The Dirichlet Integral and the Rearrangements According to a Function

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Abstract

We compare the Dirichlet integrals of a function u on $\mathbf{R}^n$ and of an "approximately spherical" rearrangement v of u according to a function f with "approximately straight" lines of steepest variation, proving that the Dirichlet integral of v is $\leq$ of the product of a suitable constant times the Dirichlet integral of u; we give an application for the elliptic rearrangements.

Key words: Rearrangements, Image of a measure, Distribution function, Comparison results.

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Riassunto

Si confrontano gli integrali di Dirichlet di una funzione u su $\mathbf{R}^n$ e di un suo riordinamento "approssimativamente sferico" secondo una funzione f avente le linee di massima variazione "approssimativamente rettilinee", provando che l'integrale di Dirichlet del riordinamento è $\leq$ del prodotto di un'opportuna costante per l'integrale di Dirichlet della funzione; si dà una applicazione ai riordinamenti ellittici.

Introduction

Following G. Talenti (see [13]), we consider the rearrangements according to a real function f. Roughly speaking, this means that as for the spherical rearrangements the rearranged function increases (or decreases) with the hypersurfaces $|x| = t$, so for the rearrangements according to f the rearranged function increases (or decreases) with the hypersurfaces $f(x) = t$. In [9] the author deals with such rearrangements, giving definitions and pointing out some proprieties.

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In [13], G. Talenti has proved that the Dirichlet integral of an increasing rearrangement $u^\star$ according to f of a function u is $\leq$ of the Dirichlet integral of u. He supposes f and u defined in an open set X of $\mathbf{R}^n$; on X he considers an absolutely continuous measure m; he supposes that there exists on X an isoperimetric inequality $q(m(E)) \leq p(E)$, where $p(E)$ is the perimeter of E and q is a positive function; the isoperimetric inequality is related to f by the equality

$$q(m\{x \in X; f(x) > t\}) = p(\{x \in X; f(x) > t\}) ;$$

furthermore he supposes that the lines of steepest descent of f are straight; with some other hypotheses, for $p \in [1, +\infty[$, he proves

$$\int_X |\mathrm{grad}\, u^\star|^p \, dm \leq \int_X |\mathrm{grad}\, u|^p \, dm$$

In this paper we consider on $\mathbf{R}^n$ the usual isoperimetric inequality; we rearrange a function u defined in $\mathbf{R}^n$ according to a function f; we suppose that the lines of steepest variation of f are "approximately straight"; this means that there exists $k_1, k_2 > 0$ and a positive function q such that

$$(\forall x \in \mathbf{R}^n) \; k_1 q(f(x)) \leq |\mathrm{grad}\, f(x)| \leq k_2 q(f(x)) ;$$

we suppose that the rearrangement according to f is "approximately spherical"; this means that there exists $M > 0$ such that for almost every t

$$H_{n-1}(f^{-1}\{t\}) \leq MnV_n^{\frac{1}{n}} \left(\lambda(\{x \in \mathbf{R}^n; f(x) < t\})\right)^{\frac{n-1}{n}}$$

(H_{n-1} in the $(n-1)$-Hausdorff measure, λ is the Lebesgue measure, V_n is the measure of the unit ball of $\mathbf{R}^n$); if v is an increasing or decreasing rearrangement according to f, for $p \in [1, +\infty[$, we prove

$$\int_{R^n} |\mathrm{grad}\; v|^p \, d\lambda \leq \left(\frac{Mk_2}{k_1}\right)^p \int_{R^n} |\mathrm{grad}\; u|^p \, d\lambda \; .$$

If $f(x) = |x|^2$, then $\frac{Mk_2}{k_1} = 1$; in this case we obtain the classical result for the spherical rearrangements. If $f(x) = \sum_{i=1}^n \frac{x_i^2}{a_i^2}$ we find a result for the elliptic rearrangements.

Regard to the hypotheses of regularity, we take u in $\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$ and the rearranging function f locally lipschitzian.

1 Monotone functions and rearrangements

We recall some of the definitions and results of [9]. We refer only to increasing functions, implying the analogous notions for decreasing functions.

Let $a, b, c, d \in \overline{\mathbf{R}}$, with $a \le b$, $c \le d$; let $f : [a, b] \longrightarrow [c, d]$ increasing; let $x \in [a, b]$; we put $f(x+) = \inf_{t \in]x, b]} f(t)$ and $f(x-) = \sup_{t \in [a, x[} f(t)$, where sup and inf are valued respect the ordered set $[c, d]$.

If $g : [a, b] \longrightarrow [c, d]$ is another increasing function, we say that f is monotonically equivalent to g if, for all $x \in [a, b]$, $f(x+) = g(x+)$, $f(x-) = g(x-)$.

We have so defined an equivalence relation on the set of the increasing functions $f : [a, b] \longrightarrow [c, d]$. Each equivalence class $[f]_m$ has least and greatest element.

The class $[f]_m$ has only one element f if and only if f is continuous and surjective.

Let $f : [a, b] \longrightarrow [c, d]$ increasing; let $g : [c, d] \longrightarrow [a, b]$ increasing; we say that g is monotone inverse of f if for all $x \in [a, b]$ and for all $y \in [c, d]$

$$(f(x) < y \Rightarrow x \le g(y)) \text{ and } (y < f(x) \Rightarrow g(y) \le x) \ .$$

The relation of inverse monotone is compatible with the equivalence relation of monotone functions; so we can define the inverse monotone $[f]_m^{-1}$ of a monotone class $[f]_m$.

There is an important not symmetric case: if g is monotone inverse of f, then f is strictly monotone if and only if g is continuous and surjective; in this case g is the only monotone inverse of f and $g \circ f$ is the identity. We note this monotone function g by $f^{-1,m}$.

Let $f : [a, b] \longrightarrow [c, d]$ increasing; let $g : [c, d] \longrightarrow [a, b]$ monotone inverse of f; then $\max([g]_m)$ is the greatest element of the set of the functions which are monotone inverse of f and $\min([g]_m)$ is the least element of this set.

We have the following symmetric result.

Let $f : [a, b] \longrightarrow [c, d]$ increasing; let $g : [c, d] \longrightarrow [a, b]$ monotone inverse of f; let $f_1 = \min([f]_m)$, $f_2 = \max([f]_m)$, $g_1 = \min([g]_m)$, $g_2 = \max([g]_m)$; then

$$(\forall x \in [a, b])(\forall y \in [c, d])\, f_1(x) \le y \le f_2(x) \Leftrightarrow g_1(y) \le x \le g_2(y) \ . \tag{1}$$

Suppose $g : [c, d] \longrightarrow [a, b]$ the greatest monotone inverse of f; then

$$(\forall x \in [a, b]) g(f(x)) \ge x \ . \tag{2}$$

Suppose $g : [c, d] \longrightarrow [a, b]$ the least monotone inverse of f; then

$$(\forall x \in [a, b]) g(f(x)) \le x \ . \tag{3}$$

The notion of rearrangement is stated in terms of image of a measure.

We recall that if (X, S, μ) is a measure space, if (Y, T) a is measurable space, if $u : X \longrightarrow Y$ is such that for all $A \in T$, $u^{-1}(A) \in S$, then the image $u(\mu)$ of the measure μ is the positive measure on T such that for all $A \in T$, $(u(\mu))(A) = \mu(u^{-1}(A))$.

If $f : Y \longrightarrow \overline{\mathbf{R}}$ is positive and measurable respect the measurable space (Y, T), then we have in $\overline{\mathbf{R}}$

$$\int_Y f(y)\, d(u(\mu))(y) = \int_X f(u(x))\, d\mu(x) \ .$$

Let (X,S,μ), (Y,T,ν) measure spaces; let I a closed interval of $\overline{\mathbf{R}}$; let us consider on I the σ-algebra $\mathcal{B}_I$ of the Borel sets; let $u : X \longrightarrow I$ and $v : Y \longrightarrow I$ measurable functions; we say that u is a rearrangement of v if

$$u(\mu) = v(\nu) \ .$$

If Y is an interval of $\overline{\mathbf{R}}$ (on which we consider the Lebesgue measure λ, i.e. the induced measure on Y of the image of the Lebesgue measure on $\mathbf{R}$ by $\mathbf{R} \longrightarrow \overline{\mathbf{R}}, x \longrightarrow x$) and if v is increasing, we obtain the definition of increasing rearrangement. If Y is a ball of $\mathbf{R}^n$ with centre 0 or $\mathbf{R}^n$ and if v satisfies $|x| \leq |y| \Rightarrow v(x) \leq v(y)$ we obtain the definition of increasing spherical rearrangement.

Let $f : [a,b] \longrightarrow [c,d]$ increasing; the distributional derivative $D_{\text{meas}}f$ of f is the only positive measure on the σ-algebra of the borelian $\mathcal{B}_{[a,b]}$, such that for all $x, x' \in [a,b]$, $x \leq x'$,

$$(D_{\text{meas}}f)([x,x']) = f(x'+) - f(x-) \ .$$

It is interesting the relation between the distributional derivative and the image of a measure: the distributional derivative of f is the image of the Lebesgue measure on $[c,d]$ by a monotone inverse g of f, i. e.

$$D_{\text{meas}}f = g(\lambda) \ .$$

We consider a positive measure $\nu : \mathcal{B}_{[a,b]} \longrightarrow \overline{\mathbf{R}}$; we say that ν is an exact measure if there exists a closed interval J of $\overline{\mathbf{R}}$, and $f : [a,b] \longrightarrow J$ increasing such that

$$D_{\text{meas}}f = \nu \ .$$

Let (X,S,μ) a measure space; let $a,b \in \overline{\mathbf{R}}$; let $a \leq b$; let $u : X \longrightarrow [a,b]$ measurable; we say that u is a distributional function if the image measure $u(\mu)$ is an exact measure.

For a distributional function $u : X \longrightarrow [a,b]$, a function $m : [a,b] \longrightarrow [c,d]$ increasing, such that

$$D_{\text{meas}}m = u(\mu)$$

is defined as an increasing distribution function of u.

From this, it follows at once that the increasing rearrangements of a distributional function f are the monotone inverse functions of the increasing distribution functions of f.

So the distributional functions are the functions u for which there exist increasing rearrangements of u or equivalently for which we can (significantly) define a distribution function of u.

We recall ([9], [3], [5]) that if $a_1 = \text{ess.inf}(u)$, $b_1 = \text{ess.sup}(u)$, then u is distributional if and only if

1. $(\forall x, x' \in]a_1, b_1[\,, x \leq x')\ \mu(u^{-1}([x,x'])) < +\infty$;
2. $\mu(u^{-1}(\{b_1\})) = 0$ or $(\forall x \in]a_1, b_1[)\ \mu(u^{-1}([x, b_1[)) < +\infty$;

3. $\mu(u^{-1}(\{a_1\})) = 0$ or $(\forall x \in]a_1, b_1[)\ \mu(u^{-1}(]a_1, x])) < +\infty$.

We say that $u : X \longrightarrow [a, b]$ is left (resp. right) convergent distributional if there exists an increasing distribution function of u, $m : [a, b] \longrightarrow [c, d]$, with $c \in \mathbf{R}$ (resp. $d \in \mathbf{R}$).

The left (resp. right) convergent distributional functions are the functions u for which there exist spherical increasing (resp. decreasing) rearrangements of u.

We recall [9] that if $b_1 = \text{ess.sup}(u)$ then u is a left (resp. right) convergent exact measure if and only if

1. $(\forall x \in [a, b_1[)\ \mu(u^{-1}([a, x])) < +\infty$ (resp. $(\forall x \in]a_1, b])\ \mu(u^{-1}([x, b])) < +\infty$);
2. $\mu(u^{-1}(\{b_1\})) = 0$ or $\mu(u^{-1}([a, b_1[)) < +\infty$ (resp. $\mu(u^{-1}(\{a_1\})) = 0$ or $\mu(u^{-1}(]a_1, b])) < +\infty$).

Following Talenti [13], we consider the rearrangements according to a function f. We need some condition on f.

Let (Y, T, ν) a measure space; let $f : Y \longrightarrow [\alpha, \beta]$ measurable; we say that f is a rearranging function if f is distributional and if

$$(\forall t \in [\alpha, \beta])\, \nu(f^{-1}(\{t\})) = 0\ .$$

Let (X, S, μ), (Y, T, ν) measure spaces; let $u : X \longrightarrow [a, b]$ μ-measurable; let $v : Y \longrightarrow [a, b]$ ν-measurable; let $f : Y \longrightarrow [\alpha, \beta]$ ν-measurable; let f a rearranging function; we say that v is an increasing rearrangement of u according to f if

$$(\forall y, y' \in Y)\, (f(y) \leq f(y') \Rightarrow v(y) \leq v(y'))$$

and if v is a rearrangement of u.

2 Ordinary derivative of monotone functions

Let $a, b, c, d \in \overline{\mathbf{R}}$; let $a < b$ and $c \leq d$; let $f : [a, b] \longrightarrow [c, d]$ increasing (resp. decreasing); let $f(]a, b[) \subset \mathbf{R}$; let f' the ordinary derivative of f defined almost everywhere in $]a, b[$; then $f'\lambda$ is the absolutely continuous part of the measure $D_{\text{meas}} f|]a, b[$; from this it follows that if $g : [a, b] \longrightarrow [c, d]$ is monotonically equivalent to f, than $g'(x) = f'(x)$ for almost every $x \in]a, b[$. More precisely we can find $A \subset]a, b[$ such that $\lambda(]a, b[-A) = 0$ and such that every $g \in [f]_m$ is derivable on A with the same derivative; it follows from the following theorem.

Theorem 2.1 *Let $a, b, c, d \in \overline{\mathbf{R}}$; let $a < b$ and $c \leq d$; let $f : [a, b] \longrightarrow [c, d]$ increasing (resp. decreasing); let $f(]a, b[) \subset \mathbf{R}$; let $f_1 = \min([f]_m)$ and $f_2 = \max([f]_m)$; let $x_0 \in]a, b[$; let f_1 and f_2 derivable at x_0 and $f_1'(x_0) = f_2'(x_0)$; then f is derivable at x_0 and $f'(x_0) = f_1'(x_0) = f_2'(x_0)$.*

Proof. We have $f(x_0) = f_1(x_0) = f_2(x_0)$. Let $x \in]a, b[$, $x > x_0$; we have

$$\frac{f_1(x) - f_1(x_0)}{x - x_0} \leq \frac{f(x) - f(x_0)}{x - x_0} \leq \frac{f_2(x) - f_2(x_0)}{x - x_0} \ ;$$

from this it follows that f is derivable on the right and that $f'_+(x_0) = f'_1(x_0) = f'_2(x_0)$. Analogously f is derivable on the left and $f'_-(x_0) = f'_1(x_0) = f'_2(x_0)$.
From this, the thesis.

It seems of some importance to precise when we can extend the usual rule for the derivative of the inverse function to the derivative of an inverse monotone of a monotone function f. We obtain this aim in the hypothesis of f strictly monotone. At the same time, we give another propriety of the set A, that we have considered above.

Theorem 2.2 *Let $a, b, c, d \in \overline{\mathbf{R}}$; let $a < b$ and $c \leq d$; let $f : [a, b] \longrightarrow [c, d]$ strictly increasing (resp. strictly decreasing); let $f(]a, b[) \subset \mathbf{R}$; let $f_1 = \min([f]_m)$ and $f_2 = \max([f]_m)$; let $x_0 \in]a, b[$; let f_1 and f_2 derivable at x_0 and $f'_1(x_0) = f'_2(x_0)$; then $f^{-1,m}$ has derivative in $\overline{\mathbf{R}}$ at $f(x_0)$ and*

$$(f^{-1,m})'(f(x_0)) = \frac{1}{f'(x_0)} \ .$$

Proof. Let $g = f^{-1,m}$. We have $g(f(x_0)) = x_0$. Let $y > f(x_0)$; by (2), (3) we have

$$\frac{g(y) - x_0}{f_2(g(y)) - f(x_0)} \leq \frac{g(y) - x_0}{y - f(x_0)} \leq \frac{g(y) - x_0}{f_1(g(y)) - f(x_0)} \ ;$$

since g is continuous, we have

$$\lim_{y \to f(x_0)+} \frac{g(y) - x_0}{f_2(g(y)) - f(x_0)} = \lim_{x \to x_0+} \frac{x - x_0}{f_2(x) - f(x_0)} = \frac{1}{f'_2(x_0)}$$

and analogously $\lim_{y \to f(x_0)+} \frac{g(y)-x_0}{f_1(g(y))-f(x_0)} = \frac{1}{f'_1(x_0)}$.
From theorem (2.1) it follows that g has derivative in $\overline{\mathbf{R}}$ on the right at $f(x_0)$ and that $g'_+(f(x_0)) = \frac{1}{f'(x_0)}$.
Analogously we see that g has derivative in $\overline{\mathbf{R}}$ on the left at $f(x_0)$ and that $g'_-(f(x_0)) = \frac{1}{f'(x_0)}$.
From this, the thesis.

3 Distribution function for $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$

In this section we consider a distributional function $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$ (Ω connected); let m a distribution function of u; we shall prove that m is strictly increasing; if $\operatorname{grad} u(x) \neq 0$ a. e., we shall prove that m is locally absolutely continuous.

Let Ω an open set of $\mathbf{R}^n$; let $A \subset \Omega$; we denote by φ_A the characteristic function of A defined on Ω. We denote by λ the Lebesgue measure on $\mathbf{R}^n$, on $\overline{\mathbf{R}}$ and on a measurable set of $\mathbf{R}^n$ or $\overline{\mathbf{R}}$.

Let $u : \Omega \longrightarrow \overline{\mathbf{R}}$; let $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$; we recall the following expression of the Radon measure $|\mathrm{grad}\, u|$ as integral of positive measures (*Coarea formula*):

$$|\mathrm{grad}\, u| = \int_{-\infty}^{+\infty} |\mathrm{grad}\, \varphi_{\{x\in\Omega;\, u(x)<t\}}|\, d\lambda(t) \; . \tag{4}$$

Let $a, b \in \overline{\mathbf{R}}$, $a \leq b$; if $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$, then (4) becomes

$$\varphi_{\{x\in\Omega;\, a\leq u(x)\leq b\}} \cdot |\mathrm{grad}\, u| = \varphi_{\{x\in\Omega;\, a<u(x)<b\}} \cdot |\mathrm{grad}\, u| = \int_a^b |\mathrm{grad}\, \varphi_{\{x\in\Omega;\, u(x)<t\}}|\, d\lambda(t) \; . \tag{5}$$

Let $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$; changing eventually u in in set of measure 0, we can always suppose that $\mathrm{ess.sup}(u) = \sup(u)$, and $\mathrm{ess.inf}(u) = \inf(u)$.

Theorem 3.1 *Let Ω a connected open set of $\mathbf{R}^n$; let $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$; let $\mathrm{ess.sup}(u) = \sup(u)$ and $\mathrm{ess.inf}(u) = \inf(u)$; let $a = \inf(u)$ and $b = \sup(u)$; let us consider $u : \Omega \longrightarrow [a, b]$; let u distributional; let $c, d \in \overline{\mathbf{R}}$, with $c \leq d$; let $m : [a, b] \longrightarrow [c, d]$ an increasing (resp. decreasing) distribution function of u; then m is strictly increasing (resp. strictly decreasing).*

Proof. Let $y_1, y_2 \in [a, b]$, with $y_1 < y_2$. Suppose by contradiction that $m(y_1) = m(y_2)$. Then we have $m(y_1+) = m(y_2-)$. By definition of distributional function, we have $\lambda(\{x \in \Omega;\, y_1 < u(x) < y_2\}) = m(y_2-) - m(y_1+) = 0$. Then we have

$$\int_{\{x\in\Omega;\, y_1<u(x)<y_2\}} |\mathrm{grad}\, u|\, d\lambda = 0 \; . \tag{6}$$

By (5) we have

$$\int_{\{x\in\Omega;\, y_1<u(x)<y_2\}} |\mathrm{grad}\, u|\, d\lambda = \int_{y_1}^{y_2} \left(\int_\Omega d|\mathrm{grad}\, \varphi_{\{x\in\Omega;\, u(x)<t\}}| \right) d\lambda(t) \; . \tag{7}$$

For all t_1, t_2 such that $y_1 < t_1 < t_2 < y_2$, we have

$$\{y \in \Omega;\; \varphi_{\{x\in\Omega;\, u(x)<t_1\}}(y) \neq \varphi_{\{x\in\Omega;\, u(x)<t_2\}}(y)\} \subset \{y \in \Omega;\; t_1 \leq u(y) < t_2\} \; ;$$

then we have

$$\varphi_{\{x\in\Omega;\, u(x)<t_1\}}(y) = \varphi_{\{x\in\Omega;\, u(x)<t_2\}}(y)$$

for almost every $y \in \Omega$; then we have

$$\mathrm{grad}\, \varphi_{\{x\in\Omega;\, u(x)<t_1\}} = \mathrm{grad}\, \varphi_{\{x\in\Omega;\, u(x)<t_2\}} \; ;$$

then the function $t \longrightarrow \int_\Omega d|\mathrm{grad}\, \varphi_{\{x\in\Omega;\, u(x)<t\}}|$ is a constant C on $]y_1, y_2[$; by (6) and (7) we have $C(y_2 - y_1) = 0$; then $C = 0$; then for all $t \in]y_1, y_2[$ the vectorial measure

$\mathrm{grad}\,\varphi_{\{x\in\Omega;\,u(x)<t\}}$ is 0; then $\varphi_{\{x\in\Omega;\,u(x)<t\}}$ is almost every constant; let $t \in]y_1, y_2[$; we have $\varphi_{\{x\in\Omega;\,u(x)<t\}}(y) = 0$ for almost every $y \in \Omega$ or $\varphi_{\{x\in\Omega;\,u(x)<t\}}(y) = 1$ for almost every $y \in \Omega$; then we have $t \leq \mathrm{ess.inf}(u)$ or $t \geq \mathrm{ess.sup}(u)$; this is absurd.

Let $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$; let $b = \mathrm{ess.sup}(u) = \sup(u)$ and $a = \mathrm{ess.inf}(u) = \inf(u)$; let us consider $u : \Omega \longrightarrow [a, b]$; let u distributional; let $c, d \in \overline{\mathbf{R}}$, with $c \leq d$; let $m : [a, b] \longrightarrow [c, d]$ an increasing (resp. decreasing) distribution function of u; then $m(]a, b[) \subset \mathbf{R}$; so we can consider the derivative of m at $t \in]a, b[$, which exists for almost every $t \in]a, b[$.

Theorem 3.2 *Let Ω a connected open set of $\mathbf{R}^n$; let $u : \Omega \longrightarrow \mathbf{R}$; let $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$; suppose $(\mathrm{grad}\,u)(x) \neq 0$ for almost every $x \in \Omega$; let $\mathrm{ess.sup}(u) = \sup(u)$ and $\mathrm{ess.inf}(u) = \inf(u)$; let $a = \inf(u)$ and $b = \sup(u)$; let us consider $u : \Omega \longrightarrow [a, b]$; let u distributional; let $c, d \in \overline{\mathbf{R}}$, with $c \leq d$; let $m : [a, b] \longrightarrow [c, d]$ an increasing (resp. decreasing) distribution function of u; then we have*

1. *u is a rearranging function;*
2. *m is continuous and surjective;*
3. *$m|]a, b[$ is locally absolutely continuous;*
4. *$m'(t) = \int_\Omega \frac{1}{|\mathrm{grad}\,u|} d|\mathrm{grad}\,\varphi_{\{x\in\Omega;u(x)<t\}}|$ for almost every $t \in]a, b[$;*
5. *$D_{\mathrm{meas}} m = m' \cdot \lambda$;*
6. *m is a homeomorphism;*
7. *if M is the set of $t \in]a, b[$, such that m derivable at t and $m'(t) = 0$, then $\lambda(u^{-1}(M)) = 0$.*

Proof. Let $t \in [a, b]$; since $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$, we have $\mathrm{grad}\,u(x) = 0$ for almost every $x \in \Omega$ such that $u(x) = t$; then $\lambda(\{x \in \Omega;\, u(x) = t\}) = 0$; then u in an arranging function.
Let $t \in [a, b]$; we have

$$m(t+) - m(t-) = \lambda(\{x \in \Omega; u(x) = t\}) = 0 \ .$$

then m in continuous and surjective.
Let $t_0, t \in]a, b[$; suppose $t_0 \leq t$; we have

$$m(t) - m(t_0) = m(t+) - m(t_0-) = \int_{\{x\in\Omega; t_0 \leq u(x) \leq t\}} d\lambda =$$

$$\int_{\{x\in\Omega; t_0 \leq u(x) \leq t\}} \frac{1}{|\mathrm{grad}\,u|} |\mathrm{grad}\,u| \, d\lambda = \int_{t_0}^{t} \left(\int_\Omega \frac{1}{|\mathrm{grad}\,u|} | \, d|\mathrm{grad}\,\varphi_{\{x\in\Omega;\,u(x)<s\}}| \right) d\lambda(s) \ ;$$

analogously we prove that if $t < t_0$ we have still

$$m(t) - m(t_0) = \int_{t_0}^{t} \left(\int_\Omega \frac{1}{|\mathrm{grad}\,u|} | \, d|\mathrm{grad}\,\varphi_{\{x\in\Omega;\,u(x)<s\}}| \right) d\lambda(s) \ .$$

From this it follows that $m|]a,b[$ is locally absolutely continuous and

$$m'(t) = \int_\Omega \frac{1}{|\mathrm{grad}\, u|}\, d|\mathrm{grad}\, \varphi_{\{x\in\Omega;\, u(x)<t\}}|$$

for almost every $t \in]a,b[$.
It follows also that $D_{\mathrm{meas}} m = m' \cdot \lambda$ on $]a,b[$. Since $(D_{\mathrm{meas}} m)(\{a\}) = (D_{\mathrm{meas}} m)(\{b\}) = 0$, we have also $D_{\mathrm{meas}} m = m' \cdot \lambda$ on $[a,b]$.
By theorem (3.1) m is strictly increasing; since m is continuous and strictly increasing, m is a homeomorphism.
We have

$$\lambda(u^{-1}(M)) = (u(\lambda))(M) = (D_{\mathrm{mis}}(m))(M) = \int_M m'(t)\, dt = 0\ .$$

4 Some inequalities for the Dirichlet integrals

The results of theorems (4.1) and (4.2) are classical; we expose them in the particulars owing to some differences in the hypotheses and in order to show that the statement of [9] permits straight proofs, essentially based on easy calculations.

Let $V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$ the measure of the unit ball of $\mathbf{R}^n$. We recall that if $E \subset \mathbf{R}^n$, E measurable, $\lambda(E) < +\infty$ and $\varphi_E \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^n)$ then

$$\int_{R^n} d|\mathrm{grad}\, \varphi_E| \geq n V_n^{\frac{1}{n}} (\lambda(E))^{\frac{n-1}{n}}$$

(*De Giorgi isoperimetric inequality*).

Let $u : \mathbf{R}^n \longrightarrow \overline{\mathbf{R}}$; let $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$; let $\sup(u) = \mathrm{ess.sup}(u) = b$ and $\inf(u) = \mathrm{ess.inf}(u) = a$; let us consider $u : \mathbf{R}^n \longrightarrow [a,b]$; let u distributional; let $c, d \in \mathbf{R}$, with $c \leq d$; let $m : [a,b] \longrightarrow [c,d]$ an increasing (resp. decreasing) distribution function of u; let $r : [c,d] \longrightarrow [a,b]$ an increasing (resp. decreasing) rearrangement of u; then we have $r(y) \in \mathbf{R}$ for every $y \in]c,d[$; so we can consider the derivative of r at y, which exists for almost every $y \in]c,d[$.

Let us suppose u left convergent distributional and the distribution function $m : [a,b] \longrightarrow [0,+\infty]$; then m is monotonically equivalent to

$$[a,b] \longrightarrow [0,+\infty], y \longrightarrow \lambda(\{x \in \mathbf{R}^n; u(x) < y\})\ ;$$

then by the isoperimetric inequality for almost every $y \in]a,b[$ we have

$$\int_{R^n} d|\mathrm{grad}\, \varphi_{\{x\in R^n;\, u(x)<y\}}| \geq n V_n^{\frac{1}{n}} (m(y))^{\frac{n-1}{n}}\ . \tag{8}$$

In the following lemma we give an extension of (8) in the hypothesis $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$.

Theorem 4.1 *Let* $u : \mathbf{R}^n \longrightarrow \overline{\mathbf{R}}$*; let* $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$*; let* $\sup(u) = \mathrm{ess.sup}(u)$ *and* $\inf(u) = \mathrm{ess.inf}(u)$*; let* $a = \inf(u)$ *and* $b = \sup(u)$*; let us consider* $u : \mathbf{R}^n \longrightarrow [a, b]$*; let us suppose* u *left (resp. right) convergent distributional; let* $m : [a, b] \longrightarrow [0, +\infty]$ *an increasing (resp. decreasing) distribution function; let* $p \in [1, +\infty[$*; then for almost every* $y \in]a, b[$ *we have*

$$\int_{R^n} |\mathrm{grad}\, u|^{p-1} d|\mathrm{grad}\, \varphi_{\{x\in R^n;\, u(x)<y\}}| \geq nV_n^{\frac{1}{n}} (m(y))^{p\frac{n-1}{n}} (m'(y))^{1-p} \ . \tag{9}$$

$$(\textit{resp.} \ \int_{R^n} |\mathrm{grad}\, u|^{p-1} d|\mathrm{grad}\, \varphi_{\{x\in R^n;\, u(x)<y\}}| \geq nV_n^{\frac{1}{n}} (m(y))^{p\frac{n-1}{n}} (-m'(y))^{1-p}) \ .$$

Proof. Let $y \in]a, b[$; let $r > 0$ such that $a < y - r$, $y + r < b$.
By Hölder inequality we have

$$\int_{u^{-1}([y-r,y+r])} |\mathrm{grad}\, u|^p \, d\lambda \geq \left(\int_{u^{-1}([y-r,y+r])} |\mathrm{grad}\, u| \, d\lambda \right)^p \left(\int_{u^{-1}([y-r,y+r])} d\lambda \right)^{p-1} \ .$$

By the coarea formula (5) and by (8) we have

$$\int_{u^{-1}([y-r,y+r])} |\mathrm{grad}\, u| \, d\lambda = \int_{y-r}^{y+r} \left(\int_{R^n} d|\mathrm{grad}\, \varphi_{\{x\in R^n;\, u(x)<t\}}| \right) d\lambda(t) \geq$$

$$nV_n^{\frac{1}{n}} \int_{y-r}^{y+r} (m(t))^{\frac{n-1}{n}} \, d\lambda(t) \ .$$

By definition of distributional function we have

$$\int_{u^{-1}([y-r,y+r])} d\lambda = D_{\mathrm{meas}} m([y - r, y + r]) = m((y + r)+) - m((y - r)-) \ .$$

By the coarea formula (5)we have

$$\int_{u^{-1}([y-r,y+r])} |\mathrm{grad}\, u|^p \, d\lambda = \int_{y-r}^{y+r} \left(\int_{R^n} |\mathrm{grad}\, u|^{p-1} \, d|\mathrm{grad}\, \varphi_{\{x\in R^n;\, u(x)<t\}}| \right) d\lambda(t) \ .$$

Then we have

$$\int_{y-r}^{y+r} \left(\int_{R^n} |\mathrm{grad}\, u|^{p-1} \, d|\mathrm{grad}\, \varphi_{\{x\in R^n;\, u(x)<t\}}| \right) d\lambda(t) \geq$$

$$n^p V_n^{\frac{p}{n}} \left(\int_{y-r}^{y+r} (m(t))^{\frac{n-1}{n}} \, d\lambda(t) \right)^p (m((y + r)+) - m((y - r)-))^{1-p} \ ;$$

then

$$\frac{\int_{y-r}^{y+r} \left(\int_{R^n} |\mathrm{grad}\, u|^{p-1} \, d|\mathrm{grad}\, \varphi_{\{x\in R^n;\, u(x)<t\}}| \right) d\lambda(t)}{2r} \geq$$

$$n^p V_n^{\frac{p}{n}} \left(\frac{\int_{y-r}^{y+r} (m(t))^{\frac{n-1}{n}} \, d\lambda(t)}{2r} \right)^p \left(\frac{m((y + r)+) - m((y - r)-)}{2r} \right)^{1-p} \ .$$

For $r \to 0$, we find (9).

The following theorem is an easy consequence of theorems (2.2) and (4.1).

Theorem 4.2 *Let* $u:\mathbf{R}^n \longrightarrow \overline{\mathbf{R}}$*; let* $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$*; let* $\sup(u) = \mathrm{ess.sup}(u)$ *and* $\inf(u) = \mathrm{ess.inf}(u)$*; let* $a = \inf(u)$ *and* $b = \sup(u)$*; let us consider* $u : \mathbf{R}^n \longrightarrow [a,b]$*; let us suppose* u *left (resp. right) convergent distributional; let* $r : [0,+\infty] \longrightarrow [a,b]$ *an increasing (resp. decreasing) rearrangement of* u*; let* $p \in [1,+\infty[$*; then we have*

$$\int_{R^n} |\mathrm{grad}\ u|^p \, d\lambda \geq n^p V_n^{\frac{p}{n}} \int_0^{+\infty} t^{p\frac{n-1}{n}} (r'(t))^p \, d\lambda(t)$$

$$\textit{(resp.}\ \int_{R^n} |\mathrm{grad}\ u|^p \, d\lambda \geq n^p V_n^{\frac{p}{n}} \int_0^{+\infty} t^{p\frac{n-1}{n}} (-r'(t))^p \, d\lambda(t)\ \textit{)}\ .$$

Proof. Let $m : [a,b] \longrightarrow [0,+\infty]$ a monotone inverse of r; m is an increasing distributional function of u. By theorem 4.1 we have

$$\int_{R^n} |\mathrm{grad}\, u|^{p-1} d|\mathrm{grad}\, \varphi_{\{x\in R^n;\, u(x)<y\}}| \geq n V_n^{\frac{1}{n}} (m(y))^{p\frac{n-1}{n}} (m'(y))^{1-p}\ .$$

By the coarea formula (5) we have

$$\int_{R^n} |\mathrm{grad}\, u|^p \, d\lambda \geq n^p V_n^{\frac{p}{n}} \int_a^b (m(y))^{p\frac{n-1}{n}} (m'(y))^{1-p} \, d\lambda(y)\ .$$

By theorem 2.2 for almost every $y \in]a,b[$ we have $m'(y) = \frac{1}{r'(m(y))}$; then we have

$$\int_a^b (m(y))^{p\frac{n-1}{n}} (m'(y))^{1-p} \, d\lambda(y) = \int_a^b (m(y))^{p\frac{n-1}{n}} (r'(m(y)))^{p-1} \, d\lambda(y)\,.$$

By definition of image of a measure and since $m(\lambda) = D_{\mathrm{meas}} r$, we have

$$\int_a^b (m(y))^{\frac{pn-p}{n}} (r'(m(y)))^{p-1} \, d\lambda(y) = \int_0^{+\infty} t^{\frac{pn-p}{n}} (r'(t))^{p-1} \, d(m(\lambda))(t) =$$

$$\int_0^{+\infty} t^{p\frac{n-1}{n}} (r'(t))^{p-1} \, d(D_{\mathrm{meas}} r)(t)\ .$$

As $D_{\mathrm{meas}} r \geq r' \cdot \lambda$, we have

$$\int_0^{+\infty} t^{p\frac{n-1}{n}} (r'(t))^{p-1} \, d(D_{\mathrm{meas}} r)(t) \geq \int_0^{+\infty} t^{p\frac{n-1}{n}} (r'(t))^p \, d\lambda(t)\ .$$

From this it follows at once the thesis.

Analogously:

Theorem 4.3 *Let* $u:\mathbf{R}^n \longrightarrow \overline{\mathbf{R}}$*; let* $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$*; let* $\sup(u) = \mathrm{ess.sup}(u)$ *and* $\inf(u) = \mathrm{ess.inf}(u)$*; let* $a = \inf(u)$ *and* $b = \sup(u)$*; let us consider* $u : \mathbf{R}^n \longrightarrow [a,b]$*; let us suppose* u *right (resp. left) convergent distributional; let* $r : [-\infty,0] \longrightarrow [a,b]$ *an increasing (resp. decreasing) rearrangement of* u*; let* $p \in [1,+\infty[$*; then we have*

$$\int_{R^n} |\mathrm{grad}\ u|^p \, d\lambda \geq n^p V_n^{\frac{p}{n}} \int_{-\infty}^{0} t^{p\frac{n-1}{n}} (r'(t))^p \, d\lambda(t)$$

$$\textit{(resp.}\ \int_{R^n} |\mathrm{grad}\ u|^p \, d\lambda \geq n^p V_n^{\frac{p}{n}} \int_{-\infty}^{0} t^{p\frac{n-1}{n}} (-r'(t))^p \, d\lambda(t)\ \textit{)}\ .$$

In the following theorem we prove that if f is a rearranging function with "approximately spherical" level hypersurfaces and with "approximately straight" lines of steepest variation, introducing a suitable constat, we can reverse the inequality of theorem (4.2). We need some lemmas.

If $A \subset \mathbf{R}^n$, we denote the topological frontier of A by $\mathrm{Fr}(A)$.

Lemma 4.1 *Let Ω an open set of $\mathbf{R}^n$; let $f : \Omega \longrightarrow \mathbf{R}$; let f continuous; let $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^n)$; let $\alpha = \inf(f)$ and $\beta = \sup(f)$; let $t \in [\alpha, \beta]$; let $\varphi_{\{x\in\Omega;\, f(x)<t\}} \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$; then*

1. $\mathrm{Supp}(|\mathrm{grad}\, \varphi_{\{x\in\Omega;\, f(x)<t\}}|) \subset \{x \in \Omega; f(x) = t\}$;
2. $f(x) = t$ *for* $|\mathrm{grad}\, \varphi_{\{x\in\Omega;\, f(x)<t\}}|$*-almost every* $x \in \mathbf{R}^n$.

Proof. We have

$$\mathrm{Supp}(|\mathrm{grad}\, \varphi_{\{x\in\Omega;\, f(x)<t\}}|) \subset \mathrm{Fr}(\{x \in \Omega; f(x) < t\}) \ .$$

From the continuity of f we have

$$\mathrm{Fr}(\{x \in \Omega; f(x) < t\}) \subset \overline{\{x \in \Omega; f(x) = t\}}) = \{x \in \Omega; f(x) = t\} \ .$$

The second affirmation follows from the first.

Lemma 4.2 *Let Ω an open set of $\mathbf{R}^n$; let $f : \Omega \longrightarrow \mathbf{R}$; let $f \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$; let $\mathrm{ess.sup}(f) = \sup(f)$ and $\mathrm{ess.inf}(f) = \inf(f)$; let $\alpha = \inf(f)$ and $\beta = \sup(f)$; let $A \subset \mathbf{R}^n$; let $\lambda(A) = 0$; then for almost every $t \in [\alpha, \beta]$ we have*

$$|\mathrm{grad}\, \varphi_{\{x\in R^n;\, f(x)<t\}}|(A) = 0 \ .$$

Proof. We have $(|\mathrm{grad}\, f| \cdot \lambda)(A) = 0$; then the thesis follows from (5).

Let $f : \mathbf{R}^n \longrightarrow [\alpha, \beta]$ a rearranging function; let $m : [\alpha, \beta] \longrightarrow [0, +\infty]$ an increasing distribution function of f; let $u : \mathbf{R}^n \longrightarrow [a, b]$ a left convergent distributional function; let $r : [0, +\infty] \longrightarrow [a, b]$ an increasing rearrangement of u; then (see [13] and [9]) $r \circ m \circ f$ is an increasing rearrangement of u according to f.

Theorem 4.4 *Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}$; let f locally lipschitzian; suppose $(\mathrm{grad}\, f)(x) \neq 0$ for almost every $x \in \mathbf{R}^n$; let $\alpha = \inf(f)$ and $\beta = \sup(f)$; let us consider $f : \mathbf{R}^n \longrightarrow [\alpha, \beta]$; let f left convergent distributional; let $m : [\alpha, \beta] \longrightarrow [0, +\infty]$ an increasing distribution function of f; let $u : \mathbf{R}^n \longrightarrow \mathbf{R}$; let $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$; let $\mathrm{ess.sup}(u) = \sup(u)$ and $\mathrm{ess.inf}(u) = \inf(u)$; let $a = \inf(u)$ and $b = \sup(u)$; let us consider $u : \mathbf{R}^n \longrightarrow [a, b]$; let u left (resp. right) convergent distributional; let $r : [0, +\infty] \longrightarrow [a, b]$ an increasing (resp. decreasing) rearrangement of u; let $v = r \circ m \circ f$ the related increasing (resp. decreasing) rearrangement of u according to f; let $p \in [1, +\infty[$; let $k_1, k_2 \in \mathbf{R}$, $0 < k_1 \leq k_2$; suppose* *there exists a continuous positive function* $q : [\alpha, \beta] \longrightarrow \mathbf{R}$ *such that for almost every* $x \in \mathbf{R}^n$

$$k_1 q(f(x)) \leq |\mathrm{grad}\, f(x)| \leq k_2 q(f(x)) ;$$

let $M \in [1, +\infty[$*; suppose that for almost every* $t \in [\alpha, \beta]$

$$\int_{R^n} d|\mathrm{grad}\, \varphi_{\{y \in R^n;\, f(y) < t\}}| \leq M n V_n^{\frac{1}{n}} \left(\lambda(\{y \in \mathbf{R}^n; f(y) < t\})\right)^{\frac{n-1}{n}} ;$$

then

$$\int_{R^n} |\mathrm{grad}\, v|^p \, d\lambda \leq \frac{M^p n^p V_n^{\frac{p}{n}} k_2^p}{k_1^p} \int_0^{+\infty} t^{p \frac{n-1}{n}} (r'(t))^p \, d\lambda(t)$$

$$\text{(resp. } \int_{R^n} |\mathrm{grad}\, v|^p \, d\lambda \leq \frac{M^p n^p V_n^{\frac{p}{n}} k_2^p}{k_1^p} \int_0^{+\infty} t^{p \frac{n-1}{n}} (-r'(t))^p \, d\lambda(t) \text{) .}$$

Proof. Let A_1 the complement respect to $[\alpha, \beta]$ of the set of $t \in]\alpha, \beta[$ such that $\varphi_{\{y \in \Omega, f(y) < t\}} \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, $m(t) = \lambda(\{y \in \Omega; f(y) < t\})$, m is derivable at t, $m'(t) = \int_{R^n} \frac{1}{|\mathrm{grad}\, f|} d|\mathrm{grad}\, \varphi_{\{y \in R^n;\, f(y) < t\}}|$ and $r \circ m$ is derivable at t.
We have $\lambda(A_1) = 0$; let $B_1 = f^{-1}(A_1)$; by theorem (3.2) (5) we have

$$\lambda(B_1) = \int_{A_1} d(f(\lambda)) = \int_{A_1} D_{\mathrm{meas}} m = \int_{A_1} m'(p) \, d\lambda(p) = 0 .$$

Let A_2 the set of $t \in]\alpha, \beta[$ such that m in derivable at t and $m'(t) = 0$; let $B_2 = f^{-1}(A_2)$; by theorem (3.2) (7) we have $\lambda(B_2) = 0$.
Let B_3 the set of $x \in \mathbf{R}^n$ such that f in not differentiable at x; we have $\lambda(B_3) = 0$.
Let B_4 the set of $x \in \mathbf{R}^n$ such that f differentiable at x and $|\mathrm{grad}\, f(x)| > k_2 q(f(x))$; we have $\lambda(B_4) = 0$.
Let $B = B_1 \cup B_2 \cup B_3 \cup B_4$; we have $\lambda(B) = 0$; for every $x \in \mathbf{R}^n - B$ we have $\varphi_{\{y \in \Omega, f(y) < f(x)\}} \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, $m(f(x)) = \lambda(\{y \in \Omega; f(y) < f(x)\})$, m and $r \circ m$ derivable at $f(x)$, $m'(t) = \int_{R^n} \frac{1}{|\mathrm{grad}\, f|} d|\mathrm{grad}\, \varphi_{\{y \in R^n;\, f(y) < f(x)\}}|$, $m'(t) \neq 0$, $r \circ m$ derivable at $f(x)$, f differentiable at x and $|\mathrm{grad}\, f(x)| \leq k_2 q(f(x))$.
Let $x \in \mathbf{R}^n - B$; let us prove that r is derivable at $m(f(x))$; let $z \in]\alpha, \beta[$, $z \neq m(f(x))$; since m is a homeomorphism, there exists $\lim_{z \to m(f(x))} \frac{r(z) - r(m(f(x)))}{z - m(f(x))}$ if and only if there exists $\lim_{t \to f(x)} \frac{r(m(t)) - r(m(f(x)))}{m(t) - m(f(x))}$; since

$$\frac{r(m(t)) - r(m(f(x)))}{m(t) - m(f(x))} = \frac{r(m(t)) - r(m(f(x)))}{t - f(x)} \frac{t - f(x)}{m(t) - m(f(x))} ,$$

the limit exists; then r is derivable at $m(f(x))$.
From this it follows that for all $x \in \mathbf{R}^n - B$ there exists $(\mathrm{grad}\, v)(x)$ and

$$\mathrm{grad}\, v(x) = r'(m(f(x))) m'(f(x)) \mathrm{grad}\, f(x) .$$

By lemmas (4.2), (4.1) and by the hypotheses, for every $x \in \mathbf{R}^n - B$ we have

$$m'(f(x)) = \int_{R^n} \frac{1}{|\mathrm{grad}\, f(z)|}\, d|\mathrm{grad}\, \varphi_{\{y\in R^n;\, f(y)<f(x)\}}|(z) \leq$$

$$\int_{R^n} \frac{1}{k_1 q(f(z))}\, d|\mathrm{grad}\, \varphi_{\{y\in R^n;\, f(y)<f(x)\}}(z) =$$

$$\int_{R^n} \frac{1}{k_1 q(f(x))}\, d|\mathrm{grad}\, \varphi_{\{y\in R^n;\, f(y)<f(x)\}}|(z) = \frac{1}{k_1 q(f(x))} \int_{R^n} d|\mathrm{grad}\, \varphi_{\{y\in R^n;\, f(y)<f(x)\}}| \leq$$

$$\frac{1}{k_1 q(f(x))} MnV_n^{\frac{1}{n}}\, (\lambda(\{y \in \mathbf{R}^n; f(y) < f(x)\}))^{\frac{n-1}{n}} = \frac{1}{k_1 q(f(x))} MnV_n^{\frac{1}{n}} (m(f(x)))^{\frac{n-1}{n}}\ ;$$

from this it follows

$$|\mathrm{grad}\, v(x)| \leq \frac{MnV_n^{\frac{1}{n}} k_2}{k_1} r'(m(f(x)))(m(f(x)))^{\frac{n-1}{n}}\ .$$

Then by definition of image of a measure and by theorem (3.2), being $m(m' \cdot \lambda_{[\alpha,\beta]}) = \lambda_{[0,+\infty]}$, we have

$$\int_{R^n} |\mathrm{grad}\, v|^p\, d\lambda \leq \frac{M^p n^p V_n^{\frac{p}{n}} k_2^p}{k_1^p} \int_{R^n} (r'(m(f(x))))^p (m(f(x)))^{p\frac{n-1}{n}}\, d\lambda(x) =$$

$$\frac{M^p n^p V_n^{\frac{p}{n}} k_2^p}{k_1^p} \int_\alpha^\beta (r'(m(y)))^p (m(y))^{p\frac{n-1}{n}} m'(y) d\lambda(t) = \frac{M^p n^p V_n^{\frac{p}{n}} k_2^p}{k_1^p} \int_0^{+\infty} (r'(u))^p u^{p\frac{n-1}{n}}\, d\lambda(u)\ .$$

Analogously:

Theorem 4.5 *Let* $f : \mathbf{R}^n \longrightarrow \mathbf{R}$*; let* f *locally lipschitzian; suppose* $(\mathrm{grad}\, f)(x) \neq 0$ *for almost every* $x \in \mathbf{R}^n$*; let* $\alpha = \inf(f)$ *and* $\beta = \sup(f)$*; let us consider* $f : \mathbf{R}^n \longrightarrow [\alpha, \beta]$*; let* f *right convergent distributional function; let* $m : [\alpha, \beta] \longrightarrow [0, +\infty]$ *an increasing distribution function of* f*; let* $u : \mathbf{R}^n \longrightarrow \mathbf{R}$*; let* $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$*; let* $\mathrm{ess.sup}(u) = \sup(u)$ *and* $\mathrm{ess.inf}(u) = \inf(u)$*; let* $a = \inf(u)$ *and* $b = \sup(u)$*; let us consider* $u : \mathbf{R}^n \longrightarrow [a, b]$*; let* u *right (resp. left) convergent distributional; let* $r : [-\infty, 0] \longrightarrow [a, b]$ *an increasing rearrangement of* u*; let* $v = r \circ m \circ f$ *the related increasing (resp. decreasing) rearrangement of* u *according to* f*; let* $p \in [1, +\infty[$*; let* $k_1, k_2 \in \mathbf{R}$*,* $0 < k_1 \leq k_2$*; suppose there exists a positive continuous function* $q : [\alpha, \beta] \longrightarrow \mathbf{R}$ *such that for almost every* $x \in \mathbf{R}^n$

$$(\forall x \in \mathbf{R}^n)\ k_1 q(f(x)) \leq |\mathrm{grad}\, f(x)| \leq k_2 q(f(x))\ ;$$

let $M \in [1, +\infty[$*; suppose that for almost every* $t \in [\alpha, \beta]$

$$\int_{R^n} d|\mathrm{grad}\, \varphi_{\{y\in R^n;\, f(y)<t\}}| \leq MnV_n^{\frac{1}{n}}\, (\lambda(\{y \in \mathbf{R}^n; f(y) < t\}))^{\frac{n-1}{n}}\ ;$$

then

$$\int_{R^n} |\mathrm{grad}\ v|^p\, d\lambda \leq \frac{M^p n^p V_n^{\frac{p}{n}} k_2^p}{k_1^p} \int_{-\infty}^0 t^{p\frac{n-1}{n}} (r'(t))^p\, d\lambda(t)\ .$$

$$\textit{(resp.}\ \int_{R^n} |\mathrm{grad}\ v|^p\, d\lambda \leq \frac{M^p n^p V_n^{\frac{p}{n}} k_2^p}{k_1^p} \int_{-\infty}^0 t^{p\frac{n-1}{n}} (-r'(t))^p\, d\lambda(t)\ \textit{)}\ .$$

5 Comparison theorem for the Dirichlet integrals

Theorem 5.1 *Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}$; let f locally lipschitzian; suppose $(\mathrm{grad}\, f)(x) \neq 0$ for almost every $x \in \mathbf{R}^n$; let $\alpha = \inf(f)$ and $\beta = \sup(f)$; let us consider $f : \mathbf{R}^n \longrightarrow [\alpha, \beta]$; let f left or right convergent distributional; let $u : \mathbf{R}^n \longrightarrow \mathbf{R}$; let $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$; let $\mathrm{ess.sup}(u) = \sup(u)$ and $\mathrm{ess.inf}(u) = \inf(u)$; let $a = \inf(u)$ and $b = \sup(u)$; let us consider $u : \mathbf{R}^n \longrightarrow [a, b]$; suppose there exists $v : \mathbf{R}^n \longrightarrow [a, b]$ increasing (resp. decreasing) rearrangement of u according to f; let $p \in [1, +\infty[$; let $k_1, k_2 \in \mathbf{R}$, $0 < k_1 \leq k_2$; suppose there exists a positive continuous function $q : [\alpha, \beta] \longrightarrow \mathbf{R}$ such that for almost every $x \in \mathbf{R}^n$*

$$k_1 q(f(x)) \leq |\mathrm{grad}\, f(x)| \leq k_2 q(f(x)) \ ;$$

let $M \in [1, +\infty[$; suppose that for almost every $t \in [\alpha, \beta]$

$$\int_{R^n} d|\mathrm{grad}\, \varphi_{\{y \in R^n; f(y) < t\}}| \leq M n V_n^{\frac{1}{n}} \left(\lambda(\{y \in \mathbf{R}^n; f(y) < t\})\right)^{\frac{n-1}{n}} \ ;$$

then

$$\int_{R^n} |\mathrm{grad}\ v|^p \, d\lambda \leq \left(\frac{M k_2}{k_1}\right)^p \int_{R^n} |\mathrm{grad}\ u|^p \, d\lambda \ .$$

Proof. Let us suppose f left convergent distributional; then u is left convergent distributional; there exists $m : [\alpha, \beta] \longrightarrow [0, +\infty[$ increasing distribution function of f and there exists $h : [a, b] \longrightarrow [0, +\infty[$ increasing distribution function of u.
By theorem (3.1) h is strictly increasing; let $r = h^{-1,m}$. Let us prove that we have $v = r \circ m \circ f$.
Let $x \in \mathbf{R}^n$. We have

$$\{y \in \mathbf{R}^n; f(y) \leq f(x)\} \subset \{y \in \mathbf{R}^n; v(y) \leq v(x)\} \ ;$$

then we have $m(f(x)+) \leq h(v(x)+)$.
We have

$$\{y \in \mathbf{R}^n; v(y) < v(x)\} \subset \{y \in \mathbf{R}^n; f(y) < f(x)\} \ ;$$

then we have $h(v(x)-) \leq m(f(x)-)$.
Being m continuous, we have

$$h(v(x)-) \leq m(f(x)) \leq h(v(x)+) \ .$$

If $h_1 = \min([h]_m)$ and $h_2 = \max([h]_m)$ we have

$$h_1(v(x)) \leq m(f(x)) \leq h_2(v(x)) \ .$$

By (1) we have

$$r(m(f(x))) \leq v(x) \leq r(m(f(x))) \ .$$

So $r(m(f(x))) = v(x)$.
Since $v = r \circ m \circ f$ the thesis follows from theorems (4.2) and (4.4).

In particular, for $f(x) = |x|^2$, v is a spherical rearrangement of u; in this case we may choose $M = k_1 = k_2 = 1$ and we find for $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$, u left or right distributional, the classical inequality

$$\int_{R^n} |\mathrm{grad}\ v|^p \, d\lambda \leq \int_{R^n} |\mathrm{grad}\ u|^p \, d\lambda \ .$$

6 Application to elliptic rearrangements

Let H_{n-1} the $n-1$-Hausdorff measure on $\mathbf{R}^n$.

We find a result for the elliptic rearrangements.

Theorem 6.1 *Let $c \in \mathbf{R}^n$; let $c_i > 0$ for all $i = 1, 2, \ldots, n$; let*

$$K_2 = \max(\{c_i; i = 1, 2, \ldots, n\}) \text{ and } K_1 = \min(\{c_i; i = 1, 2, \ldots, n\}) \ ;$$

let

$$f : \mathbf{R}^n \longrightarrow \mathbf{R}, x \longrightarrow \sum_{i=1}^{n} \frac{x_i^2}{c_i^2} \ ;$$

let $u : \mathbf{R}^n \longrightarrow \mathbf{R}$; let $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$; let $\mathrm{ess.sup}(u) = \sup(u)$ and $\mathrm{ess.inf}(u) = \inf(u)$; let $a = \inf(u)$ and $b = \sup(u)$; let us consider $u : \mathbf{R}^n \longrightarrow [a, b]$; let u left convergent distributional; let $v : \mathbf{R}^n \longrightarrow \mathbf{R}$ an increasing (resp. decreasing) rearrangement of u according to f; let $p \in [1, +\infty[$; then

$$\int_{R^n} |\mathrm{grad}\ v|^p \, d\lambda \leq \left(\frac{K_2}{(c_1 c_2 \ldots c_n)^{\frac{1}{n}}} \right)^{pn-p} n^p V_n^{\frac{p}{n}} \frac{K_2^p}{K_1^p} \int_{R^n} |\mathrm{grad}\ u|^p \, d\lambda \ .$$

Proof. For all $x \in \mathbf{R}^n$ we have

$$|\mathrm{grad}\, f(x)| = \sqrt{\sum_{i=1}^{n} \frac{4x_i^2}{c_i^4}} \leq \sqrt{\sum_{i=1}^{n} \frac{4x_i^2}{c_i^2 K_1^2}} = \frac{2}{K_1} \sqrt{f(x)}$$

and analogously $|\mathrm{grad}\, f(x)| \geq \frac{2}{K_2} \sqrt{f(x)}$.
For all $t > 0$ we have

$$\int_{R^n} d|\mathrm{grad}\, \varphi_{x \in R^n; f(x) < t\}}| = H_{n-1} \left(\left\{ x \in \mathbf{R}^n ; \sum_{i=1}^{n} \frac{x_i^2}{t c_1^2} = 1 \right\} \right) \leq$$

$$H_{n-1} \left(\left\{ x \in \mathbf{R}^n ; \sum_{i=1}^{n} \frac{x_i^2}{t K_2^2} = 1 \right\} \right) = (\sqrt{t} K_2)^{n-1} n V_n =$$

$$K_2^{n-1} n V_n (\sqrt{t} c_1 \sqrt{t} c_2 \ldots \sqrt{t} c_n)^{\frac{n-1}{n}} \frac{1}{(c_1 c_2 \ldots c_n)^{\frac{n-1}{n}}} =$$

$$\left(\frac{K_2}{(c_1 c_2 \ldots c_n)^{\frac{1}{n}}} \right)^{n-1} n V_n^{\frac{1}{n}} \lambda \left(\left\{ x \in \mathbf{R}^n ; \sum_{i=1}^{n} \frac{x_i^2}{t c_i^2} < 1 \right\} \right)^{\frac{n-1}{n}} =$$

$$\left(\frac{K_2}{(c_1 c_2 \ldots c_n)^{\frac{1}{n}}} \right)^{n-1} n V_n^{\frac{1}{n}} \lambda \left(\{ x \in \mathbf{R}^n ; f(x) < t \} \right)^{\frac{n-1}{n}} \ .$$

From this it follows the thesis.

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