $p_0 \geq p_1 \geq \cdots \geq p_r \geq 1$ and $p_0 + p_1 + \cdots + p_r = N$. The blocks "*" can be arbitrary;

iii) If we put

$$C(t) = \int_0^t E(s)AE^T(s)ds,$$  \hspace{1cm} (2.3)

then

$$C(t) > 0 \quad \text{for every} \ t > 0;$$

iv) (Hörmander’s condition, see [3]) If we set $Y = \langle x, BD \rangle - \partial_t$, then

$$\text{rank} \left( \mathcal{L}(\partial_{x_1}, \ldots, \partial_{x_{p_0}}, Y) \right)(z) = N + 1 \quad \forall z \in \mathbb{R}^{N+1},$$

where $\mathcal{L}(\partial_{x_1}, \ldots, \partial_{x_{p_0}}, Y)$ is the Lie algebra generated by $\partial_{x_1}, \ldots, \partial_{x_{p_0}}, Y$.

A3 [4, Proposition 2.2]. The operator $L_0$ is invariant with respect to the dilation group $\mathcal{G} = \left( (D(\lambda), \lambda^2) \right)_{\lambda > 0} \ (\text{where} \ D(\lambda) \ \text{is defined in (1.4)} \ if \ \text{(and only if) the "*" blocks of} \ B \ \text{in (2.2) are zero matrices}.$

A4 [4, Proposition 2.3 and Remark 2.1]. If $L_0$ is invariant with respect to the dilation group $\mathcal{G}$, then the following identities hold:

$$C(\lambda^2 t) = D(\lambda)C(t)D(\lambda), \quad E(\lambda^2 t) = D(\lambda)E(t)D(1/\lambda)$$  \hspace{1cm} (2.4)

for every $t \in \mathbb{R}, \lambda > 0$. Moreover the fundamental solution $\Gamma$ of (2.1) with pole at $(\xi, \tau) \in \mathbb{R}^{N+1}$ is given by

$$\Gamma(x, t; \xi, \tau) = \Gamma(x - E(t - \tau)\xi, t - \tau),$$
where \( \Gamma(x,t) = \Gamma(x,t;0,0) = 0 \) if \( t \leq 0 \) and

\[
\Gamma(x,t) = c_N t^{-Q/2} \exp \left( -\frac{1}{4} \left< C^{-1} \left( t^{-1/2} x, D \left( t^{-1/2} x \right) \right) \right> \right) \text{ if } t > 0,
\]

with

\[
c_N = \frac{(4\pi)^{-N/2}}{\sqrt{\det C(1)}} \quad \text{and} \quad Q = p_0 + 3p_1 + \ldots + (2r + 1)p_r.
\]

The result A2 justifies the claim that the operator \( L_\zeta \) in (1.2) is hypoelliptic. Furthermore, A1 and A3 ensure that \( L_\zeta \) is invariant with respect to the translation and dilation groups defined in (1.3) and (1.4). Finally A4 yields an explicit expression for \( \Gamma \) when \( A \) is constant.

In [7] the existence of a fundamental solution \( \Gamma(z, \zeta) \) of \( L \) in (1.1) was proved, when hypotheses H1 and H2 are satisfied. Moreover, the following local estimate was obtained.

**Theorem B** [7, Theorem 1.2]. Let \( L \) be as in (1.1), verifying H1 and H2. Then for every \( \epsilon > 0 \) there exists \( K > 0 \) such that

\[
(1 - \epsilon)Z(z, \zeta) \leq \Gamma(z, \zeta) \leq (1 + \epsilon)Z(z, \zeta)
\]

for any \( z, \zeta \in \mathbb{R}^{N+1} \) such that \( Z(z, \zeta) \geq K \).

Here \( Z(z, \zeta) \) denotes the parametrix, which is the fundamental solution of \( L_\zeta \) with pole at \( \zeta \).

The following theorem states an invariant Harnack inequality for non-negative solutions of \( Lu = 0 \). We first introduce some notation. For every \( \rho > 0 \) set

\[
H_\rho = \{(x,t) \in \mathbb{R}^{N+1}; \: -\rho^2 \leq t \leq 0; \: |D(\rho)x| \leq 1\}
\]

\[
H_\rho^- = \{(x,t) \in \mathbb{R}^{N+1}; \: t = -\rho^2; \: |D(\rho)x| \leq 1\}
\]

\[
H_\rho(z_0) = z_0 \circ H_\rho, \quad H_\rho^-(z_0) = z_0 \circ H_\rho^-.
\]

**Theorem C** [7, Theorem 1.3]. Let \( L \) be as in (1.1), verifying H1 and H2, and let \( \Omega \) be an open subset of \( \mathbb{R}^{N+1} \). Then there exist constants \( c, r_0 > 0 \) and \( \theta \in (0,1) \), depending only on the constants in H1 and H2, such that

\[
\sup_{z \in H_{r_0}^-(z_0)} u(z) \leq cu(z_0)
\]

for every non-negative solution \( u \) of \( Lu = 0 \) in \( \Omega \), for every \( z_0 \in \Omega \) such that \( H_\rho(z_0) \subset \Omega \) and for every \( r \in (0,r_0) \).

### 3. The Harnack principle

In this section we shall prove our main Harnack principle. To this end, let

\( I = (T_0,T_1), I' = (T_0 - \omega, T_1) \subset \mathbb{R}, \omega \in \mathbb{R}^+ \), we denote by \( S_I \) and \( S_{I'} \) the strips

\[
S_I = \mathbb{R}^N \times I, \quad S_{I'} = \mathbb{R}^N \times I'.
\]
Proposition 3.1 - Harnack principle. Fix $\omega \in (0, (1 - \theta)r_0)$, and let $u$ be a non-negative solution of $Lu = 0$ in $S_I$. Then, for every $\tilde{z} \in S_I$ and for every $z = (x, -t) \in \mathbb{R}^{N+1}$, with $t > 0$ and $\tilde{z} \circ z \in S_I$, we have

$$u(\tilde{z} \circ z) \leq a \exp \left( b |D \left( \frac{t^{-1/2}}{|x|} \right) | + \frac{b't}{\omega} \right) u(\tilde{z}),$$

(3.1)

where $a, b, b'$ are positive constants depending only on the quantities in $H1$ and $H2$ and on the matrix $B$.

In order to prove Proposition 3.1 we shall make use of a method introduced by Aronson and Serrin for parabolic operators, consisting of the repeated application of the invariant Harnack inequality stated in Theorem C.

Definition 3.1. Let $\Omega$ be an open subset of $\mathbb{R}^{N+1}$, and let $c > 0$ be the constant of the (invariant) Harnack inequality of Theorem C. A set $\{z_0, z_1, ..., z_m\} \subset \Omega$ is called a Harnack chain of length $m$ if

$$u(z_j) \leq c u(z_{j-1}) \quad \forall j = 1, ..., m.$$ 

We say that two points $(\zeta, z) \in \Omega \times \Omega$ are connected by a Harnack chain if there exists a Harnack chain $\{z_0, z_1, ..., z_m\}$ such that $z_0 = \zeta, z_m = z$.

Remark 3.1. If two points $(\zeta, z) \in \Omega \times \Omega$ are connected by a Harnack chain of length $m$, then

$$u(z) \leq c^m u(\zeta).$$

(3.2)

Remark 3.2. If $L$ is a “classical” parabolic operator (non-degenerate, $p_0 = N$) and if $S_I$ and $S_{IF}$ are the strips of Proposition 3.1, then for every pair $\zeta = (\xi, \tau)$ and $z = (x, t)$, with $t < \tau$, we can easily construct a Harnack chain of length $m$ that connects the pair $(\zeta, z)$ by selecting $z_j$ in the segment $[\xi, z]$. Moreover, we can bound the length $m$ of such a chain by

$$m \leq c' \frac{|\xi - x|^2}{\tau - t} + 1,$$

(3.3)

where $c'$ only depends on $S_I$ and $S_{IF}$. The Harnack principle of Aronson and Serrin [1, Theorem 5] easily follows from (3.2) and (3.3).

In order to adapt this method to the more general context of operators (1.1), we first note that the (canonical) parameterization $\gamma : [0, 1] \rightarrow \mathbb{R}^{N+1}$ of the segment $[\zeta, z]$ is the solution of the Cauchy problem

$$\frac{d}{ds} \gamma(s) = (\xi - x, \tau - t), \quad \gamma(0) = (\xi, \tau).$$

In other words, given the vector fields $\partial_{x_1}, ..., \partial_{x_N}, -\partial_t$, the solution $\gamma$ can be characterized as the trajectory of a certain linear combination of the fields $\partial_{x_1}, ..., \partial_{x_N}, -\partial_t$ ($= Y$). 

6
In the general case (1.1), due to Hörmander’s condition (hypothesis H1) we are able to construct a Harnack chain connecting $\zeta$ and $z$ by choosing the points $z_j$ on the trajectory $\gamma$ of a certain linear combination of the vector fields $\partial_{x_1}, \ldots, \partial_{x_{n_0}}, Y = \langle x, BD \rangle - \partial_t$.

We also provide a sharp upper bound for this chain length, thus generalizing the estimate (3.3).

For $j = 1, \ldots, p_0$, denote by $X_j$ the vector field $\partial_{x_j}$ and, for $k = 1, \ldots, r$, let

$$X_j^0 = X_j, \quad X_j^k = [X_j^{k-1}, Y].$$

For $k = 0, 1, \ldots, r$, define the following subspaces of $\mathbb{R}^N$

$$V_k = \text{span}\left\{X_j^h : j = 1, \ldots, p_0; h = 1, \ldots, k\right\} = \text{span}\left\{e_1, e_2, \ldots, e_{p_0+\ldots+p_k}\right\},$$

$$W_k = V_k^\perp,$$

where $e_j = (0, \ldots, 1, \ldots, 0)$. Note that

$$D(\lambda)V_k = V_k, \quad D(\lambda)W_k = W_k,$$

while from H1 we obtain

$$\{0\} \subsetneq V_0 \subsetneq \cdots \subsetneq V_{r-1} \subsetneq V_r = \mathbb{R}^N$$

$$\mathbb{R}^N \supsetneq W_0 \supsetneq \cdots \supsetneq W_{r-1} \supsetneq W_r = \{0\}.$$

For any $\xi \in V_0$, with $|\xi| \leq 1$ and for $\sigma \in (0, \omega \theta/(1-\theta))$, let

$$\zeta = D\left(\sqrt{\sigma}\right)\xi, \quad \tilde{\zeta} = (-D\left(\sqrt{\sigma}\right)\xi, -\sigma),$$

and define $(\zeta^k)_{k \in \mathbb{N}}$ and $(\tilde{\zeta}^k)_{k \in \mathbb{N}}$ through the induction relations

$$\zeta^1 = \zeta, \quad \zeta^{k+1} = \zeta^k \circ \zeta, \quad \tilde{\zeta}^1 = \tilde{\zeta}, \quad \tilde{\zeta}^{k+1} = \tilde{\zeta}^k \circ \tilde{\zeta}.$$

Then it can easily be shown that

$$\zeta^k = \left(\sum_{j=0}^{k-1} E(-j\sigma)D\left(\sqrt{\sigma}\right)\xi, -k\sigma\right),$$

$$\tilde{\zeta}^k = \left(-\sum_{j=0}^{k-1} E(-j\sigma)D\left(\sqrt{\sigma}\right)\xi, -k\sigma\right).$$

Using the second identity of (2.4) and setting

$$F_k = \sum_{j=0}^{k-1} E(-j), \quad (3.4)$$

7
we have
\[ \zeta^k = (D \left( \sqrt{\sigma} \right) F_k \xi, -k \sigma), \quad \tilde{\zeta}^k = (-D \left( \sqrt{\sigma} \right) F_k \xi, -k \sigma). \]

Next, for \( k \in \mathbb{N} \) and \( \xi \) and \( \sigma \) as above, we put
\[ w_0 = \zeta^k, \quad \tilde{w}_0 = \tilde{\zeta}^k, \quad (3.5) \]
and, by induction,
\[ w_j = \tilde{w}_{j-1} \circ w_{j-1}, \quad \tilde{w}_j = w_{j-1} \circ \tilde{w}_{j-1}. \quad (3.6) \]

A direct computation proves that
\[ w_j = (G_j(-k \sigma) D \left( \sqrt{\sigma} \right) F_k \xi, -k2^j \sigma), \]
where, for every \( s \in \mathbb{R} \),
\[ G_j(s) = \sum_{l=0}^{2^j - 1} \beta_l E(ls). \quad (3.7) \]
The coefficients \( \beta_l \) are defined by induction, namely
\[ \beta_0 = 1, \quad \beta_1 = -1, \quad (3.8) \]
and, for \( j = 1, 2, ..., r \),
\[ \beta_{2^j+l} = -\beta_l \quad \text{for} \quad l = 0, 1, ..., 2^j - 1. \quad (3.9) \]

Letting \( G_j = G_j(-1) \), from the second identity (2.4) we see that
\[ w_j = \left( D \left( \sqrt{\sigma} \right) D \left( \sqrt{k} \right) G_j D \left( 1/\sqrt{k} \right) F_k \xi, -k2^j \sigma \right). \quad (3.10) \]
Clearly \( w_j \) depends on \( \xi \), on \( \sigma \), and on \( k \). In the sequel, to avoid ambiguity, we will often write
\[ w_j = w_j(\xi, \sigma, k). \]

In order to find “shortest” Harnack chains we need the following lemma, whose proof is postponed to the Appendix.

**Lemma 3.1.** Denote by \( \pi_j \) the orthogonal projection of \( \mathbb{R}^N = V_r \) on \( V_j \). Then
(i) For every \( \xi \in V_0 \) and \( k \in \mathbb{N} \)
\[ \pi_0(F_k \xi) = k \xi. \quad (3.11) \]
(ii) There exists \( \tilde{c} = \tilde{c}(B) > 0 \) such that, for every \( \xi \in V_0 \) and for every \( k \in \mathbb{N} \)
\[ \left| D \left( k^{-1/2} \right) F_k \xi \right| \leq \tilde{c}\sqrt{k} |\xi|. \quad (3.12) \]
(iii) For every \( j, m \in \{0, ..., r \} \)
\[
y \in W_m \Rightarrow G_j y \in W_{m+j}, \quad y \in \mathbb{R}^N \Rightarrow G_j y \in W_{j-1}
\] (3.13)

(we agree to set \( W_{-1} = \mathbb{R}^N \)).

(iv) For every \( j \in \{0, ..., r \} \) and for every \( y \in W_{j-1} \cap V_j \), there exists \( \eta \in V_0 \)

such that \( G_j \eta - y \in W_j, \quad |\eta| \leq \tilde{c}_j |y| \),

(3.14)

where \( \tilde{c}_j, j = 0, ..., r \), are positive constants depending only on matrix B.

**Proof of Proposition 3.1.** Let \( u \) be a non-negative solution of \( Lu = 0 \) in \( S_T \).

Also let \( \tilde{z} = (\tilde{x}, \tilde{t}) \in S_T \) and \( z = (x, -t) \in \mathbb{R}^{N+1} \) be two points such that \( t > 0 \) and \( \tilde{z} \circ z = S_T \).

We shall construct a Harnack chain connecting \( \tilde{z} \) to \( \tilde{z} \circ z \). Due to the complexity of the proof, we carry it out in four main steps A, B, C and D.

**Step A.** Suppose that
\[
t \leq \omega \theta \frac{r + 1}{1 - \theta}
\] (3.15)

and put
\[
s = \frac{t}{r + 1}.
\] (3.16)

We claim that for every \( j \in \{0, 1, ..., r \} \) there exists
\[
\xi_j \in V_0 : |\xi_j| \leq 1, \quad \sigma_j \in \left(0, \frac{\omega \theta}{1 - \theta}\right], \quad k_j \in \mathbb{N},
\] (3.17)

such that, letting
\[
z_0 = z \circ w_0^{-1}, \quad z_j = z_{j-1} \circ w_j^{-1},
\] (3.18)

where \( z_j = (x_j, -t_j) \) and \( w_j = w_j(\xi_j, \sigma_j, k_j) \) as in (3.10), the following relation holds:
\[
x_j \in W_j, \quad s = k_j^2 \sigma_j, \quad t_j = t - (j + 1)s.
\] (3.19)

From (3.10) we get
\[
w_j = (v_j, -s), \quad \text{with } v_j \in \mathbb{R}^N.
\]

In order to prove the claim, we put
\[
k_0 = \min \left\{ m \in \mathbb{N} : m \geq \left| D \left(1/\sqrt{s}\right)x\right|, \frac{\sigma_0}{s/k_0}, \frac{\xi_0}{\pi_0(x)/(k_0 \sqrt{s})}, \pi_0(x)/(k_0 \sqrt{s})\right\},
\] (3.20)

Clearly \( \xi_0 \in V_0 \) and, from (3.15), \( \sigma_0 \in \left(0, \omega \theta/(1 - \theta)\right] \). Moreover, using the expression (1.4) for \( D(\lambda) \), we find
\[
|\xi_0|^2 = (1/k_0^2 \sigma_0) |\pi_0(x)|^2 = (1/k_0^2) |\pi_0(x)|^2
\]
\[
= (1/k_0) |\pi_0 \left( D \left(1/\sqrt{s}\right)x\right)|^2 \leq (1/k_0) \left| D \left(1/\sqrt{s}\right)x\right|^2 \leq 1.
\]

9
Therefore, if \( j = 0 \), conditions (3.17) hold together with the second equation in (3.19).

In order to prove the first condition of (3.19), we express \( z_0 \) explicitly. Using the composition law “\( \circ \)” defined in (1.3), we have

\[
\begin{align*}
  w_0^{-1} &= (v_0, -s)^{-1} = (-E(s)v_0, s), \\
  \text{and, by (3.18),} & \quad z_0 = (x_0, -t_0) = (E(s)(x - v_0), -t + s),
\end{align*}
\]

from which we obtain the last equality in (3.19). On the other hand, from Lemma 3.1 (i) and from the definition of \( w_0 \),

\[
  \pi_0(v_0) = \pi_0 \left( D \left( \sqrt{\sigma_0} F_{k_0} \xi_0 \right) \right) = \sqrt{\sigma_0} \pi_0 \left( F_{k_0} \xi_0 \right) = \sqrt{\sigma_0 k_0} \xi_0 = \pi_0(x);
\]

thus \( x - v_0 \in W_0 \). Finally, since \( E(s) \) is a lower triangular matrix, it follows from (3.21) that also \( x_0 \in W_0 \). Hence (3.19) holds if \( j = 0 \).

The expression (3.21) also yields the following estimate: there exists a positive constant \( c_0 = c_0(B) \), such that

\[
  |D (1/\sqrt{s}) x_0| \leq c_0 \left( |D (1/\sqrt{s}) x| + 1 \right). \tag{3.22}
\]

Indeed, since \( s = \sigma_0 k_0 \), (3.10) yields

\[
  v_0 = D \left( \sqrt{s} \right) D \left( 1/\sqrt{k_0} \right) F_{k_0} \xi_0;
\]

therefore, from Lemma 3.1 (ii) and (3.20),

\[
  |D (1/\sqrt{s}) v_0| = \left| D \left( 1/\sqrt{k_0} \right) F_{k_0} \xi_0 \right| \leq \varepsilon \sqrt{k_0} \leq \varepsilon \left( |D (1/\sqrt{s}) x| + 1 \right). \tag{3.23}
\]

On the other hand, from (3.21) and (2.4), with \( \lambda = \sqrt{s} \) and \( t = 1 \), we obtain

\[
  D \left( 1/\sqrt{s} \right) x_0 = E(1) \left[ D \left( 1/\sqrt{s} \right) x - D \left( 1/\sqrt{s} \right) v_0 \right]. \tag{3.24}
\]

Since \( E(1) \) is a constant matrix, (3.22) follows directly from (3.23) and (3.24).

We now proceed by induction. Suppose that (3.17) and (3.19) hold for \( j \in \{0, 1, \ldots, r - 1\} \). Moreover assume that

\[
  |D (1/\sqrt{s}) x_j| \leq c_j \left( |D (1/\sqrt{s}) x| + 1 \right). \tag{3.25}
\]

with \( c_j = c_j(B) > 0 \). We want to show that (3.17) and (3.19) also hold for \( j + 1 \). Let

\[
  y = D \left( \sqrt{2^{j+1}/s} \right) (\pi_{j+1}(x_j)). \tag{3.26}
\]

From the induction hypothesis \( x_j \in W_j \), we get \( y \in W_j \cap V_{j+1} \). Hence, from Lemma 3.1 (iv), there exists \( \eta \in V_0 \) such that

\[
  G_{j+1} \eta - y \in W_{j+1}, \quad |\eta| \leq \overline{c}_{j+1} |y|. \tag{3.27}
\]
Let
\[ k_{j+1} = \min \left\{ m \in \mathbb{N} : m \geq \tilde{c}_{j+1}^2 c_{j+1}^2 2^{(j+1)(2\gamma+1)} \left( 1 + |D (1/\sqrt{s}) x_j|^{2} \right) \right\}, \]
(3.28)
\[ \sigma_{j+1} = s / (2^{j+1} k_{j+1}), \quad \xi_{j+1} = \eta / \sqrt{k_{j+1}}. \]
Then \( \xi_{j+1} \in V_0 \) and \( \sigma_{j+1} \in \left( 0, \omega \theta / (1 - \theta) \right] \). Moreover, from (3.26) and (3.27) we find
\[ |\xi_{j+1}|^2 = |\eta|^2 / k_{j+1} \leq (\tilde{c}_{j+1}^2 / k_{j+1}) |y|^2 \leq (\tilde{c}_{j+1}^2 / k_{j+1}) \left| D \left( 2^{(j+1)/2} \right) D (1/\sqrt{s}) x_j \right|^2 \]
\[ \leq (\tilde{c}_{j+1}^2 / k_{j+1}) 2^{(j+1)(2\gamma+1)} \left| D (1/\sqrt{s}) x_j \right|^2 \]
\[ \leq (\tilde{c}_{j+1}^2 / k_{j+1}) \tilde{e}_{j+1}^2 2^{(j+1)(2\gamma+1)} \left( 1 + |D (1/\sqrt{s}) x_j|^{2} \right)^2 \leq 1. \]
This proves conditions (3.17) together with the second and third equalities in (3.19).
Next, we prove that \( x_{j+1} \in W_{j+1} \). From Lemma 3.1 (i) and from (3.28) we obtain
\[ \pi_0 \left( D \left( 1/\sqrt{k_{j+1}} \right) F_{k_{j+1}} \xi_{j+1} \right) = D \left( 1/\sqrt{k_{j+1}} \right) \pi_0 \left( F_{k_{j+1}} \xi_{j+1} \right) \]
\[ = \sqrt{k_{j+1}} \xi_{j+1} = \eta, \]
or, equivalently
\[ D \left( 1/\sqrt{k_{j+1}} \right) F_{k_{j+1}} \xi_{j+1} - \eta \in W_0. \]
Then, from Lemma 3.1 (iii),
\[ G_{j+1} \left( D \left( 1/\sqrt{k_{j+1}} \right) F_{k_{j+1}} \xi_{j+1} - \eta \right) \in W_{j+1}, \]
which implies, together with (3.27),
\[ G_{j+1} D \left( 1/\sqrt{k_{j+1}} \right) F_{k_{j+1}} \xi_{j+1} - y \in W_{j+1}. \]
(3.29)
Relations (3.10), (3.26) and (3.29) give \( v_{j+1} - \pi_{j+1}(x_j) \in W_{j+1} \); therefore
\[ x_j - v_{j+1} = (x_j - \pi_{j+1}(x_j)) - (v_{j+1} - \pi_{j+1}(x_j)) \in W_{j+1}. \]
(3.30)
Finally, by (3.18)
\[ z_{j+1} = (x_{j+1}, -t_{j+1}) = (E(s)(x_j - v_{j+1}), -t_j + s). \]
(3.31)
Since \( E(s) \) is a lower triangular matrix, then (3.30) yields \( x_{j+1} \in W_{j+1} \). This proves (3.19), subject to the assumption (3.25).
It remains to prove (3.25) for \( x_{j+1} \). Since \( s = \sigma_{j+1} 2^{j+1} k_{j+1} \), by Lemma 3.1 (ii) we find
\[ |D (1/\sqrt{s}) v_{j+1}| = \left| D \left( 2^{-(j+1)/2} \right) G_{j+1} D \left( 1/\sqrt{k_{j+1}} \right) F_{k_{j+1}} \xi_{j+1} \right| \]
\[ \leq \left| D \left( 2^{-(j+1)/2} \right) \right| \left| G_{j+1} \right| \left| D \left( 1/\sqrt{k_{j+1}} \right) F_{k_{j+1}} \xi_{j+1} \right| \leq \left| G_{j+1} \right| \tilde{e} \sqrt{k_{j+1}} \]
\[ \leq \tilde{e} \left| G_{j+1} \right| \left[ 1 + \tilde{c}_{j+1} \tilde{c}_{j+1}^2 2^{(j+1)(2\gamma+1)/2} \left( 1 + |D (1/\sqrt{s}) x_j|^{2} \right) \right]. \]
(3.32)
On the other hand, using equation (2.4), we get from (3.31)

\[ D \left( 1/\sqrt{s} \right) x_{j+1} = E(1) \left[ D \left( 1/\sqrt{s} \right) x_j - D \left( 1/\sqrt{s} \right) v_{j+1} \right]. \]

Hence, by (3.32) and (3.25) (relative to \( x_j \)), we see that there exists a constant \( c_{j+1} = c_{j+1}(B) > 0 \) such that

\[ |D \left( 1/\sqrt{s} \right) x_{j+1}| \leq c_{j+1} \left( |D \left( 1/\sqrt{s} \right) x_j| + 1 \right). \]

This completes the proof of relations (3.17) and (3.19).

Note that equality (3.31) yields \( t_r = t - (r + 1)s = 0 \) and \( x_r \in W_r = \{0\} \). As a consequence \( z_r = (0,0) \), from which, together with (3.18), it results

\[ z = w_r \circ ... \circ w_1 \circ w_0. \]  

(3.33)

**Step B.** There exists a Harnack chain of length \( m_r = k_r2^r \) connecting the points

\[ (\tilde{z}, \tilde{z} \circ w_r) \]  

(3.34)

and, for any \( j \in \{0, ..., r - 1\} \), there exists a Harnack chain of length \( m_j = k_j2^j \) connecting the points

\[ (\tilde{z} \circ w_r \circ ... \circ w_{j+1}, \tilde{z} \circ w_r \circ ... \circ w_{j+1} \circ w_j). \]  

(3.35)

We prove only (3.35), the proof of (3.34) being similar. If we put

\[ \zeta = \zeta_j = (\sqrt{\sigma_j} \tilde{\zeta}_j, -\sigma_j), \quad \tilde{\zeta} = \tilde{\zeta}_j = (-\sqrt{\sigma_j} \tilde{\zeta}_j, -\sigma_j), \]  

then, since \( w_j = w_j(\xi_j, \sigma_j, k_j) \), we get from (3.5) and (3.6) that

\[ w_j = \zeta^{(1)} \circ \zeta^{(2)} \circ ... \circ \zeta^{(k_j2^j)}, \]

with either \( \zeta^{(i)} = \zeta \) or \( \zeta^{(i)} = \tilde{\zeta} \). It is now easy to show that

\[ K = \left\{ \tilde{z} \circ w_r \circ ... \circ w_{j+1}, \tilde{z} \circ w_r \circ ... \circ w_{j+1} \circ \zeta^{(1)}, ..., \tilde{z} \circ w_r \circ ... \circ w_{j+1} \circ \zeta^{(1)}, ..., \zeta^{(k_j2^j)} \right\} \]

is a Harnack chain of length \( m_j = k_j2^j \) for the pair (3.35).

Indeed, let \( z' = (x', t') \in K, z' \neq \tilde{z} \circ z \), and let \( z'' = (x'', t'') \) be the element following \( z' \) in \( K \). Then either \( z'' = z' \circ \zeta \) or \( z'' = z' \circ \tilde{\zeta} \), with \( \zeta \) and \( \tilde{\zeta} \) defined in (3.36). To show that \( K \) is a Harnack chain it is sufficient to apply Theorem C with \( z_0 = z' \) and \( z = z'' \). To this end it is enough to show that

\[ \delta < r_0, \quad H_\delta(z') \subset S_T, \quad z'' \in H_{\delta\theta}(z'), \]  

(3.37)

where \( \delta = \sigma_j/\theta \). The inequality \( \delta < r_0 \) is a consequence of the second relation in (3.17). On the other hand, since \( T_0 < \tilde{t} + t \leq t'' < t' \leq \tilde{t} < T_1 \), we have

\[ t' - \delta = (t' - \theta \delta) - (1 - \theta)\delta = t'' - (1 - \theta)\delta \geq T_0 + (1 - \theta)\delta \geq T_0 - \omega, \]
and thus $H_\delta(z') \subset S_I$. The last relation in (3.37) follows directly from the definition of $H_{\delta_0}(z')$ and from (3.36) (since $|\xi_j| \leq 1$). This proves that $K$ is a Harnack chain.

**Step C.** The existence of a Harnack chain of length

$$m = \sum_{j=0}^{r} m_j = \sum_{j=0}^{r} k_j 2^j,$$

connecting the pair $(\tilde{z}, \tilde{z} \circ z)$ is a straightforward consequence of Step A, Step B and (3.33).

Moreover the inequalities (3.20) and (3.28) lead to the estimate

$$m \leq c' \left( |D (1/\sqrt{s}) x|^2 + 1 \right),$$

where $c' = c'(B)$ is a positive constant. Since $t = (r + 1)s$ in (3.16), we also have

$$|D (1/\sqrt{s}) x|^2 \leq (r + 1)^{r+1} \left| D \left( \frac{1}{\sqrt{t}} \right) x \right|^2,$$

from which, substituting (3.38) and (3.39) in (3.2), we get

$$u(\tilde{z} \circ z) \leq c'(r+1)^{r+1} \left( |D(1/\sqrt{t})x|^2 + 1 \right) u(\tilde{z}).$$

Under the additional hypothesis (3.15), we obtain (3.1) by setting

$$a = c', \quad b = c'(r + 1)^{r+1} \log(c), \quad b' = 0.$$

**Step D.** We next remove hypothesis (3.15). Thus, suppose that $z = (x, -t)$ satisfies

$$t > \omega \theta \frac{r+1}{1 - \theta}.$$

Define $\bar{z} = (\bar{x}, -\bar{t}), \hat{z} = (\hat{x}, -\hat{t})$ where

$$\bar{x} = x, \quad \bar{t} = \omega \theta \frac{r+1}{1 - \theta},$$

$$\hat{x} = 0, \quad \hat{t} = t - \bar{t}.$$

Clearly $\hat{t} > 0$ and $\bar{z} \circ z = (\bar{z} \circ \hat{z}) \circ z$. Since $\bar{z} \circ \hat{z} \in S_I$ and $\bar{z}$ verifies (3.15), using the result of Step C we obtain

$$u(\bar{z} \circ z) \leq a \exp \left( b \left| D \left( \frac{1}{\sqrt{\bar{t}}} \right) \bar{x} \right|^2 \right) u(\bar{z} \circ \hat{z}).$$

We next construct a Harnack chain connecting the pair $(\bar{z}, \bar{z} \circ \hat{z})$. Let

$$\hat{k} = \min \left\{ m \in \mathbb{N}: m \omega \theta \geq \hat{t}(1 - \theta) \right\}, \quad \sigma = \hat{t}/\hat{k}, \quad \hat{\zeta} = (0, -\sigma),$$

13
and, for any \( j = 1, \ldots, k - 1 \), define
\[
\hat{\zeta}^1 = \zeta, \quad \hat{\zeta}^{j+1} = \hat{\zeta}^j \circ \hat{\zeta}.
\]
Clearly \( \hat{\sigma} \in \left( 0, \omega \theta/(1 - \theta) \right) \). Then, proceeding as above, we easily prove that
\[
\hat{K} = \left\{ \tilde{z}, \tilde{z} \circ \hat{\zeta}, \ldots, \tilde{z} \circ \hat{\zeta}^k = \tilde{z} \circ \hat{\zeta} \right\}
\]
is a Harnack chain of length
\[
\hat{m} = k \leq 1 + \hat{t}(1 - \theta)/(\theta \omega).
\]
From (3.2) we get
\[
u(\tilde{z} \circ \hat{\zeta}) \leq c^{1 + \hat{t}(1 - \theta)/(\theta \omega)} u(\tilde{z}),
\]
so that, by (3.40),
\[
u(\tilde{z} \circ z) \leq a c^{1 + \hat{t}(1 - \theta)/(\theta \omega)} \exp \left( b \left| D \left( \tilde{t}^{-1/2} \right) \tilde{x} \right|^2 \right) u(\tilde{z}). \tag{3.41}
\]
Thus relation (3.2) follows from (3.41), since \( \left| D \left( \tilde{t}^{-1/2} \right) x \right| \leq \left| D \left( t^{-1/2} \right) x \right| \) and \( 0 < \hat{t} < t \), provided we now choose
\[
a = c\hat{t} + 1, \quad b = c'(r + 1)^{r+1} \log(c), \quad b' = \log(c)(1 - \theta)/\theta.
\]
This completes the proof of Proposition 3.1.

4. A global bound for the fundamental solution

The aim of this section is to prove our main theorem, namely that
\[
c^{-\Gamma^{-}}(z; \zeta) \leq \Gamma(z; \zeta).
\]

**Lemma 4.1.** For every \( \xi \in \mathbb{R}^N \), \( \delta, s > 0 \), let
\[
P(\delta, s) = \left\{ x \in \mathbb{R}^N : \left| D \left( s^{-1/2} \right) x \right| \leq \delta \right\}.
\]
Then there exist three positive constants \( \delta, \gamma, s_0 \), depending only on the operator \( L \), such that
\[
\int_{P(\delta, s)} \Gamma(((\xi, \tau) \circ (x, s); \xi, \tau) dx \geq \gamma \tag{4.1}
\]
for every \( \xi = (\xi, \tau) \in \mathbb{R}^{N+1} \) and for every \( s \in (0, s_0] \).

**Proof.** Let \( \tilde{\Gamma} \) be the fundamental solution of
\[
\tilde{L} = \Lambda^{-1} \Delta_{p_0} + \langle x, BD \rangle - \partial_t,
\]
where \( \Lambda \) is the constant in H1. From the explicit expressions for \( \tilde{\Gamma}(z; \zeta) \) and \( Z(z; \zeta) \), we see that there exists a positive constant \( \bar{c} \) such that
\[
\bar{c} \tilde{\Gamma}(z, \zeta) \leq Z(z, \zeta) \quad \forall z, \zeta \in \mathbb{R}^{N+1}.
\] (4.2)

(A detailed proof of this result can be found in [7, Proposition 2.4]). Then, by Theorem B with \( \varepsilon = 1/2 \), there exists \( \bar{s} > 0 \) such that for every \( s \in (0, \bar{s}] \),
\[
\frac{1}{2} Z(z, \zeta) \leq \Gamma(z, \zeta) \quad \forall z, \zeta \in \mathbb{R}^{N+1} \text{ such that } \Gamma(z; \zeta) \geq 1/s.
\] (4.3)

Hence, for any \( s \in (0, \bar{s}] \) and for every \( \zeta \in \mathbb{R}^{N+1} \),
\[
\left\{ \zeta \in \mathbb{R}^{N+1} : \tilde{\Gamma}(z; \zeta) \geq 2/(\bar{c}s) \right\} \subset \left\{ \zeta \in \mathbb{R}^{N+1} : \Gamma(z; \zeta) \geq 1/s \right\}.
\]

On the other hand, again using the expression (2.5) for \( \tilde{\Gamma} \), it is easy to show that there exist two positive constants \( \delta, s_0 \) such that
\[
\left\{ z \in \mathbb{R}^{N+1} : z = \zeta \circ (x, s), x \in P(\delta, s) \right\} \subset \left\{ z \in \mathbb{R}^{N+1} : \tilde{\Gamma}(z; \zeta) \geq 2/(\bar{c}s) \right\}
\]
for every \( s \in (0, s_0] \) and for every \( \zeta \in \mathbb{R}^{N+1} \). Therefore, for any \( s \in (0, s_0] \), the relations (4.2) and (4.3) yield
\[
\int_{P(\delta, s)} \Gamma(\xi, \tau) \circ (x, s) dx \geq \frac{\bar{c}}{2} \int_{P(\delta, s)} \tilde{\Gamma}(\xi, \tau) \circ (x, s) dx
\]
\[
= \frac{\bar{c}}{2} \int_{P(\delta, 1)} \tilde{\Gamma}(y, 1) dy.
\]

This proves (4.1), with \( \gamma = \frac{\bar{c}}{2} \int_{P(\delta, 1)} \tilde{\Gamma}(y, 1) dy \).

**Proof of the Main Theorem.** Let \( T > 0 \), and let \( z = (x, t) \), \( \zeta = (\xi, \tau) \in \mathbb{R}^{N+1} \) be such that \( 0 < t - \tau < T \). Furthermore, let \( \hat{z} = (\hat{x}, \hat{t}) = \zeta^{-1} \circ z \) and set \( s = \min\{\frac{t}{3}, s_0\} \). Note that \( \hat{t} = t - \tau > 0 \), from which we obtain \( s > 0 \). By the mean value theorem, there exists \( y \in P(\delta, s) \) such that
\[
\Gamma(\zeta \circ (y, s), \zeta) = \frac{1}{m(\delta)s^{d/2}} \int_{P(\delta, s)} \Gamma(\zeta \circ (x, s), \zeta) dx,
\]
where \( m(\delta) \) is the measure of the set \( P(\delta, 1) \).

Since \( 0 < s \leq s_0 \) and \( y \in P(\delta, s) \), we see from Lemma 4.1 that
\[
\Gamma(\zeta \circ (y, s), \zeta) \geq \frac{\gamma}{m(\delta)s^{-d/2}}.
\] (4.4)
Let $I' = (\tau, \tau + T)$ and $I = (\tau + \omega, \tau + T)$, where $\omega = \min \{s/2, (1 - \theta) r_0\}$. Applying Proposition 3.1 to the function $u = \Gamma(\cdot; \zeta)$, with $\tilde{z} = \zeta \circ (0, 2s)$ and $z = (y, s)$, we get
\[
\Gamma(\zeta \circ (y, s); \zeta) \leq a \exp \left( b \left| D \left( s^{-1/2} \right) y \right|^2 + b' s / \omega \right) \Gamma(\zeta \circ (0, 2s); \zeta). \tag{4.5}
\]
Having chosen $s$ and $\omega$ as above, we now have
\[
\frac{b' s}{\omega} = \max \left\{ 2b', \frac{b' s}{(1 - \theta) r_0} \right\} \leq \max \left\{ 2b', \frac{b' s_0}{(1 - \theta) r_0} \right\} \overset{\text{def}}{=} c. \tag{4.6}
\]
Since $y \in P(\delta, s)$, the inequality (4.5) then gives
\[
\Gamma(\zeta \circ (y, s); \zeta) \leq a e^{c + b' s} \Gamma(\zeta \circ (0, 2s); \zeta). \tag{4.7}
\]
The inequalities (4.7) and (4.4) ensure that there exists $c' > 0$, depending only on $L$, such that
\[
\Gamma(\zeta \circ (0, 2s); \zeta) \geq c' s^{-Q/2}. \tag{4.8}
\]
Once more applying Proposition 3.1 to the function $u = \Gamma(\cdot; \zeta)$, with $\tilde{z} = \zeta \circ \tilde{z}$ and $z = (-E(2s - \ell)\hat{x}, 2s - \ell)$, yields
\[
\Gamma(\zeta \circ \tilde{z} \circ z; \zeta) = \Gamma(\zeta \circ (0, 2s); \zeta) \leq a \exp \left( b \left| D \left( (\ell - 2s)^{-1/2} \right) E(2s - \ell) \hat{x} \right|^2 + b' (\ell - 2s) / \omega \right) \Gamma(\zeta \circ \tilde{z}; \zeta). \tag{4.9}
\]
Using the second identity in (2.4), we have
\[
\left| D \left( (\ell - 2s)^{-1/2} \right) E(2s - \ell) \hat{x} \right| = \left| E(-1) D \left( (\ell - 2s)^{-1/2} \right) \hat{x} \right|. \tag{4.10}
\]
Then, since $\ell/3 \leq \ell - 2s \leq \ell$, there exists a positive constant $\overline{c} = \overline{c}(B)$ such that
\[
\left| D \left( (\ell - 2s)^{-1/2} \right) E(2s - \ell) \hat{x} \right| \leq \overline{c} \left| D \left( \ell^{-1/2} \right) \hat{x} \right|. \tag{4.10}
\]
On the other hand, from (4.6),
\[
\frac{b' (\ell - 2s)}{\omega} = \frac{b' s (\ell - 2s)}{\omega s} \leq c \frac{(\ell - 2s)}{s} \leq c \left( 1 + \frac{\ell}{s_0} \right). \tag{4.11}
\]
By substituting (4.10) and (4.11) in (4.9), we then find
\[
\Gamma(\zeta \circ (0, 2s); \zeta) \leq a \exp \left( b \overline{c}^2 \left| D \left( \ell^{-1/2} \right) \hat{x} \right|^2 + c \left( 1 + \ell / s_0 \right) \right) \Gamma(\zeta \circ \tilde{z}; \zeta). \tag{4.12}
\]
Finally, since $s \leq \ell$, it follows from (4.8) and (4.12) that
\[
\Gamma(\zeta \circ \tilde{z}; \zeta) \geq \frac{c'}{a} e^{-c(1 + \ell / s_0)} \ell^{-Q/2} \exp \left( -b \overline{c}^2 \left| D \left( \ell^{-1/2} \right) \hat{x} \right|^2 \right). \tag{4.13}
\]
Thus there exist two positive constants $k'$, depending on $L$ only, and $k''$, depending on $L$ and $T$, such that

$$\Gamma (\zeta \circ \hat{z}; \zeta) \geq k'' \hat{t}^{-Q/2} \exp \left( -k' \left| D \left( \hat{t}^{-1/2} \hat{x} \right) \right|^2 \right).$$

(4.13)

Using (4.13) we are now able to determine a positive constant $\lambda$ such that the fundamental solution of the operator

$$L_\lambda = \lambda \Delta_{p_0} + \langle x, BD \rangle - \partial_t$$

satisfies (1.6). Indeed, the expression (2.5) for $\Gamma_\lambda$ gives

$$\Gamma_\lambda (\zeta \circ \hat{z}; \zeta) = \Gamma_\lambda (\hat{z}; (0, 0))$$

$$= c_N \lambda^{-N/2} \hat{t}^{-Q/2} \exp \left( - \left( 1/(4\lambda) \right) \left( C^{-1} D \left( \hat{t}^{-1/2} \hat{x} \right), D \left( \hat{t}^{-1/2} \hat{x} \right) \right) \right),$$

where $C = C(1)$ is the matrix (2.3) corresponding to $\lambda = 1$. Moreover, there exists $\lambda > 0$ such that

$$\langle C^{-1} \eta, \eta \rangle \geq 4k' \lambda |\eta|^2$$

for every $\eta \in \mathbb{R}^N$; therefore

$$\Gamma_\lambda (\zeta \circ \hat{z}; \zeta) \leq c_N \lambda^{-N/2} \hat{t}^{-Q/2} \exp \left( -k' \left| D \left( \hat{t}^{-1/2} \hat{x} \right) \right|^2 \right).$$

(4.14)

Since $z = \zeta \circ \hat{z}$, the inequality (1.6) follows from (4.13) and (4.14) by letting $c^* = c_N \lambda^{-N/2}/k''$ and $\Gamma^* = \Gamma_\lambda$.

**Appendix: proof of Lemma 3.1**

**Lemma A1.** For $j, k \in \mathbb{N} \cup \{0\}$, let

$$\gamma_{k,j} = \sum_{l=0}^{2j-1} \beta_l l^k,$$

(A1)

where $(\beta_l)_{l \in \mathbb{N}}$ is the sequence defined in (3.8), (3.9). Then for every $k \in \mathbb{N} \cup \{0\}$ we have

$$\gamma_{k,k} \neq 0, \quad \gamma_{k,k+m} = 0 \quad \forall m \in \mathbb{N}.$$

(A2)

**Proof.** We proceed by induction on $k$. If $k = 0$ then (A2) is a trivial consequence of (3.8), since $\gamma_{0,0} = \beta_0 = 1$. Moreover $\gamma_{0,1} = \beta_0 + \beta_1 = 0$ and also, if $m > 1$, by (3.9) we find

$$\gamma_{0,m} = \sum_{l=0}^{2m-1} \beta_l = \sum_{l=0}^{2m-1-1} (\beta_l + \beta_{l+2m-1}) = 0.$$
Suppose that (A2) holds for any \( j = 0, \ldots, k \). Then, by (3.9), we see that for every \( m \in \mathbb{N} \cup \{0\} \)

\[
\gamma_{k+1,k+1+m} = \sum_{t=0}^{2^{k+1+m}-1} \beta_t l^{k+1} = \sum_{t=0}^{2^{k+m}-1} \beta_t \left[ l^{k+1} - (l + 2^k + m)^{k+1} \right]
\]

\[
= \sum_{t=0}^{2^{k+m}-1} \beta_t \left[ l^{k+1} - \sum_{j=0}^{k+1} \binom{k+1}{j} l^j (2^k+m)^{k+1-j} \right]
\]

\[
= - \sum_{j=0}^{k} \binom{k+1}{j} (2^k+m)^{k+1-j} \sum_{l=0}^{2^{k+m}-1} \beta_l l^j
\]

\[
= - \sum_{j=0}^{k} \binom{k+1}{j} (2^k+m)^{k+1-j} \gamma_{j,k+m}. \tag{A3}
\]

(if \( k = 0 \) and \( m = 0 \) equality (A3) is given by (3.8)).

If in the last term of (A3) we have \( k + m > j \) then, by our induction hypothesis there results \( \gamma_{k+1,k+1+m} = 0 \). On the other hand, if \( m = 0 \), then \( \gamma_{k+1,k+1} = -(k+1)2^k \gamma_{k,k} \neq 0 \) since from the induction hypothesis we have \( \gamma_{j,k} = 0 \) for \( j < k \) and \( \gamma_{k,k} \neq 0 \). This proves the claim for \( k + 1 \).

**Proof of Lemma 3.1.** In the sequel, for every \( N \times N \) matrix \( M \) we shall use the decomposition \( M = (M_{i,j})_{i,j=0, \ldots, r} \), where \( M_{i,j} \) is a \( p_i \times p_j \) block matrix.

Since \( D(\lambda) \) has the form (1.4), the second identity (2.4) can be written

\[
(E(\lambda^2 t))_{i,j} = \lambda^{2(j-i)} (E(t))_{i,j} \quad \text{for } i, j = 0, \ldots, r; \tag{A4}
\]

and for every \( \lambda > 0, t \in \mathbb{R} \). Consider the vector \( v = (v_0, v_1, \ldots, v_r) \in \mathbb{R}^N \), with \( v_j \in \mathbb{R}^{p_j} \). Then

\[
v \in V_m \iff v_j = 0 \text{ for } j = m+1, \ldots, r
\]

\[
v \in W_m \iff v_j = 0 \text{ for } j = 0, 1, \ldots, m. \tag{A5}
\]

(i) Using the identity (A4), we can write the matrix \( F_k \) in (3.4) as

\[
(F_k)_{i,j} = \sum_{h=0}^{k-1} (E(-h))_{i,j} = \left( \sum_{h=0}^{k-1} h^{i-j} \right) (E(-1))_{i,j}. \tag{A6}
\]

Hence \( (F_k)_{0,0} = k (E(-1))_{0,0} = k I_{p_0} \), and \( \xi \in V_0 \) implies

\[
(F_k \xi)_0 = k \xi_0,
\]

18
which is equivalent to (3.11).

(ii) We first note that
\[
\sum_{h=0}^{k-1} t^m \leq \int_0^k t^m dt = \frac{k^{m+1}}{m+1}
\]
for every \( m \in \mathbb{N} \cup \{0\} \). Hence, using (A6), we get
\[
\left| D \left( k^{-1/2} F_k \xi \right) \right| = k^{-2j+1/2} \left| (F_k \xi)_j \right| \leq \sqrt{k} \left| (E(-1) \xi)_j \right| / (j + 1)
\]
for every \( \xi \in V_0 \) and for \( j = 0, 1, ..., r \). Thus (3.12) holds, with \( \bar{c} = \|E(-1)\| \).

(iii) Let \( j, m \in \{0, 1, ..., r\} \) and \( y \in W_m \). Then (A5) gives
\[
y_j = 0 \quad \text{for} \quad j = 0, 1, ..., m.
\]
For any \( i \geq n \), by using (A4) and (A1) we can write the blocks of \( G_j \) in (3.7) in the form
\[
(G_j)_{i,n} = (G_j(-1))_{i,n} = \sum_{l=0}^{2^j-1} (\beta l E(-l))_{i,n}
\]
\[
\left( \sum_{l=0}^{2^j-1} \beta l \right) (E(-1))_{i,n} = \gamma_{i-n,j} (E(-1))_{i,n}.
\]
(A7)
Since \( E(t) \) is a lower triangular matrix, it is clear that \( G_j(s) \) is also lower triangular. Then \( (G_j)_{i,n} = 0 \) if \( i < n \), and therefore
\[
(G_j y)_i = \sum_{n=0}^{i} (G_j)_{i,n} y_n = \sum_{n=0}^{i} \gamma_{i-n,j} (E(-1))_{i,n} y_n.
\]
On the other hand, if \( y \in W_m \) then \( y_n = 0 \) for \( n \leq m \), from which we obtain
\[
(G_j y)_i = \sum_{n=0}^{i} \gamma_{i-n,j} (E(-1))_{i,n} y_n.
\]
(A8)
Moreover, if \( i \leq m + j \) and \( m \leq n \leq i \), then \( i-n < j \) and, by Lemma A1, \( \gamma_{i-n,j} = 0 \). Hence (A8) gives
\[
(G_j y)_i = 0 \quad \text{for} \quad i \leq m + j \quad (\text{if} \quad y \in W_m).
\]
This proves the first statement in (3.13). The second claim can be proved similarly.

(iv) If \( j = 0 \) then (3.14) is trivially true. Suppose then that \( j \in \{1, ..., r\} \) and let \( y \in W_{j-1} \cap V_j \). By (A5) we have \( y_k = 0 \) for \( k \neq j \). From the second relation in (3.13) we obtain
\[
G_j \eta \in W_{j-1} \quad \forall \eta \in V_0.
\]
Thus in order to prove (3.14) it is sufficient to solve the relation
\[(G_j \eta)_j = y_j \quad \text{with} \quad \eta \in V_0, \quad |\eta| \leq \tilde{c}_j |y|.
\] (A9)

Note that if \( \eta \in V_0 \), then (A7) gives
\[(G_j \eta)_j = \gamma_{j,j} (E(-1))_{j,0} \eta_0.
\] (A10)

By Lemma A1 we get \( \gamma_{j,j} \neq 0 \), while it can be shown directly that
\[(E(-1))_{j,0} = B^T_J B^T_{j-1} ... B^T_1.
\]

From H1 the matrix \((E(-1))_{j,0}\) has maximum rank. Thus the map
\[\mathcal{E}_j : \mathbb{R}^{p_0} \rightarrow \mathbb{R}^{p_j}, \quad \mathcal{E}_j : \xi \mapsto (E(-1))_{j,0} \xi\]
is surjective, since \( p_0 \geq p_j \). Hence there exists a unique \( \tilde{\xi} \in \mathbb{R}^{p_0} \) for which
\[\tilde{\xi} \in \left( \text{Ker} \mathcal{E}_j \right)^\perp, \quad (E(-1))_{j,0} \tilde{\xi} = y_j / \gamma_{j,j}.
\]
Moreover there exists a positive constant \( \tilde{c}_j = \tilde{c}_j(B) \) such that \( |\tilde{\xi}| \leq \tilde{c}_j |y| \). Finally from (A10) it is clear that \( \eta = (\tilde{\xi}, 0) \) solves (A9). This gives (3.14) for \( j > 0 \), completing the proof of Lemma 3.1.

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References


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20