Rearrangements and the Image of a Measure

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Abstract

We tie the notion of rearrangement with that of image of a measure; for not necessary bounded measures, we give necessary and sufficient conditions for the existence of increasing and decreasing rearrangements, of spherical rearrangements and of rearrangements according to a real function f.

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Riassunto

Si lega la nozione di riordinamento con quella di immagine di una misura; per misure non necessariamente limitate, si danno condizioni necessarie e sufficienti per l'esistenza di riordinamenti crescenti e decrescenti, di riordinamenti sferici e di riordinamenti secondo una funzione reale f.

Introduction

Starting from the works of many authors, as A. Alvino, C. Bandle, K. M. Chong, P. W. Day, P. L. Lions, S. Matarasso, J. Mossino, N. M. Rice, G. Talenti, G. Trombetti (mentioning only someone) the notion of rearrangement of a function and related applications have been developed very much and in many directions.

In spite of the large amount of works, it seems that the elementary tie between rearrangements and the image of a measure has not been sufficiently displayed. From this it follows in particular that the notion of distribution function is significant not for all measurable functions, but only for someone, which we have called distributional functions. Really this aspect of existence of the distribution function (or equivalently, as we shall see, of an increasing or decreasing rearrangement) has been investigated by P. W. Day [5] and by K. M. Chang-N. M. Rice [3] from a different point of view; in particular P. W. Day [5] has found the condition for the existence of the distribution function.

In this work we purpose

- to show that the notion of image of a measure is the basis of the notion of rearrangement,
- to supply the tools of the development of the theory,
- to study the problem of existence of increasing and decreasing rearrangements and of increasing and decreasing spherical rearrangements,
- to state precisely the notion of rearrangement according to a real function f,
- to study the problem of existence of increasing and decreasing rearrangements according to a real function f.

Using the notion of image of a measure, we consider the following **general definition of rearrangement** (definitions 2.2.1, 2.2.2).

Let (X, S, μ), (Y, T, ν) measure spaces; let I a closed interval of $\overline{\mathbf{R}}$; let $u : X \longrightarrow I$ and $v : Y \longrightarrow I$ measurable functions; we say that u is a rearrangements of v if

$$u(\mu) = v(\mu) \ .$$

If Y is an interval of $\overline{\mathbf{R}}$ and if v is increasing (resp. decreasing), we obtain the definition of increasing (resp. decreasing) rearrangement. If Y is a ball of $\mathbf{R}^n$ with centre 0 or $\mathbf{R}^n$ and if v satisfies $|x| \leq |y| \Rightarrow v(x) \leq v(y)$ (resp. $|x| \leq |y| \Rightarrow v(x) \geq v(y)$) we obtain the definition of increasing (resp. decreasing) spherical rearrangement.

With this process many theorems of the classical expositions of the theory of the rearrangements get a more true meaning (see [8] chapter 1, theorem 1.1, for instance): this may be a check of the exactness of the process.

Among the **tools** of the theory we recall:

- The notion of monotone inverse function (definition 1.2.1).

  Let $f : [a, b] \longrightarrow [c, d]$ increasing; let $g : [c, d] \longrightarrow [a, b]$ increasing; we say that g is monotone inverse of f if for all $x \in [a, b]$ and for all $y \in [c, d]$ we have

  $$(f(x) < y \Rightarrow x \leq g(y)) \text{ and } (y < f(x) \Rightarrow g(y) \leq x) \ .$$

- The notion of distributional derivative of a monotone function (definition 3.1.1).

  Let $f : [a, b] \longrightarrow [c, d]$ increasing; the distributional derivative $D_{\text{meas}} f$ of f is the only positive measure on the σ-algebra of the borelian, $\mathcal{B}_{[a,b]}$, such that for all $x, x' \in [a, b]$, $x \leq x'$,

  $$(D_{\text{meas}} f)([x, x']) = f(x'+) - f(x-) \ .$$

  It is interesting the tie between the distributional derivative and the image of a measure (theorem 3.3.1): the distributional derivative of f is the image of the Lebesgue measure on $[c, d]$ by a monotone inverse of f, g, i. e.

  $$D_{\text{meas}} f = g(\lambda) \ .$$

- The notion of exact measure (definition 4.1.1).

  We consider a positive measures $\nu : \mathcal{B}_{[a,b]} \longrightarrow \overline{\mathbf{R}}$; we say that ν is an exact measure if there exists a closed interval of $\overline{\mathbf{R}}$, J, and $f : [a,b] \longrightarrow J$ increasing such that

$$D_{\text{meas}} f = \nu \ .$$

- The notion of distributional function (definition 5.1.1).

  Let (X, S, μ) a measure space; let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a,b]$ measurable; we say that u is a distributional function if the image measure $u(\mu)$ is an exact measure.

- The notion of increasing distribution function (definition 5.2.1).

  For a distributional function $u : X \longrightarrow [a,b]$ the increasing distribution function of u is a function $m : [a,b] \longrightarrow [c,d]$ increasing, such that

$$D_{\text{meas}} m = u(\mu) \ .$$

- The notion of left convergent distributional function (definitions 4.4.1, 5.3.1).

  We say that $u : X \longrightarrow [a,b]$ is left convergent distributional if there exists an increasing distribution function of u, $m : [a,b] \longrightarrow [c,d]$, with $c \in \mathbf{R}$.

As regards the problem of the **existence of rearrangements**, if one supposes that the measure μ is bounded, than there always exists (in obvious hypotheses) the viewed rearrangements; on the contrary, we do not consider only bounded measures μ; so problems of existence of the rearrangements arise.

- Let us consider the increasing and decreasing rearrangements: we have the following result on the tie among increasing rearrangement, distribution function and monotone inverse (theorem 6.2.1).

  Let (X, S, μ) a measure space; let $u : X \longrightarrow [a,b]$ measurable; let $m : [a,b] \longrightarrow [c,d]$ increasing; let $r : [c,d] \longrightarrow [a,b]$ a monotone inverse of m; then r is an increasing rearrangement of u if and only if m is an increasing distribution function of u.

  So there exists an increasing rearrangement of u if and only if u is distributional.

  So we find again the condition of P. W. Day [5](theorems 6.2.2, 5.1.1):

  Let $a_1 = \text{ess.inf}(u)$ and $b_1 = \text{ess.sup}(u)$; then there exists an increasing rearrangement of u if and only if u satisfies the three conditions

  1. $(\forall x, x' \in]a_1, b_1[\, , x \leq x')\ \mu(u^{-1}([x, x'])) < +\infty$;
  2. $\mu(u^{-1}(\{b_1\})) = 0$ or $(\forall x \in]a_1, b_1[)\ \mu(u^{-1}([x, b_1[)) < +\infty$;
  3. $\mu(u^{-1}(\{a_1\})) = 0$ or $(\forall x \in]a_1, b_1[)\ \mu(u^{-1}(]a_1, x])) < +\infty$.

- Let us consider the increasing spherical rearrangements. We find that there exists an increasing spherical rearrangement of u if and only if u is left convergent distributional (theorem 7.2.3) .

  If $b_1 = \text{ess.sup}(u)$ this means that u satisfies the two conditions (theorem 5.3.1):

  1. $(\forall x \in [a, b_1[)\ \mu(u^{-1}([a, x])) < +\infty$;
  2. $\mu(u^{-1}(\{b_1\})) = 0$ or $\mu(u^{-1}([a, b_1[)) < +\infty$.

Following G. Talenti (see [12]), we consider also the **rearrangements according to a real function** f. Roughly speaking, this means that as for the spherical rearrangements the rearranged function increases (or decreases) with the hypersurfaces $|x| = t$, so for the rearrangements according to f the rearranged function increases (or decreases) with the hypersurfaces $f(x) = t$.

- Evidently for the consideration of such rearrangements we need some condition on f; we call these functions f, rearranging functions (definition 8.1.1).

  Let (Y, T, ν) a measure space; let $f : Y \longrightarrow [\alpha, \beta]$ measurable; we say that f is a rearranging function if f is distributional and if

$$(\forall t \in [\alpha, \beta])\, \nu(f^{-1}(\{t\})) = 0\ .$$

- We give the definition of rearrangement according to a rearranging function f (definition 8.1.2):

  Let (X, S, μ), (Y, T, ν) measure spaces; let $u : X \longrightarrow [a, b]$ μ-measurable; let $v : Y \longrightarrow [a, b]$ ν-measurable; let $f : Y \longrightarrow [\alpha, \beta]$ ν-measurable; let f a rearranging function; we say that v is an increasing rearrangement of u according to f if

$$(\forall y, y' \in Y)\, (f(y) \leq f(y') \Rightarrow v(y) \leq v(y'))$$

  and if v is a rearrangement of u.

  Following [12] a rearrangement v according to f may be obtained as

$$v = r \circ \tau_K \circ m \circ f\ ;$$

  where $m : [\alpha, \beta] \longrightarrow [\gamma, \delta]$ is an increasing distribution function of f, $r : [a, b] \longrightarrow [c, d]$ is an increasing rearrangement of u, and $\tau_K : [\gamma, \delta] \longrightarrow [c, d], x \longrightarrow x + K$ is bijective (theorem 8.1.1).

Supposing $\mu(X) = \nu(Y)$, the **existence of an increasing rearrangement of** u **according to** f is tied to the existence of the bijective translation τ_K; if $\mu(X) < +\infty$ there exists τ_K and consequently an increasing rearrangement of u according to f (theorem 8.2.4); if $\mu(X) = \mu(Y) = +\infty$, $[\gamma, \delta]$ and $[c, d]$ must be $[-\infty, +\infty]$ or of the type $[M, +\infty]$, with $M \in \mathbf{R}$, of the type $[-\infty, N]$, with $N \in \mathbf{R}$; we prove that there exists an increasing rearrangement of u according to f if and only if we can choose $[\gamma, \delta]$ and $[c, d]$ of the same type (theorems 8.2.1, 8.2.2, 8.2.3).

Although in the applications mainly decreasing rearrangements are used, for sake of logical simplicity we have privileged in the exposition the increasing rearrangements; however all the results are translated for decreasing rearrangements.

1 Monotone functions on closed intervals of $\overline{\mathbf{R}}$

1.1 Monotonically equivalent functions

Let us consider monotone functions $f : I \longrightarrow J$, with I,J closed intervals of $\overline{\mathbf{R}}$. The essential hypothesis is that the intervals are closed; this permits to consider the least upper bound and the greatest lower bound of every subset A of I or J, even if $A = \emptyset$.

Let $a, b, c, d \in \overline{\mathbf{R}}$; let $a \leq b$ and $c \leq d$; let $f : [a, b] \longrightarrow [c, d]$; let f increasing; if $x \in [a, b]$, we put

$$f(x+) = \inf(\{f(t); t \in [a, b], t > x\})$$

and

$$f(x-) = \sup(\{f(t); t \in [a, b], t < x\}) \ ,$$

where the least upper bound and the greatest lower bound are done respect the ordered set $[c, d]$. We observe explicitly that if $x = b$, we have $f(b+) = \inf(\emptyset) = d$ and if $x = a$, we have $f(a-) = \sup(\emptyset) = c$; if $x < b$ we have $f(x+) = \lim_{t \to x, t > x} f(t)$, while if $x > a$ we have $f(x-) = \lim_{t \to x, t < x} f(t)$.

It follows at once that f is continuous and surjective if and only if $(\forall x \in [a, b])$ $f(x+) = f(x-)$.

Analogically, if f is decreasing, we put

$$f(x+) = \sup(\{f(t); t \in [a, b], t > x\})$$

and

$$f(x-) = \inf(\{f(t); t \in [a, b], t < x\}) \ .$$

Definition 1.1.1 *Let $a, b, c, d \in \overline{\mathbf{R}}$; let $a \leq b$ and $c \leq d$; let $f, g : [a, b] \longrightarrow [c, d]$; let f, g increasing (resp. decreasing); we say that f is monotonically equivalent to g if*

$$(\forall x \in [a, b])\, f(x+) = g(x+) \ \text{and}\ f(x-) = g(x-) \, .$$

We note by $f \sim_m g$ the equivalence relation on the set of the increasing (resp. decreasing) functions from $[a, b]$ to $[c, d]$ so defined; we note by $[f]_m$ the equivalent class of a function f.

$[f]_m$ has least element and which is

$$[a, b] \longrightarrow [c, d], x \longrightarrow f(x-)$$

$$(\text{resp. } [a, b] \longrightarrow [c, d], x \longrightarrow f(x+)\ ) \ .$$

$[f]_m$ has greatest element which is

$$[a,b] \longrightarrow [c,d], x \longrightarrow f(x+)$$

$$(\text{resp. } [a,b] \longrightarrow [c,d], x \longrightarrow f(x-)\ ) .$$

If f is continuous and surjective, then $[f]_m$ has an only element f; we identify $[f]_m$ with f.

1.2 Monotone inverse of a monotone function

We introduce[1] the notion of function that is monotone inverse of another; this relation is symmetric and is compatible with the equivalence relation of the monotonically equivalent functions. The hypothesis of closed intervals permits to obtain results of existence and uniqueness, as easy relations between a function and its monotone inverse.

Definition 1.2.1 *Let $a, b, c, d \in \overline{\mathbf{R}}$; let $a \leq b$ and $c \leq d$; let $f : [a,b] \longrightarrow [c,d]$ increasing (resp. decreasing); let $g : [c,d] \longrightarrow [a,b]$ increasing (resp. decreasing); we say that g is monotone inverse of f if for all $x \in [a,b]$ and for all $y \in [c,d]$ we have*

$$(f(x) < y \Rightarrow x \leq g(y)) \text{ and } (y < f(x) \Rightarrow g(y) \leq x)$$

$$\textit{(resp. } (f(x) < y \Rightarrow x \geq g(y)) \text{ and } (y < f(x) \Rightarrow g(y) \geq x)\textit{)} .$$

This means that for all $y \in [c,d]$

$$\sup(\{x \in [a,b]; f(x) < y\}) \leq g(y) \leq \inf(\{x \in [a,b]; f(x) > y\})$$

$$(\text{resp. } \sup(\{x \in [a,b]; f(x) > y\}) \leq g(y) \leq \inf(\{x \in [a,b]; f(x) < y\})) ,$$

where the least upper bound and the greatest lower bound are made respect the ordered set $[a,b]$.

It is easy to see that g is monotone inverse of f if and only if f is monotone inverse of g.

The definition is equivalent also to

$$(f(x-) < y \Rightarrow x \leq g(y)) \text{ and } (y < f(x+) \Rightarrow g(y) \leq x) \tag{1}$$

$$(\text{resp. } (f(x-) < y \Rightarrow x \geq g(y)) \text{ and } (y < f(x+) \Rightarrow g(y) \geq x)) .$$

If in the definition of inverse monotone we substitute the strict inequalities $f(x) < y$ and $y < f(x)$ by $f(x) \leq y$ and $y \leq f(x)$, we have

$$(f(x) \leq y \Rightarrow x \leq g(y+)) \text{ and } (y \leq f(x) \Rightarrow g(y-) \leq x) \tag{2}$$

[1]In spite of the simplicity, the author is not acquainted in mathematical texts with the use of the following topic, at least in the exposed terms.

$$\text{(resp. } (f(x) \leq y \Rightarrow x \geq g(y+)) \text{ and } (y \leq f(x) \Rightarrow g(y-) \geq x)) \ .$$

From this it follows at once

$$g(f(x)-) \leq x \leq g(f(x)+) \quad \text{(resp. } g(f(x)+) \leq x \leq g(f(x)-)), \tag{3}$$

which substitutes the usual $g(f(x)) = x$, when g is the inverse of the injective function f.

From elementary proprieties of monotone functions and from from the axiom of choice if follows at once the theorem on the existence of the monotone inverse.

Theorem 1.2.1 *Let $a, b, c, d \in \overline{\mathbf{R}}$; let $a \leq b$ and $c \leq d$; let $f : [a, b] \longrightarrow [c, d]$ increasing (resp. decreasing); then there exists $g : [c, d] \longrightarrow [a, b]$ increasing (resp. decreasing) monotone inverse of f.*

We observe that for this theorem is necessary the hypothesis of closed intervals: the increasing function $f : [0, 1] \longrightarrow [0, 2]$ $x \longrightarrow 1$ has as monotone inverse for example

$$g : [0, 2] \longrightarrow [0, 1], x \longrightarrow \begin{cases} 0 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 < x \leq 2 \end{cases} ,$$

while the function $f : [0, 1[\longrightarrow [0, 2]$, $x \longrightarrow 1$ has no inverse monotone (extending definition (1.2.1) to consider not necessary closed intervals).

The bounds $g(x+)$ and $g(x-)$ for a monotone inverse of an increasing (resp. decreasing) function f are given by:

$$g(y+) = \inf(\{x \in [a, b]; f(x) > y\}) \ ,$$

$$g(y-) = \sup(\{x \in [a, b]; f(x) < y\})$$

$$\text{(resp. } g(y+) = \sup(\{x \in [a, b]; f(x) > y\}) \ ,$$

$$g(y-) = \inf(\{x \in [a, b]; f(x) < y\})) \ .$$

From this it follows at once that the relation "g is monotone inverse of f" is compatible with the equivalence relations of monotone equivalent functions on $[a, b]$ and on $[c, d]$; so we may consider the relation between classes "$[g]_m$ is monotone inverse of $[f]_m$"; moreover, given a class $[f]_m$, there exists one and only one class $[g]_m$ such that g is monotone inverse of f. We put

$$[f]_m^{-1} = [g]_m \ .$$

1.3 Monotone inverse of a strictly monotone function

In view of the applications there is an important not symmetric case: if g is monotone inverse of f, then f is strictly monotone if and only if g is continuous and surjective; in this case g is the only monotone inverse of f and $g \circ f$ is the identity. Indeed this is the case in which the notion of monotone inverse is more close to that of inverse of a function.

Theorem 1.3.1 *Let $a, b, c, d \in \overline{\mathbf{R}}$; let $a \leq b$ and $c \leq d$; let $f : [a, b] \longrightarrow [c, d]$ increasing (resp. decreasing); let $g : [c, d] \longrightarrow [a, b]$ increasing (resp. decreasing); let g a monotone inverse of f; then f is strictly increasing (resp. strictly decreasing) if and only if g is continuous and surjective.*

Proof. Suppose f strictly increasing; let us prove that g is continuous and surjective; this is equivalent to prove that $(\forall y \in [c, d])\ g(y-) = g(y+)$; let $y \in [c, d]$; we have

$$g(y-) = \sup(\{x \in [a, b]; f(x) < y\})$$

and

$$g(y+) = \inf(\{x \in [a, b]; f(x) > y\})\ ;$$

assume by contradiction that $g(y-) < g(y+)$; let $x \in [a, b]$ such that $g(y-) < x < g(y+)$; as $g(y-) < x$, from (1) we have $y \leq f(x)$; as $x < g(y+)$, we have $f(x) \leq y$; so we have $f(x) = y$; then f è constant on $]g(y-), g(y+)[$; this is a contradiction.
Suppose g continuous and surjective; let us prove that f is strictly increasing; let $x, x' \in [a, b]$ with $x < x'$; let us prove that $f(x) < f(x')$; let us assume by contradiction that $f(x) = f(x')$; let $y = f(x) = f(x')$; we have

$$g(y-) = \sup(\{t \in [a, b]; f(t) < y\}) \leq x$$

and

$$g(y+) = \inf(\{t \in [a, b]; f(t) > y\}) \geq x'\ ;$$

then we have $g(y-) < g(y+)$; then g is not continuous and surjective; this is a contradiction.

If f is strictly increasing (resp. strictly decreasing), then there exists only a monotone inverse of f; if g is this function, following the above notations, we have $[f]_m^{-1} = [g]_m = g$; we also note this function g by $f^{-1,m}$.

Let $f : [a, b] \longrightarrow [c, d]$ strictly increasing (resp. strictly decreasing); from (3) and from theorem 1.3.1, it follows $f^{-1,m} \circ f = 1_{[a,b]}$.

2 Rearrangements and image of a measure

2.1 Image of a measure

When we say (X, S) measurable space, we mean that X is a set and S is a σ-algebra of subsets of X; when we say (X, S, μ) measure space, we mean that X is a set, S is a σ-algebra of subsets of X, and $\mu : S \longrightarrow \overline{\mathbf{R}}$ a positive measure on X.

The notion of image of a positive measure is a well-known subject that belongs to the basic topic of the measure theory.

In order to conform the language, we briefly recall the definitions and some proprieties.

Let (X, S, μ) a measure space; let (Y, T) a measurable space; let $u : X \longrightarrow Y$; suppose that for all $A \in T$, $u^{-1}(A) \in S$; then the image $u(\mu)$ is the positive measure on T such that for all $A \in T$ $(u(\mu))(A) = \mu(u^{-1}(A))$.

We have $(u(\mu))(Y) = \mu(X)$. So, $u(\mu)$ is a bounded measure if and only if μ is a bounded measure.

We have $u(\mu) = 0$ if and only if $\mu = 0$.

Let $c \in Y$ and $k \in \overline{\mathbf{R}}$; then we have $u(\mu) = k\delta_c$ if and only if for almost every $x \in X$ we have $u(x) = c$ and $k = \mu(X)$.

In particular u is the constant function c, then $u(\mu) = \mu(X)\delta_c$.

Let $f : Y \longrightarrow \overline{\mathbf{R}}$ positive and measurable respect the measurable space (Y, T); then we have in $\overline{\mathbf{R}}$

$$\int_Y f(y)\, d(u(\mu))(y) = \int_X f(u(x))\, d\mu(x) \ . \tag{4}$$

2.2 Equimeasurable functions and rearrangements

Definition 2.2.1 *Let (X, S, μ), (X_1, S_1, μ_1) measure spaces; let (Y, T) a measurable space; let $u : X \longrightarrow Y$; suppose that for all $A \in T$, $u^{-1}(A) \in S$; let $u_1 : X_1 \longrightarrow Y$; suppose that for all $A \in T$, $u_1^{-1}(A) \in S_1$; we say that u is equimeasurable with u_1 if $u(\mu) = u_1(\mu_1)$.*

Clearly if u is equimeasurable with u_1 then $\mu(X) = \mu_1(X_1)$.

If I is an closed interval of $\overline{\mathbf{R}}$, we consider on I the σ-algebra $\mathcal{B}_I$ of the Borel sets of I. If (X, S, μ) is a measure space and if $u : X \longrightarrow I$ the condition "for all $A \in \mathcal{B}_I$, $u_1^{-1}(A) \in S$" becomes "u μ-measurable".

Definition 2.2.2 *Let (X, S, μ), (X_1, S_1, μ_1) measure spaces; let I a closed interval of $\overline{\mathbf{R}}$; let $u : X \longrightarrow I$ μ-measurable; let $u_1 : X_1 \longrightarrow I$ μ_1-measurable; we say that u is a rearrangements of u_1 if u is equimeasurable with u_1.*

The following theorem is immediate consequence of the definitions and of (4).

Theorem 2.2.1 *Let (X, S, μ), (X_1, S_1, μ_1) measure spaces; let I a closed interval of $\overline{\mathbf{R}}$; let $u : X \longrightarrow I$ measurable; let $u_1 : X_1 \longrightarrow I$ measurable; let u a rearrangements of u_1; let $f : I \longrightarrow \overline{\mathbf{R}}$ borelian and positive; then we have in $\overline{\mathbf{R}}$*

$$\int_X f(u(x))\, d\mu(x) = \int_{X_1} f(u_1(x))\, d\mu_1(x) \ .$$

3 Distributional derivative of a monotone function

3.1 Distributional derivative of a monotone function

In $\overline{\mathbf{R}}$, besides the usual conventions for operations, we put $(+\infty) + (-\infty) = 0$.

Let $(\mathbf{R}, L_R, \lambda)$ the measure space of Lebesgue on $\mathbf{R}$; let $i : \mathbf{R} \longrightarrow \overline{\mathbf{R}}$, $x \longrightarrow x$; the Lebesgue measure space $(\overline{\mathbf{R}}, L_{\overline{\mathbf{R}}}, \lambda_{\overline{\mathbf{R}}})$ on $\overline{\mathbf{R}}$ is the image by i of $(\mathbf{R}, L_R, \lambda)$; we have $A \in L_{\overline{\mathbf{R}}}$ if and only if $i^{-1}(A) \in L_R$ and, in this case, we have $\lambda_{\overline{\mathbf{R}}}(A) = \lambda(i^{-1}(A))$.

Let J a closed interval of $\overline{\mathbf{R}}$; we consider on J the measure space induced on J by $(\overline{\mathbf{R}}, L_{\overline{\mathbf{R}}}, \lambda_{\overline{\mathbf{R}}})$. We note still by λ_J (or simply by λ) the related measure, that we call the Lebesgue measure on J.

Let $f : [a, b] \longrightarrow [c, d]$ increasing; then there exists one and only one positive measure ν on $\mathcal{B}_{[a,b]}$ that for all $x, x' \in [a, b]$, $x \leq x'$, $\nu([x, x']) = f(x'+) - f(x-)$; in fact, although we work with intervals of $\overline{\mathbf{R}}$, and with not necessary σ-finite measures, the classical topics can still be used.

Definition 3.1.1 *Let $a, b, c, d \in \overline{\mathbf{R}}$; let $a \leq b$ and $c \leq d$; let $f : [a, b] \longrightarrow [c, d]$ increasing; the distributional derivative $D_{\text{meas}} f$ of f is the only positive measure on $\mathcal{B}_{[a,b]}$ such that for all $x, x' \in [a, b]$, $x \leq x'$,*

$$(D_{\text{meas}} f)([x, x']) = f(x'+) - f(x-) .$$

Let $x, x' \in [a, b]$, $x < x'$; we have

$$(D_{\text{meas}} f)([x, x'[) = f(x'-) - f(x-) ,$$

$$(D_{\text{meas}} f)(]x, x']) = f(x'+) - f(x+)$$

and

$$(D_{\text{meas}} f)(]x, x'[) = f(x'-) - f(x+) .$$

If $f, g : [a, b] \longrightarrow [c, d]$ are increasing monotonically equivalent functions, we have $D_{\text{meas}} f = D_{\text{meas}} g$; so we may put $D_{\text{meas}}[f]_m = D_{\text{meas}} f$.

We observe that $D_{\text{meas}} f$ depends not only on the graphic G_f of f, but also by $[c, d]$; in other word in this context we do not identify the function $f = (G_f, [a, b], [c, d])$ with its graphic G_f.

For instance if $f : [0, 1] \longrightarrow [0, 2]$, $x \longrightarrow 1$, we have $D_{\text{meas}} f = \delta_0 + \delta_1$.

So the only functions functions $f : [a, b] \longrightarrow [c, d]$ increasing such that $D_{\text{meas}} f = 0$ are those for which $c = d$.

As $(D_{\text{meas}} f)([a, b]) = d - c$, $D_{\text{meas}} f$ is bounded if and only if $d - c \in \mathbf{R}$; this means $c, d \in \mathbf{R}$ or $c = d = +\infty$ or $c = d = -\infty$.

Let $x_0 \in [a, b]$ and $k \in \overline{\mathbf{R}}$; we have $D_{\text{meas}} f = k\delta_{x_0}$ if and only if $k = d - c$, $(\forall x \in [a, x_0[)\, f(x) = c$ and $(\forall x \in]x_0, b])\, f(x) = d$.

If f is decreasing, we put $D_{\text{meas}} f = -D_{\text{meas}}(-f)$, where

$$-f : [a, b] \longrightarrow [-d, -c], x \longrightarrow -f(x) .$$

3.2 Functions with equal distributional derivative

Let $a, b, c, d, c', d' \in \overline{\mathbf{R}}$; let $a \leq b$, $c \leq d$, $c' \leq d'$; let $f : [a, b] \longrightarrow [c, d]$ and $g : [a, b] \longrightarrow [c', d']$; let f, g increasing. Suppose $D_{\mathrm{meas}} f = D_{\mathrm{meas}} g$. As $d - c = (D_{\mathrm{meas}} f)([a, b])$ and $d' - c' = (D_{\mathrm{meas}} g)([a, b])$, a fist consequence is $d - c = d' - c'$. But what about the usual relation $g = f + k$? Unfortunately, because of the presence of $\overline{\mathbf{R}}$, this relation is not always true.

We point out two cases of this situation.

The fist is $f : [a, b] \longrightarrow [+\infty, +\infty]$, $x \longrightarrow +\infty$, and $g : [a, b] \longrightarrow [1, 1]$, $x \longrightarrow 1$; we have $D_{\mathrm{meas}} f = D_{\mathrm{meas}} g = 0$, but we have $g \neq f + k$, for any $k \in \overline{\mathbf{R}}$.

The second is

$$f : [-1, 1] \longrightarrow [0, +\infty], x \longrightarrow \begin{cases} 0 & \text{if } -1 \leq x \leq 0 \\ +\infty & \text{if } 0 < x \leq 1 \end{cases}$$

and

$$g : [-1, 1] \longrightarrow [-\infty, 1], x \longrightarrow \begin{cases} -\infty & \text{if } -1 \leq x \leq 0 \\ 1 & \text{if } 0 < x \leq 1 \end{cases} \quad ;$$

we have $D_{\mathrm{meas}} f = D_{\mathrm{meas}} g = +\infty \delta_0$, but we have $g \neq f + k$, for any $k \in \overline{\mathbf{R}}$.

Fortunately this two examples individualize the only exceptions of the customary rule, as we shall see later; but firstly we precise what means $f + k$.

Definition 3.2.1 *Let $a, b, c, d \in \overline{\mathbf{R}}$; let $a \leq b$; let $c \leq d$; let $f : [a, b] \longrightarrow [c, d]$; let $k \in \mathbf{R}$; we put*

$$f + k : [a, b] \longrightarrow [c + k, d + k], x \longrightarrow f(x) + k \; .$$

We have $D_{\mathrm{meas}} f = D_{\mathrm{meas}}(f + k)$.

Let $a, b \in \overline{\mathbf{R}}$, with $a \leq b$; let $\nu : \mathcal{B}([a, b]) \longrightarrow \overline{\mathbf{R}}$ a positive measure; the convex support of ν is the least closed interval $I \subset [a, b]$ such that $\nu([a, b] - I) = 0$.

Let $c < d$; let $a_1 = \sup(f^{-1}(\{c\}))$; let $b_1 = \inf(f^{-1}(\{d\}))$; then the convex support of $D_{\mathrm{meas}} f$ is $[a_1, b_1]$.

Now we see that with some exceptions, if two increasing functions have the same distributional derivative, one is monotonically equivalence to a left translated of the other.

Theorem 3.2.1 *Let $a, b, c, d, c', d' \in \overline{\mathbf{R}}$; let $a \leq b$, $c \leq b$, $c' \leq d'$; let $\nu : \mathcal{B}_{[a,b]} \longrightarrow \overline{\mathbf{R}}$ a positive measure; let $\nu \neq 0$; let $(\forall x \in [a, b])\ \nu \neq +\infty \delta_x$; let $f : [a, b] \longrightarrow [c, d]$ increasing (resp. decreasing); let $g : [a, b] \longrightarrow [c', d']$ increasing (resp. decreasing); suppose $D_{\mathrm{meas}} f = D_{\mathrm{meas}} g = \nu$; then there exists $k \in \mathbf{R}$ such that f is monotonically equivalent a $g + k$.*

Proof. Let us prove at first that there exists $x_0 \in [a, b]$ such that $f(x_0-), g(x_0-) \in \mathbf{R}$.
Let $a_1, b_1 \in [a, b]$, with $a_1 \leq b_1$, such that $[a_1, b_1]$ is the convex support of ν.
Suppose $a_1 < b_1$. Let $x_0 \in [a, b]$ such that $a_1 < x_0 < b_1$; we have $a_1 = \sup(f^{-1}(\{c\})$ and $b_1 = \inf(f^{-1}(\{d\}))$; from this it follows that $c < f(x_0-) < d$; then $f(x_0-) \in \mathbf{R}$; analogously we see that $g(x_0-) \in \mathbf{R}$.

Suppose $a_1 = b_1$; let $x_0 = a_1$; as $\nu \neq +\infty\delta_{x_0}$, there exists $k \in]0, +\infty[$ such that $D_{\text{meas}}f = k\delta_{x_0}$; we have $k = (D_{\text{meas}}f)([x_0, x_0]) = f(x_0+) - f(x_0-)$; if $f(x_0-) = \pm\infty$, as $k \in \mathbf{R}$, we have $f(x_0+) = f(x_0-) = \pm\infty$; then $k = 0$; this is a contradiction; so we have $f(x_0-) \in \mathbf{R}$; analogously we see that $g(x_0-) \in \mathbf{R}$.
Let $x \in [a, b]$. If $x \geq x_0$ we have $(D_{\text{meas}}f)([x_0, x]) = (D_{\text{meas}}g)([x_0, x])$; then we have $f(x+)-f(x_0-) = g(x+)-g(x_0-)$; if $x < x_0$ we have $(D_{\text{meas}}f)(]x, x_0[) = (D_{\text{meas}}g)(]x, x_0[)$; then we have still $f(x+) - f(x_0-) = g(x+) - g(x_0-)$. Then we have

$$(f(x+) - f(x_0-)) + f(x_0-) = (g(x+) - g(x_0-)) + f(x_0-) ;$$

since $f(x_0-), g(x_0-) \in \mathbf{R}$, we have $f(x+) = g(x+) + (-g(x_0-) + f(x_0-))$.
Analogously we prove $f(x-) = g(x-) + (-g(x_0-) + f(x_0-))$.
From this, it follows the thesis.

As pointed above, if $\nu = 0$, then for every $c \in \overline{\mathbf{R}}$, if

$$f : [a, b] \longrightarrow [c, c], x \longrightarrow c$$

we have $D_{\text{meas}}f = \nu = 0$; the thesis of theorem 3.2.1 is not true.

Suppose $p \in [c, d]$, and $\nu = +\infty\delta_p$; then for all $c \in [-\infty, +\infty[$ if

$$f : [a, b] \longrightarrow [c, +\infty], x \longrightarrow \begin{cases} c & \text{if } x \leq p \\ +\infty & \text{if } x > p \end{cases} , \tag{5}$$

we have $D_{\text{meas}}f = \nu = +\infty\delta_p$; moreover for all $d \in]-\infty, +\infty]$, if

$$f : [a, b] \longrightarrow [-\infty, d], x \longrightarrow \begin{cases} -\infty & \text{if } x \leq p \\ d & \text{if } x > p \end{cases} , \tag{6}$$

we have still $D_{\text{meas}}f = \nu = +\infty\delta_p$; the thesis of theorem 3.2.1 is not true.

3.3 Distributional derivative and monotone inverse

The distributional derivative of a monotone function f is closely related to the notion of monotone inverse and of image of a measure, as stated by the following theorem.

Theorem 3.3.1 *Let $a, b, c, d \in \overline{\mathbf{R}}$; let $a \leq b$ and $c \leq d$; let $f : [a, b] \longrightarrow [c, d]$ increasing (resp. decreasing); let $g : [c, d] \longrightarrow [a, b]$ increasing (resp. decreasing); let g monotone inverse of f; then*

$$D_{\text{meas}}f = g(\lambda_{[c,d]}) \quad (\textit{resp.} \ - D_{\text{meas}}f = g(\lambda_{[c,d]} \ ) .$$

Proof. Let $x, x' \in [a, b]$ with $x \leq x'$; from (1) and (2) it follows at once

$$]f(x-), f(x'+)[\subset g^{-1}([x, x']) \subset [f(x-), f(x+)] .$$

Then we have

$$(g(\lambda))([x,x']) = \lambda(g^{-1}([x,x'])) = f(x'+) - f(x-) = (D_{\mathrm{meas}}f)([x,x']) \ .$$

If f is increasing (resp. decreasing) and if $h : [a,b] \longrightarrow \overline{\mathbf{R}}$ is a positive borelian function, we have in particular in $\overline{\mathbf{R}}$

$$\int_{[c,d]} h(g(y))\, d\lambda(y) = \int_{[a,b]} h(x)\, d(D_{\mathrm{meas}}f)(x)$$

$$(\text{resp. } \int_{[c,d]} h(g(y))\, d\lambda(y) = \int_{[a,b]} h(x)\, d(-D_{\mathrm{meas}}(f))(x)) \ .$$

4 Exact measures

4.1 Exact measures

Now let ν a positive measure on $[a,b]$; the problem that arises is the condition on ν for the existence of an increasing function f such that $D_{\mathrm{meas}}f = \nu$.

Definition 4.1.1 *Let $a,b \in \overline{\mathbf{R}}$; let $a \le b$; let $\nu : \mathcal{B}_{[a,b]} \longrightarrow \overline{\mathbf{R}}$ a positive measure; we say that ν is an exact measure if there exists a closed interval of $\overline{\mathbf{R}}$, J, and $f : [a,b] \longrightarrow J$ increasing such that $D_{\mathrm{meas}}f = \nu$.*

If ν is an exact measure and if $D_{\mathrm{meas}}f = \nu$, then we call f a primitive of ν.

If $\nu = 0$, then ν is an exact measure and, if $c \in \overline{\mathbf{R}}$, $f : [a,b] \longrightarrow [c,c]$ is a primitive of ν.

A Dirac measure $k\delta_p$, with $k \in \overline{\mathbf{R}}_+$, and $p \in [a,b]$, is an exact measure. A primitive of $k\delta_p$ is

$$f : [a,b] \longrightarrow [-\frac{k}{2}, \frac{k}{2}], x \longrightarrow \begin{cases} -\frac{k}{2} & \text{if } x \le p \\ \frac{k}{2} & \text{if } x > p \end{cases} \ .$$

If $k = +\infty$, (5) and (6) give other primitive of $k\delta_p = +\infty\delta_p$.

4.2 Integral function

Let $a,b \in \overline{\mathbf{R}}$; let $a \le b$; let $\nu : \mathcal{B}_{[a,b]} \longrightarrow \overline{\mathbf{R}}$ a positive measure; let $x_1, x_2 \in [a,b]$; we put

$$\int_{x_1}^{x_2} d\nu = \begin{cases} \frac{1}{2}\nu(\{x_1\}) + \int_{]x_1,x_2[} d\nu + \frac{1}{2}\nu(\{x_2\}) & \text{if } x_1 < x_2 \\ 0 & \text{if } x_1 = x_2 \\ -\frac{1}{2}\nu(\{x_2\}) - \int_{]x_2,x_1[} d\nu - \frac{1}{2}\nu(\{x_1\}) & \text{if } x_1 > x_2 \end{cases} \ .$$

Definition 4.2.1 *Let $a,b \in \overline{\mathbf{R}}$; let $a \le b$; let $\nu : \mathcal{B}_{[a,b]} \longrightarrow \overline{\mathbf{R}}$ a positive measure; let $x_0 \in [a,b]$; let $c = -\nu([a,x_0[) - \frac{1}{2}\nu(\{x_0\})$; let $d = \nu(]x_0,b]) + \frac{1}{2}\nu(\{x_0\})$; the function*

$$F_{x_0} : [a,b] \longrightarrow [c,d], x \longrightarrow \int_{x_0}^{x} d\nu$$

is called integral function of ν of initial point x_0.

Clearly F_{x_0} is an increasing function. In general $D_{\text{meas}}F_{x_0} = \nu$ is not verified; however, the problem of the existence of an increasing function f such that $D_{\text{meas}}f = \nu$ is strongly tied to the existence of a $x_0 \in [a, b]$ such that F_{x_0} is a primitive of ν.

We call a point $x_0 \in [a, b]$ such that F_{x_0} is a primitive of ν, an initial point for ν.

If $\nu = 0$ or if ν is a Dirac measure $k\delta_p$, with $k \in \overline{\mathbf{R}}_+$ and $p \in [a, b]$, then every $x_0 \in [a, b]$ is an initial point for ν.

One easily sees that, if x_0 in an initial point for ν, then a function as

$$f : [a, b] \longrightarrow [-\nu([a, x_0[), \nu([x_0, b])], x \longrightarrow \begin{cases} \nu([x_0, x]) & \text{if } x \geq x_0 \\ -\nu([a, x_0[& \text{if } x < x_0 \end{cases}$$

is monotonically equivalent to a left translate $k + F_{x_0}$ of F_{x_0}; so it is still a primitive of ν; in such a way, we can find other primitives of ν.

4.3 Characterization of the exact measures

In the following theorem we give the characterization of the exact measures $\nu \neq 0$.

Theorem 4.3.1 *Let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $\nu : \mathcal{B}_{[a,b]} \longrightarrow \overline{\mathbf{R}}$ a positive measure; let $\nu \neq 0$; let $a_1, b_1 \in \overline{\mathbf{R}}$, with $a_1 \leq b_1$ such that the convex support of ν is $[a_1, b_1]$; then ν is an exact measure if and only if*

1. $(\forall x, x' \in]a_1, b_1[\,, x \leq x')\ \nu([x, x']) < +\infty$;
2. $\nu(\{b_1\}) = 0$ *or* $(\forall x \in]a_1, b_1[)\ \nu([x, b_1[) < +\infty$;
3. $\nu(\{a_1\}) = 0$ *or* $(\forall x \in]a_1, b_1[)\ \nu(]a_1, x]) < +\infty$.

If ν is exact, then every $x_0 \in]a_1, b_1[$ is an initial point for ν.

Proof. If $a_1 = b_1$, we have $\nu = \nu([a, b])\delta_{a_1}$; so ν satisfies the three conditions and ν is exact; the thesis of the theorem is then true; so we may suppose $a_1 < b_1$.
Suppose that ν satisfies the three conditions. Let x_0 such that $a_1 < x_0 < b_1$; let $c = -\nu([a, x_0[) - \frac{1}{2}\nu(\{x_0\})$, let $d = \nu(]x_0, b]) + \frac{1}{2}\nu(\{x_0\})$; let $F : [a, b] \longrightarrow [c, d]$ the integral function of initial point x_0.
By condition (1) we have $\nu(\{x_0\}) = \nu([x_0, x_0]) \in \mathbf{R}$. Let us prove that for all $x \geq x_0$ we have

$$F(x+) + \frac{1}{2}\nu(\{x_0\}) = \nu([x_0, x]) \ . \tag{7}$$

For all $y > x$ we have

$$\nu([x_0, y[) \leq \nu([x_0, y[) + \frac{1}{2}\nu(\{y\}) = F(y) + \frac{1}{2}\nu(\{x_0\}) \leq \nu([x_0, y]) \ . \tag{8}$$

Then we have

$$\lim_{y \to x, y > x} \nu([x_0, y[) \leq \lim_{y \to x, y > x} \left(F(y) + \frac{1}{2}\nu(\{x_0\}) \right) \leq \lim_{y \to x, y > x} \nu([x_0, y]) \, . \tag{9}$$

If $x < b_1$ by condition (1), we have $\nu([x_0, y]), \nu([x_0, y[) \in \mathbf{R}$ for all $y \in]x, b_1[$; then

$$\lim_{y \to x, y > x} \nu([x_0, y[) = \lim_{y \to x, y > x} \nu([x_0, y]) = \nu([x_0, x]) \ ;$$

then

$$\lim_{y \to x, y > x} \left(F(y) + \frac{1}{2}\nu(\{x_0\}) \right) = \nu([x_0, x]) \ ;$$

then we have (7).
If $x \geq b_1$, then F is constant and equal to $\frac{1}{2}\nu(\{x_0\}) + \nu(]x_0, b_1])$ on $]b_1, b]$; then we have $F(x+) = \frac{1}{2}\nu(\{x_0\}) + \nu(]x_0, b_1])$; then we have still (7).
In particular we have $F(x_0+) = \frac{1}{2}\nu(\{x_0\})$.
Let us prove that for all $x \geq x_0$ we have

$$F(x-) + \frac{1}{2}\nu(\{x_0\}) = \nu([x_0, x[) \,. \tag{10}$$

Suppose $x > x_0$. For all y, $x_0 < y < x$ we have (8).
Then we have

$$\lim_{y \to x, y < x} \nu([x_0, y[) \leq \lim_{y \to x, y < x} \left(F(y) + \frac{1}{2}\nu(\{x_0\}) \right) \leq \lim_{y \to x, y < x} \nu([x_0, y]) \ .$$

We have

$$\lim_{y \to x, y < x} \nu([x_0, y[) = \lim_{y \to x, y < x} \nu([x_0, y]) = \nu([x_0, x[) \ ;$$

then

$$\lim_{y \to x, y < x} \left(F(y) + \frac{1}{2}\nu(\{x_0\}) \right) = \nu([x_0, x]) \ ;$$

then we have (10).
If $x = x_0$ for symmetry we have $F(x_0-) = -F(x_0+) = -\frac{1}{2}\nu(\{x_0\})$; then we have still (10).
Let $x, x' \in [a, b]$ with $x_0 \leq x \leq x'$; we have $[x_0, x'] = [x_0, x[\cup [x, x']$; then we have $\nu([x_0, x']) = \nu([x_0, x[) + \nu([x, x'])$. Let us prove that from this it follows in $\overline{\mathbf{R}}$

$$\nu([x, x']) = \nu([x_0, x']) - \nu([x_0, x[) \ . \tag{11}$$

If $\nu([x_0, x']) \in \mathbf{R}$, then we have $\nu([x_0, x[), \nu([x, x']) \in \mathbf{R}$ and (11) is true.
Suppose $\nu([x_0, x']) = +\infty$; if $\nu([x_0, x[) \in \mathbf{R}$, we have a $\nu([x, x']) = +\infty$; so we have still (11).
Suppose $\nu([x_0, x']) = +\infty$ and $\nu([x_0, x[) = +\infty$; from condition (1), if follow that we have $b_1 \leq x$; if $b_1 = x$ from condition (2), it follows that we have $\nu(\{b_1\}) = 0$; then we have $\nu([x, x']) = \nu([b_1, x']) = \nu(\{b_1\}) + \nu(]b_1, x']) = 0$; if $x > b_1$ we have still $\nu([x, x']) = 0$; we have $\nu([x_0, x']) - \nu([x_0, x[) = +\infty - \infty = 0$ and so (11) is true.
Being the addition commutative and associative in $] - \infty, +\infty]$, from (11) it follows $\nu([x, x']) = F(x'+) - F(x-) = (D_{\text{meas}}F)([x, x'])$.
Analogously we prove that if $x \leq x' \leq x_0$, we have still $\nu([x, x']) = F(x'+) - F(x-)$.

If $x \leq x_0 \leq x'$, since the addition is commutative and associative in $]-\infty, +\infty]$, we have

$$\nu([x, x']) = \nu([x, x_0]) + \nu([x_0, x']) - \nu(\{x_0\}) =$$

$$F(x_0+) - F(x-) + F(x'+) - F(x_0-) - \nu(\{x_0\}) = F(x'+) - F(x-) \,.$$

This proves that F is a primitive of ν.
Vice versa suppose ν an exact measure measure; let $f : [a, b] \longrightarrow [c, d]$ a primitive of ν. Let $x \in]a_1, b_1[$; f can not be equal to the constant $+\infty$ on $]x, b]$, otherwise it would be $\nu(]x, b]) = 0$ and then $b_1 \leq x$; so we have $f(x+) \in \mathbf{R}$; analogously we see that $f(x-) \in \mathbf{R}$; from this it follows propriety (1).
Suppose there exists $x \in]a_1, b_1[$ such that $\nu([x, b_1[) = +\infty$; we have $\nu([x, b_1[) = f(b_1-) - f(x-)$; then we have $f(b_1-) = +\infty$; then we have $f(b_1+) = +\infty$ and $\nu(\{b_1\}) = f(b_1+) - f(b_1-) = 0$; this proves propriety (2); we prove propriety (3) analogously.

It follows at once from the theorem that the relation ν exact does not depend on the interval $[a, b]$ containing the convex support $[a_1, b_1]$ of ν, but only on $[a_1, b_1]$.

From theorem 4.3.1 it follows at once the following result.

Theorem 4.3.2 *Let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $\nu : \mathcal{B}_{[a,b]} \longrightarrow \overline{\mathbf{R}}$ a bounded positive measure; then ν is exact.*

If $\nu : \mathbf{B}_{[a,b]}$ is an exact measure and if $f : [a, b] \longrightarrow [c, d]$ is a primitive of ν, the care is on $[c, d]$; in fact we have the condition $\nu([a, b]) = d - c$, while the interval $[a, b]$ may be equivalently replaced by another closed interval containing the convex support of ν.

4.4 Left convergent exact measures

If $f : [a, b] \longrightarrow [c, d]$, the relation $D_{\text{meas}} f = \nu$ depends not only by the graphic of f, but also by the set $[c, d]$; now the problem that arises is to find conditions on ν for the existence of increasing functions $f : [a, b] \longrightarrow [c, d]$ such that $D_{\text{meas}} f = \nu$, with $[c, d]$ satisfying special proprieties, i.e. $c \in \mathbf{R}$ or $c = -\infty$, $d \in \mathbf{R}$ or $d = +\infty$.

Definition 4.4.1 *Let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $\nu : \mathcal{B}_{[a,b]} \longrightarrow \overline{\mathbf{R}}$ a positive measure; let ν exact; we say that ν is a left (resp. right) convergent exact measure if these exists $c \in \mathbf{R}$ (resp. $c \in \overline{\mathbf{R}}$), there exists $d \in \overline{\mathbf{R}}$ (resp. $d \in \mathbf{R}$), with $c \leq d$, there exists $f : [a, b] \longrightarrow [c, d]$, such that f is a primitive of ν.*
We say that ν is a left (resp. right) divergent exact measure if these exists $d \in \overline{\mathbf{R}}$ (resp. $c \in \overline{\mathbf{R}}$), there exists $f : [a, b] \longrightarrow [-\infty, d]$ (resp. $f : [a, b] \longrightarrow [c, +\infty]$) such that f is a primitive of ν.
We say that ν is a left-right divergent distributional measure if these exists $f : [a, b] \longrightarrow [-\infty, +\infty]$ such that f is a primitive of ν.

If $\nu = 0$, ν is left convergent, right convergent, left divergent, right divergent.

If $p \in [a, b]$, $+\infty\delta_p$ is left convergent, right convergent, left divergent, right divergent, left-right divergent.

From theorem 3.2.1, if $\nu \neq 0$ and if $\nu \neq +\infty\delta_p$ for any $p \in [a,b]$, ν is left (resp right) divergent if and only if ν is not left (resp. right) convergent; more precisely let $[a_1, b_1]$ the convex support of ν; let $x_0 \in]a_1, b_1[$; then we have:
ν is left convergent if and only if $\int_a^{x_0} \in \mathbf{R}$,
ν is left divergent if and only if $\int_a^{x_0} = +\infty$,
ν is right convergent if and only if $\int_{x_0}^b \in \mathbf{R}$,
ν is left divergent if and only if $\int_{x_0}^b = +\infty$,
ν is left-right divergent if and only if $\int_a^{x_0} = \int_{x_0}^b = +\infty$.
If ν is bounded, then ν is a left convergent and a right convergente exact measure.
From theorem 4.3.1 it follows the characterization of the left (resp. right) convergent exact measures $\nu \neq 0$.

Theorem 4.4.1 *Let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $\nu : \mathcal{B}_{[a,b]} \longrightarrow \overline{\mathbf{R}}$ a positive measure; let $\nu \neq 0$; let $a_1, b_1 \in \overline{\mathbf{R}}$ with $a_1 \leq b_1$ such that $[a_1, b_1]$ is the convex support of ν; then ν is a left (resp. right) convergent exact measure if and only if*

1. $(\forall x \in [a, b_1[)\ \nu([a,x]) < +\infty$ *(resp.* $(\forall x \in]a_1, b])\ \nu([x,b]) < +\infty$*);*
2. $\nu(\{b_1\}) = 0$ *or* $\nu([a, b_1[) < +\infty$ *(resp.* $\nu(\{a_1\}) = 0$ *or* $\nu(]a_1, b]) < +\infty$*).*

If ν is a left (resp. right) convergent measure, then every $x_0 \in [a, b_1[$ (resp. $x_0 \in]a_1, b]$) is an initial point for ν.

For a left (resp. right)convergent exact measure we can consider the primitive of ν as an equivalence class of monotone functions, according to the following theorem.

Theorem 4.4.2 *Let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $\nu : \mathcal{B}_{[a,b]} \longrightarrow \overline{\mathbf{R}}$ a positive measure; let ν a left (resp. right) convergent exact measure; we have*

1. *there exists $f : [a,b] \longrightarrow [0, \nu([a,b])]$ (resp. $f : [a,b] \longrightarrow [-\nu([a,b]), 0]$) primitive of ν;*
2. *if $f : [a,b] \longrightarrow [0, \nu([a,b])]$ (resp. $f : [a,b] \longrightarrow [-\nu([a,b]), 0]$) is a primitive of ν, then $g : [a,b] \longrightarrow [0, \nu([a,b])]$ (resp. $g : [a,b] \longrightarrow [-\nu([a,b]), 0]$) is a primitive of ν if and only if g is monotonically equivalent to f.*

Proof. If $\nu = 0$ or if $\nu = +\infty\delta_p$ for some $p \in [a,b]$ the thesis is proved directly; otherwise if follows from theorems 4.4.1, 3.2.1.

So for left (resp. right) convergent exact measure, we can consider the primitive of ν as the equivalent class $[f]_m$, where f in an increasing function $f : [a,b] \longrightarrow [0, \nu([a,b])]$ (resp. $f : [a,b] \longrightarrow [-\nu([a,b]), 0]$) such that $D_{\text{meas}} f = \nu$.
We can choose as element of this class the function $f : [a,b] \longrightarrow [0, \nu([a,b])], x \longrightarrow \nu([a,x])$, (resp. $f : [a,b] \longrightarrow [-\nu([a,b]), 0]$) such that $f(x) = F_a(x)$ or $f(x) = \nu([a,x])$ or $f(x) = \nu([a,x[)$ (resp. $f(x) = F_b(x)$ or $f(x) = -\nu([x,b])$ or $f(x) = -\nu(]x,b])$).

5 Distributional Functions

5.1 Distributional functions

The distributional functions of the following definition will be the functions for which we can (significantly) define the distribution function or equivalently the function for which increasing and decreasing rearrangements exist.

Definition 5.1.1 *Let (X, S, μ) a measure space; let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; we say that u is a distributional function (as regards to μ) if $u(\mu)$ is an exact measure.*

A point x_0, initial point for $u(\mu)$, is called an initial point for the distributional function u.

If $\mu = 0$, then every measurable function is distributional.

If $\mu(X) < +\infty$, $u(\mu)$ is bounded; then $u(\mu)$ is exact; so every measurable function u is distributional.

We have seen that a Dirac measure $k\delta_x$ is exact; from this it follows that an almost every constant function is distributional.

In the following theorem we give the characterization of the distributional functions with $\mu \neq 0$.

Theorem 5.1.1 *Let (X, S, μ) a measure space; let $\mu \neq 0$; let $a, b \in \overline{\mathbf{R}}$ with $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; let $a_1 = \text{ess.inf}(u)$; let $b_1 = \text{ess.sup}(u)$; then u is distributional if and only if*

1. $(\forall x, x' \in]a_1, b_1[\,, x \leq x')\ \mu(u^{-1}([x, x'])) < +\infty$;
2. $\mu(u^{-1}(\{b_1\})) = 0$ *or* $(\forall x \in]a_1, b_1[)\ \mu(u^{-1}([x, b_1[)) < +\infty$;
3. $\mu(u^{-1}(\{a_1\})) = 0$ *or* $(\forall x \in]a_1, b_1[)\ \mu(u^{-1}(]a_1, x])) < +\infty$.

Proof. It follows from theorem 4.3.1.

From theorem 5.1.1 it follows that the propriety "u distributional" does not depend on the closed interval $[a, b]$ such that $f(X) \subset [a, b]$.

Considering on $\mathbf{R}$ and $[0, +\infty[$, the Lebesgue measure, from theorem 5.1.1 it follows that the function $\mathbf{R} \longrightarrow [-\infty, +\infty]$, $x \longrightarrow x^2$ is distributional, while $\mathbf{R} \longrightarrow [-\infty, +\infty]$, $x \longrightarrow \sin x$ is not distributional; the function

$$f : [0, +\infty[\longrightarrow [-\infty, +\infty], x \longrightarrow \begin{cases} 1/x & \text{if } x \geq 1 \\ 0 & \text{if } 0 \leq x < 1 \end{cases}$$

is not distributional (see [3] pag.10 for a comparison).

5.2 Distribution function

Definition 5.2.1 *Let (X, S, μ) a measure space; let $a, b \in \overline{\mathbf{R}}$, let $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; let $c, d \in \overline{\mathbf{R}}$, let $c \leq d$; let $m : [a, b] \longrightarrow [c, d]$ increasing (resp. decreasing); we say that m is an increasing (resp. decreasing) distribution function of u if $D_{\text{meas}} m = u(\mu)$ (resp $D_{\text{meas}} m = -u(\mu)$).*

If m is an increasing distribution function of u, then the left translated $m + c$ (c constant) is an increasing distribution function of u. If m is an increasing distribution function and if $m_1 : [a, b] \longrightarrow [c, d]$ is an increasing function that is monotone equivalent to m, then m_1 is an increasing distribution function of u.

From definitions it follows that there exists an increasing (resp. decreasing) distribution function of u if and only is u is distributional.

Evidently, if m is an increasing distribution function, then $-m$ is a decreasing distribution function.

The following theorem explains what happens when we have two distribution functions of the same distributional function u.

Theorem 5.2.1 *Let (X, S, μ) a measure space; let $\mu \neq 0$; if $\mu(X) = +\infty$ let u not almost every constant; let $a, b \in \overline{\mathbf{R}}$, let $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; let u a distributional function; let $m : [a, b] \longrightarrow [c, d]$ increasing (resp. decreasing); let $m_1 : [a, b] \longrightarrow [c_1, d_1]$ increasing (resp. decreasing); let m and m_1 increasing (resp. decreasing) distribution functions of u; then exists $k \in \mathbf{R}$ such that m_1 is monotonically equivalente to $m + k$.*

Proof. It follows at once from theorem 3.2.1.

The thesis of the theorem is not true if $\mu = 0$ or if u is constant, with $\mu(X) = +\infty$.

Theorem 5.2.2 *Let (X, S, μ) a measure space; let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; let u distributional; let $m : [a, b] \longrightarrow [c, d]$ an increasing (resp. decreasing) distribution function of u; then m is continuous and surjective if and only if*

$$(\forall y \in [a, b])\, \mu(u^{-1}(\{y\})) = 0 \ .$$

Proof. We have $\mu(u^{-1}(\{y\}) = (u(\mu))(\{y\}) = (D_{\text{meas}} m)([y, y]) = m(y+) - m(y-)$; then we have $\mu(u^{-1}(\{y\})) = 0$ if and only if $m(y+) = m(y-)$.

5.3 Left convergent distributional functions

Definition 5.3.1 *Let (X, S, μ) a measure space; let $a, b \in \mathbf{R}$, with $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; we say that u is a left (resp. right) convergent distributional function if $u(\mu)$ is a left (resp. right) convergent exact measure.*
We say that u is a left (resp. right, resp. left-right) divergent distributional function if $u(\mu)$ is a left (resp. right, resp. left-right) divergent exact measure.

If $\mu = 0$, then every measurable function u is left convergent, right convergent, left divergent, right divergent.

If $\mu(X) = +\infty$ and if u is almost every constant, then u is left convergent, right convergent, left divergent, right divergent, left-right divergent.

If $\nu \neq 0$ and if, for $\mu(X) = +\infty$, u in not almost every equal to a constant, u is left (resp right) divergent if and only if ν is not left (resp. right) convergent and vice versa; more precisely: let $a_1 = \text{ess.inf}(u)$, let $b_1 = \text{inf.ess}(u)$; let $x_0 \in]a_1, b_1[$; then we have:

u is left convergent if and only if $\mu(u^{-1}[a, x_0]) \in \mathbf{R}$,

u is left divergent if and only if $\mu(u^{-1}[a, x_0]) = +\infty$,

u is right convergent if and only if $\mu(u^{-1}[x_0, b]) \in \mathbf{R}$,

u is left divergent if and only if $\mu(u^{-1}[x_0, b]) = +\infty$,

ν is left-right divergent if and only if $\mu(u^{-1}[a, x_0]) = \mu(u^{-1}[x_0, b]) = +\infty$.

If μ is bounded then every measurable function u is left ad right convergent distributional.

From theorem 4.4.1, we have the following characterization of left (resp. right) distributional convergent measures.

Theorem 5.3.1 *Let (X, S, μ) a measure space; let $a, b \in \overline{\mathbf{R}}$, with $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; let $b_1 = \text{ess.sup}(u)$ (resp. $a_1 = \text{ess.sup}(u)$); then u is a left (resp. right) convergent exact measure if and only if*

1. $(\forall x \in [a, b_1[)\ \mu(u^{-1}([a, x])) < +\infty$ *(resp.* $(\forall x \in]a_1, b])\ \mu(u^{-1}([x, b])) < +\infty$*);*
2. $\mu(u^{-1}(\{b_1\})) = 0$ *or* $\mu(u^{-1}([a, b_1[)) < +\infty$ *(resp.* $\mu(u^{-1}(\{a_1\})) = 0$ *or* $\mu(u^{-1}(]a_1, b])) < +\infty$*).*

From theorem 4.4.2 it follows that for a left (resp. right) convergent distributional function we can consider the increasing distribution function as an equivalence class, according to the following theorem.

Theorem 5.3.2 *Let (X, S, μ) a measure space; let $a, b \in \mathbf{R}$, with $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; let u a left (resp. right) convergent distributional function; we have*

1. *there exists $m : [a, b] \longrightarrow [0, \mu(X)]$ (resp. $m : [a, b] \longrightarrow [-\mu(X), 0]$) increasing distribution function of u;*
2. *if $m : [a, b] \longrightarrow [0, \mu(X)]$ (resp. $m : [a, b] \longrightarrow [-\mu(X), 0]$) is an increasing distribution function of u, then $m_1 : [a, b] \longrightarrow [0, \mu(X)]$ (resp. $m_1 : [a, b] \longrightarrow [-\mu(X), 0]$) is an increasing distribution function of u if and only if m_1 is monotonically equivalent to m.*

So for left (resp. right) convergent distributional functions, we can consider the increasing distribution function of u as the equivalent class $[m]_m$, where m in an increasing distribution function $m : [a, b] \longrightarrow [0, \mu(X)]$ (resp. $m : [a, b] \longrightarrow [-\mu(X), 0]$).

We can choose as m a function $m : [a, b] \longrightarrow [0, \mu(X)]$ (resp. $m : [a, b] \longrightarrow [-\mu(X), 0]$) such that $m(x) = \mu(u^{-1}([a, x]))$ or $m(x) = \mu(u^{-1}([a, x[))$ (resp. $m(x) = -\mu(u^{-1}([x, b]))$ or $m(x) = -\mu(u^{-1}(]x, b])))$ which are monotonically equivalent.

Analogously for decreasing distributional function we have:

Theorem 5.3.3 *Let (X, S, μ) a measure space; let $a, b \in \mathbf{R}$, with $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; let u a left (resp. right) convergent distributional function; we have*

1. *there exists $m : [a, b] \longrightarrow [-\mu(X), 0]$ (resp. $m : [a, b] \longrightarrow [0, \mu(X)]$) decreasing distribution function of u;*

2. *if $m : [a, b] \longrightarrow [-\mu(X), 0]$ (resp. $m : [a, b] \longrightarrow [0, \mu(X)]$) is a decreasing distribution function of u, then $m_1 : [a, b] \longrightarrow [-\mu(X), 0]$ (resp. $m_1 : [a, b] \longrightarrow [0, \mu(X)]$) is a decreasing distribution function of u if and only if m_1 is monotonically equivalent to m.*

So for left (resp. right) convergent distributional functions, we can consider the decreasing distribution function of u as the equivalent class $[m]_m$, where m in a decreasing distribution function $m : [a, b] \longrightarrow [-\mu(X), 0]$ (resp. $m : [a, b] \longrightarrow [0, \mu(X)]$).

We can choose as m the function $m : [a, b] \longrightarrow [-\mu(X), 0]$ (resp. $m : [a, b] \longrightarrow [0, \mu(X)]$) such that $m(x) = -\mu(u^{-1}([a, x]))$ or $m(x) = -\mu(u^{-1}([a, x[))$ (resp. $m(x) = \mu(u^{-1}([x, b]))$ or $m(x) = \mu(u^{-1}(]x, b])))$, which are monotonically equivalent.

If $p \in [1, +\infty[$ and if $u \in \mathcal{L}^p(X; \mathbf{R})$, then $|u| : X \longrightarrow [0, +\infty]$ is right convergent distributional; as decreasing distributional function of $|u|$ we can consider the monotone class of

$$m : [0, +\infty] \longrightarrow [0, \mu(X)], t \longrightarrow \lambda(|u|^{-1}([t, +\infty]))$$

(see [11]).

If $\mu(X) < +\infty$, than every measurable function is both left and right convergent distributional. As u is left convergent we can define the increasing distribution function as a monotone class $[m]_m$, where $m : [a, b] \longrightarrow [0, \mu(X)]$. As u is right convergent we can define the decreasing distribution function as a monotone class $[m]_m$, where is still $m : [a, b] \longrightarrow [0, \mu(X)]$.

6 Increasing and decreasing rearrangements

6.1 Increasing and decreasing rearrangements

Once defined the general notion of rearrangement (definition 2.2.2), it follows the problem of the existence of rearrangements with specific proprieties. We begin with the increasing and decreasing rearrangements.

Definition 6.1.1 *Let (X, S, μ) a measure space; let $a, b \in \overline{\mathbf{R}}$, with $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; let $c, d \in \overline{\mathbf{R}}$; let $c \leq d$; let $r : [c, d] \longrightarrow [a, b]$ measurable; we say that r is an increasing (resp. decreasing) rearrangement of u if r is increasing (resp. decreasing) and if r is a rearrangement of u.*

Evidently if r is an increasing (resp. decreasing) rearrangement of u, then we have $d - c = \mu(X)$.

If $r : [c, d] \longrightarrow [a, b]$ is an increasing (resp. decreasing) rearrangement of u and if $k \in \mathbf{R}$, then the right translated

$$[c - k, d - k] \longrightarrow [a, b], x \longrightarrow r(x + k)$$

is an increasing (resp. decreasing) rearrangement of u.

6.2 Existence of increasing rearrangements

Theorem 6.2.1 *Let (X, S, μ) a measure space; let $a, b \in \mathbf{R}$ with $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; let $c, d \in \overline{\mathbf{R}}$, with $c \leq d$; let $m : [a, b] \longrightarrow [c, d]$ increasing (risp. decreasing); let $r : [c, d] \longrightarrow [a, b]$ a monotone inverse of m; then r is an increasing (resp. decreasing) rearrangement of u if and only if m is an increasing (resp. decreasing) distribution function of u.*

Proof. It follows from 3.3.1

So we find as simple corollary the result of P. W. Day [5].

Theorem 6.2.2 *Let (X, S, μ) a measure space; let $a, b \in \overline{\mathbf{R}}$ with $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; then there exists $c, d \in \overline{\mathbf{R}}$ with $c \leq d$ and $r : [c, d] \longrightarrow [a, b]$, increasing (risp. decreasing) rearrangement of u if and only if u distributional.*

Proof. It follows from theorems 6.2.1.

Let (X, S, μ) a measure space; let $a, b \in \overline{\mathbf{R}}$ with $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; let $c, d \in \overline{\mathbf{R}}$, with $c < d$; let $r : [c, d] \longrightarrow [a, b]$ increasing (resp. decreasing); since r is a rearrangement of u if and only if the restrictions $r|]a, b]$, $r|[a, b[$, $r|]a, b[$ are increasing (resp. decreasing) rearrangement of u, theorem 6.2.2 gives also a necessary and sufficient condition for the existence of an increasing rearrangement on a not closed interval of $\overline{\mathbf{R}}$.

6.3 Increasing rearrangements of the same function

Theorem 6.3.1 *Let (X, S, μ) a measure space; let $\mu \neq 0$; let $a, b \in \mathbf{R}$ with $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; if $\mu(X) = +\infty$ let u not almost every constant; let u distributional; let $c, d, c', d' \in \overline{\mathbf{R}}$ with $c \leq d$ and $c' \leq d'$; let $r : [c, d] \longrightarrow [a, b]$ and $s : [c', d'] \longrightarrow [a, b]$ increasing (resp. decreasing) rearrangements of u; then there exists $k \in \mathbf{R}$ such that $c' = c - k$, $d' = d - k$ and s is monotonically equivalent to*

$$[c - k, d - k] \longrightarrow [a, b], x \longrightarrow r(x + k)\ .$$

Proof. It follows at once from 3.2.1.

6.4 Left finite increasing rearrangements

Definition 6.4.1 *Let (X, S, μ) a measure space; let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; let $c, d \in \overline{\mathbf{R}}$ with $c \leq d$; let $r : [c, d] \longrightarrow [a, b]$ an increasing (resp. decreasing) rearrangement of u; we say that r is left finite if $c \in \mathbf{R}$; we say that r is right finite if $d \in \mathbf{R}$.*

Theorem 6.4.1 *Let (X, S, μ) a measure space; let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a, b]$ measurable; then*

1. *there exists $c, d \in \overline{\mathbf{R}}$ with $c \leq d$ and $r : [c, d] \longrightarrow [a, b]$ left finite increasing (resp. decreasing) rearrangement of u if and only if u is left (resp. right) convergent distributional*

2. *there exists $c, d \in \overline{\mathbf{R}}$ with $c \leq d$ and $r : [c, d] \longrightarrow [a, b]$ right finite increasing (resp. decreasing) rearrangement of u if and only if u is right (resp. left) convergent distributional*

Proof. If follows at once from theorem 3.3.1.

If u is left (resp. right) convergent distributional, we can consider the increasing distribution function as a monotone class $M = [m]_m$ of function $m : [a, b] \longrightarrow [0, \mu(X)]$ (resp. $m : [a, b] \longrightarrow [-\mu(X), 0]$); let M^{-1} the inverse monotone class of M; then the increasing rearrangement of u defined on $[0, \mu(X)]$ are the elements of M^{-1}. In this case we can define the increasing rearrangement of u as the monotone class M^{-1}. If M^{-1} has only one element (for instance if m is strictly increasing), we can call it **the** increasing rearrangement of u and denote it by u^*; otherwise we can denote (by abuse) by u^* a generic element of M^{-1} or an element chosen with some criterion for instance to be the least of the class M^{-1}; in the second case, then it arises the problem of which proprieties of u^* depend on this choice or on the class M^{-1}.

We have analogous considerations if u is left (resp. right) convergent distributional and we consider the decreasing distribution function as a monotone class $M = [m]_m$ of function $m : [a, b] \longrightarrow [-\mu(X), 0]$ (resp. $m : [a, b] \longrightarrow [0, \mu(X)]$).

7 Spherical rearrangements

7.1 Spherical rearrangements

We put $\mathbf{S}_n = \mathbf{R}^n \cup \{\infty\}$ and $|\infty| = +\infty$.

Let $(\mathbf{R}^n, L_{R^n}, \lambda)$ the measure space of Lebesgue on $\mathbf{R}^n$; let $i : \mathbf{R}^n \longrightarrow \mathbf{S}_n$, $x \longrightarrow x$; the Lebesgue measure space $(\mathbf{S}_n, L_{S_n}, \lambda_{S_n})$ on $\mathbf{S}_n$ is the image by i of $(\mathbf{R}^n, L_{R^n}, \lambda)$.

Definition 7.1.1 *Let $n \in \mathbf{N}$; let $R \in [0, +\infty]$; then we put*

$$B^n_R = \{x \in \mathbf{S}_n; |x| \leq R\} .$$

We also note B_R^n with B_R.

Definition 7.1.2 *Let (X,S,μ) a measure space; let $a,b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a,b]$ measurable; let $n \in \mathbf{N}$; let $R \in [0,+\infty]$; let $v : B_R^n \longrightarrow [a,b]$; we say that v is a spherical increasing (resp. decreasing) rearrangement of u if*

$$(\forall x,y \in B_R^n)\,(|x| \leq |y| \Rightarrow v(x) \leq v(y))$$

$$\textit{(resp. } (\forall x,y \in B_R)\,(|x| \leq |y| \Rightarrow v(x) \geq v(y))\textit{)}$$

and if v is a rearrangement of u.

A spherical increasing (resp. decreasing) rearrangement of u is a radial function, i.e. we have $((\forall x,y \in B_R)\ (|x| = |y|) \Rightarrow v(x) = v(y)$.

7.2 Existence of spherical rearrangements

Let $V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$ the measure of the unit ball of $\mathbf{R}^n$.

The following statements follows at once from the definitions.

Theorem 7.2.1 *Let (X,S,μ) a measure space; let $a,b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a,b]$ measurable; let $d \in [0,+\infty]$; let $r : [0,d] \longrightarrow [a,b]$ an increasing (resp. decreasing) rearrangement of u ; if $d = +\infty$, let $R = +\infty$; if $d \in \mathbf{R}$, let $R = V_n^{-\frac{1}{n}} d^{\frac{1}{n}}$; let*

$$v : B_R \longrightarrow [a,b], x \longrightarrow r(V_n|x|^n)\ ;$$

then v is a spherical increasing (resp. decreasing) rearrangement of u.

Theorem 7.2.2 *Let (X,S,μ) a measure space; let $a,b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a,b]$ measurable; let $R \in [0,+\infty]$; let $v : B_R \longrightarrow [a,b]$ a spherical increasing (resp. decreasing) rearrangement of u; if $R = +\infty$, let $d = +\infty$; if $R \in \mathbf{R}$, let $d = V_n R^n$; let*

$$r : [0,d] \longrightarrow [a,b], \longrightarrow v((\sqrt[n]{\frac{t}{V_n}}, 0, \ldots, 0))$$

then r is an increasing (resp. decreasing) rearrangement of u.

From theorems 7.2.1, 7.2.2, 6.4.1 it follows at once the condition for the existence of spherical rearrangements.

Theorem 7.2.3 *Let (X,S,μ) a measure space; let $a,b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a,b]$ measurable; there exists $R \in [0,+\infty]$ and $v : B_R \longrightarrow [a,b]$ increasing (resp. decreasing) spherical rearrangement of u if and only if u left (risp. right) convergent distributional.*

Let (X,S,μ) a measure space; let $a,b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a,b]$ measurable; let $R \in [0,+\infty]$ and $v : B_R \longrightarrow [a,b]$; since v is a rearrangement of u if and only if the restriction $v|(B_R - \{x \in B_R;\ |x| = R\})$ is a rearrangement of u, theorem 7.2.1 gives also a necessary and sufficient condition for the existence of "increasing (resp. decreasing) rearrangements on $\{x \in B_R; |x| < R\}$".

8 Rearrangement according to a real function f

8.1 Rearrangement according to a real function f

Definition 8.1.1 *Let (Y, T, ν) a measure space; let $\alpha, \beta \in \overline{\mathbf{R}}$; let $\alpha \leq \beta$; let $f : Y \longrightarrow [\alpha, \beta]$ measurable; we say that f is a rearranging function if f is distributional and if*

$$(\forall t \in [\alpha, \beta])\, \nu(f^{-1}(\{t\})) = 0 \ .$$

The function $f : \mathbf{R}^n \longrightarrow [0, +\infty]$, $x \longrightarrow |x|$ is rearranging function respect the Lebesgue measure space.

Let $m : [\alpha, \beta] \longrightarrow [c, d]$ an increasing distribution function of a rearranging function f; by theorem 5.2.2 it follows that m is continuous and surjective.

Definition 8.1.2 *Let (X, S, μ), (Y, T, ν) measure spaces; let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a, b]$ μ-measurable; let $v : Y \longrightarrow [a, b]$ ν-measurable; let $\alpha, \beta \in \overline{\mathbf{R}}$; let $\alpha \leq \beta$; let $f : Y \longrightarrow [\alpha, \beta]$ ν-measurable; let f a rearranging function; we say that v is an increasing (resp. decreasing) rearrangement of u according to f if*

$$(\forall y, y' \in Y)\, (f(y) \leq f(y') \Rightarrow v(y) \leq v(y'))$$

$$\textit{(resp. } (\forall y, y' \in Y)\, (f(y) \leq f(y') \Rightarrow v(y) \geq v(y'))\textit{)}$$

and if v is a rearrangement of u.

If $f : \mathbf{R}^n \longrightarrow [0, +\infty]$, $x \longrightarrow |x|$ or $f : \mathbf{R}^n \longrightarrow [0, +\infty]$, $x \longrightarrow |x|^2$ is the rearranging function (respect the Lebesgue measure space), then the rearrangements of u according to f are the spherical rearrangements of u on $\mathbf{R}^n$.

We can also define the elliptic rearrangements.

Let $a \in \mathbf{R}^n$; let $a_i > 0$ for all $i = 1, 2, \ldots, n$; let $f : \mathbf{R}^n \longrightarrow [0, +\infty]$, $x \longrightarrow \sum_{i=1}^{n} \frac{x_i^2}{a_i^2}$; let us refer to the Lebesgue measure space $(\mathbf{R}^n, L_{R^n}, \lambda)$; then the rearrangements of u according to f are called the elliptic rearrangements of u of coefficient a.

In the following theorem we construct an increasing rearrangement according to a function f; we follow an idea of [12].

Theorem 8.1.1 *Let (X, S, μ), (Y, T, ν) measure spaces; let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a, b]$ μ-measurable; let $\alpha, \beta \in \overline{\mathbf{R}}$; let $\alpha \leq \beta$; let $f : Y \longrightarrow [\alpha, \beta]$ ν-measurable; let f a rearranging function; let $\gamma, \delta \in \overline{\mathbf{R}}$; let $\gamma \leq \delta$; let $m : [\alpha, \beta] \longrightarrow [\gamma, \delta]$ an increasing distribution function of f; let $c, d \in \overline{\mathbf{R}}$; let $c \leq d$; let $r : [c, d] \longrightarrow [a, b]$ an increasing (resp. decreasing) rearrangement of u; suppose there exists $K \in \mathbf{R}$ such that the function*

$$\tau_K : [\gamma, \delta] \longrightarrow [c, d], x \longrightarrow x + K$$

is bijective; let

$$v = r \circ \tau_K \circ m \circ f \ ;$$

then v is an increasing (resp. decreasing) rearrangement of u according to f.

Proof. If $y, y' \in Y$ and if $f(y) \leq f(y')$ we have $v(y) \leq v(y')$.
Let us prove that v is a rearrangement of f.
Let $s : [\gamma, \delta] \longrightarrow [\alpha, \beta]$ a monotone inverse of m; as m is continuous and surjective, s is a strictly increasing rearrangement of f; so we have $m \circ s = 1_{[c,d]}$.
Let A a Borel set of $[a, b]$; we have

$$\nu(v^{-1}(A)) = \nu((r \circ \tau_K \circ m \circ f)^{-1}(A)) = \nu(f^{-1}(m^{-1}(\tau_K^{-1}(r^{-1}(A))))) =$$

$$\lambda(s^{-1}(m^{-1}(\tau_K^{-1}(r^{-1}(A))))) = \lambda((m \circ s)^{-1}(\tau_K^{-1}(r^{-1}(A)))) =$$
$$\lambda(\tau_K^{-1}(r^{-1}(A))) = \lambda(r^{-1}(A)) = \mu(u^{-1}(A)) \ ;$$

then we have $u(\mu) = v(\nu)$.

8.2 Existence of rearrangements according to a real function

From theorem 8.1.1 it follows that the existence of a rearrangement of u according to f is tied to the existence of an increasing rearrangement of u defined on a translated of $[\gamma, \delta]$; we observe that the condition $\nu(Y) = \mu(X)$ is a necessary condition for the existence of such a rearrangement; the example of spherical rearrangements shows that this condition is not sufficient.

For the question of the existence of rearrangements according to a function f we make four cases.

Theorem 8.2.1 *Let (X, S, μ), (Y, T, ν) measure spaces; let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a, b]$ μ-measurable; let $\nu(Y) = \mu(X)$; let $\alpha, \beta \in \overline{\mathbf{R}}$; let $\alpha < \beta$; let $f : Y \longrightarrow [\alpha, \beta]$ ν-measurable; let f a rearranging function; let f left-right divergent distributional; then there exists $v : Y \longrightarrow [a, b]$ ν-measurable, v increasing (resp. decreasing) rearrangement of u according to f if and only if u is left-right divergent distributional.*

Proof. Since f is left-right divergent distributional, we have $\nu(Y) = \mu(X) = +\infty$. In particular we have $\nu \neq 0$. Suppose there exists $v : Y \longrightarrow [a, b]$ ν-measurable, v increasing rearrangement of u according to f. Let $a_1, b_1 \in \overline{\mathbf{R}}$ with $a_1 \leq b_1$ such that $[a_1, b_1]$ is the convex support of $v(\nu)$. If $a_1 = b_1$ then $v(\nu) = +\infty\delta_{a_1}$; then u is left-right divergent distributional. Suppose $a_1 < b_1$; let $a_1 < y < b_1$ and let us prove that $\mu(v^{-1}([a, y])) = +\infty$. We have $a_1 = \text{ess.inf}(v)$; then we have $\nu(v^{-1}([a, y])) \neq 0$; let $\alpha_1, \beta_1 \in \overline{\mathbf{R}}$ with $\alpha_1 \leq \beta_1$ such that $[\alpha_1, \beta_1]$ is the convex support of $f(\nu)$; as $\nu(f^{-1}(\{\alpha_1\})) = 0$, we have $\nu(f^{-1}([\alpha, \alpha_1])) = 0$; we have also $\nu(v^{-1}([a, y]) - f^{-1}([\alpha, \alpha_1])) \neq 0$; then there exists $x_0 \in v^{-1}([a, y])$ such that $f(x_0) > \alpha_1$; since $v(x_0) \leq y$, we have $v^{-1}([a, v(x_0)]) \subset v^{-1}([a, y])$; from the definition of rearrangement according to f, it follows $f^{-1}([\alpha, f(x_0)]) \subset v^{-1}([a, v(x_0)])$; then we have $f^{-1}([\alpha, f(x_0)]) \subset v^{-1}([a, y])$; then we have $\nu(f^{-1}([\alpha, f(x_0)])) \leq \nu(v^{-1}([a, y]))$; as $f(x_0) > \alpha_1$, we have $\nu(f^{-1}([\alpha, f(x_0)])) = +\infty$; then we have $\mu(v^{-1}([a, y])) = \nu(v^{-1}([a, y])) = +\infty$. Analogously we prove that $\mu(v^{-1}([y, b])) = +\infty$. This proves that u is left-right divergent distributional.

Suppose u left-right divergent distributional; there exists $m : [\alpha, \beta] \longrightarrow [-\infty, +\infty]$ increasing distribution function of f; there exists $s : [a, b] \longrightarrow [-\infty, +\infty]$ increasing distribution function of u; let $r : [-\infty, +\infty] \longrightarrow [a, b]$ an monotone inverse of s; then the existence of a rearrangement v of u according to f follows from theorem 8.1.1.

Theorem 8.2.2 *Let (X, S, μ), (Y, T, ν) measure spaces; let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a, b]$ μ-measurable; let $\nu(Y) = \mu(X)$; let $\alpha, \beta \in \overline{\mathbf{R}}$; let $\alpha \leq \beta$; let $f : Y \longrightarrow [\alpha, \beta]$ ν-measurable; let f a rearranging function; let f left convergent distributional; then there exists $v : Y \longrightarrow [a, b]$ ν-measurable, v increasing (resp. decreasing) rearrangement of u according to f if and only if u is left (resp. right) convergent distributional.*

Proof. Suppose there exists $v : Y \longrightarrow [a, b]$ ν-measurable, v increasing rearrangement of u according to f. If $\nu = 0$, then u is u is left convergent distributional. Suppose $\nu \neq 0$. Let $a_1, b_1 \in \overline{\mathbf{R}}$ with $a_1 \leq b_1$ such that $[a_1, b_1]$ is the convex support of $v(\nu)$. If $a_1 = b_1$ then $u(\mu) = v(\nu) = \mu(X)\delta_{a_1}$; then u is left convergent distributional. Suppose $a_1 < b_1$; let $y < b_1$; we have $b_1 = \text{ess.sup}(v)$; then we have $\nu(v^{-1}(]y, b])) \neq 0$; let $\alpha_1, \beta_1 \in \overline{\mathbf{R}}$ with $\alpha_1 \leq \beta_1$ such that $[\alpha_1, \beta_1]$ is the convex support of $f(\nu)$; since $\nu(f^{-1}(\{\beta_1\})) = 0$, we have $\nu(f^{-1}([\beta_1, \beta])) = 0$; then we have $\nu(v^{-1}(]y, b]) - f^{-1}([\beta_1, \beta])) \neq 0$; then there exists $x_0 \in v^{-1}(]y, b])$ such that $f(x_0) < \beta_1$; since $y < v(x_0)$, we have $v^{-1}([a, y]) \subset v^{-1}([a, v(x_0)[)$; from the definition of rearrangement according to f, it follows $v^{-1}([a, v(x_0)[) \subset f^{-1}([\alpha, f(x_0)[)$; then we have $v^{-1}([a, y]) \subset f^{-1}([\alpha, f(x_0)])$; then we have $\nu(v^{-1}([a, y])) \leq \nu(f^{-1}([\alpha, f(x_0)]))$; since $f(x_0) < \beta_1$, we have $\nu(f^{-1}([\alpha, f(x_0)])) < +\infty$; then we have $\nu(v^{-1}([a, y])) < +\infty$. This proves that u is left convergent distributional.
Suppose u left convergent distributional; there exist $\gamma \in \mathbf{R}$ and $\delta \in \overline{\mathbf{R}}$ with $\gamma \leq \delta$ and there exists $m : [\alpha, \beta] \longrightarrow [\gamma, \delta]$ increasing distribution function of f; we have $\delta - \gamma = \nu(Y)$; there exists $c \in \mathbf{R}$ and $d \in \overline{\mathbf{R}}$ with $c \leq d$ and exists $s : [a, b] \longrightarrow [c, d]$ increasing distribution function of u; we have $d - c = \mu(X)$; so we have $d - c = \delta - \gamma$; let $k = c - \gamma$; we have $c = \gamma + k$ and $d = \delta + k$; so the function

$$\tau_k : [\gamma, \delta] \longrightarrow [c, d], x \longrightarrow x + k$$

is bijective; let $r : [c, d] \longrightarrow [a, b]$ an inverse monotone of s; as r is an increasing rearrangement of u, the existence or a rearrangement v of u according to f follows from theorem 8.1.1.

Analogously:

Theorem 8.2.3 *Let (X, S, μ), (Y, T, ν) measure spaces; let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a, b]$ μ-measurable; let $\nu(Y) = \mu(X)$; let $\alpha, \beta \in \overline{\mathbf{R}}$; let $\alpha \leq \beta$; let $f : Y \longrightarrow [\alpha, \beta]$ ν-measurable; let f a rearranging function; let f right convergent distributional; then there exists $v : Y \longrightarrow [a, b]$ ν-measurable, v increasing (resp. decreasing) rearrangement of u according to f if and only if u is right (resp. left) convergent distributional.*

Theorem 8.2.4 *Let X a set; let S a σ-algebra on X; let μ a positive measure on X; let $\mu(X) < +\infty$; let $a, b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a, b]$ μ-measurable; let Y a set; let*

T a σ-algebra on Y; let ν a positive measure on Y; let $\nu(Y) = \mu(X)$; let $\alpha, \beta \in \overline{\mathbf{R}}$; let $\alpha \leq \beta$; let $f : Y \longrightarrow [\alpha, \beta]$ ν-measurable; let f a rearranging function; then there exists $v : Y \longrightarrow [a, b]$ ν-measurable, v rearrangement of u according to f.

Proof. There exist $\gamma, \delta \in \mathbf{R}$ with $\gamma \leq \delta$ and there exists $m : [\alpha, \beta] \longrightarrow [\gamma, \delta]$ increasing distribution function of f; we have $\delta - \gamma = \nu(Y)$; there exists $c, d \in \mathbf{R}$ with $c \leq d$ and exists $s : [a, b] \longrightarrow [c, d]$ increasing distribution function of u; we have $d - c = \mu(X)$; so we have $d - c = \delta - \gamma$; let $k = c - \gamma$; we have $c = \gamma + k$ and $d = \delta + k$; so the function

$$\tau_k : [\gamma, \delta] \longrightarrow [c, d], x \longrightarrow x + k$$

is bijective; let $r : [c, d] \longrightarrow [a, b]$ an inverse monotone of s; as r is an increasing rearrangement of u, the existence of a rearrangement v of u according to f follows from theorem 8.1.1.

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