Dynamics of water waves

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These lectures will be concerned with the study of the complex dynamics of the water waves equations describing the evolution of a bi-dimensional fluid under the action of gravity and capillary forces at the free surface, with space periodic boundary conditions, i.e. $x \in \mathbb{T}$. For small amplitude waves the dynamics is in first approximation described by the linearized equations, whose solutions are time periodic, or quasi-periodic, or almost-periodic. In such a case also the dynamics of the non-linear equations is expected to have a complex time recurrent behavior, in contrast to the case $x \in \mathbb{R}$, with data decaying at infinity, where the linear flow exhibits dispersive properties. A major difficulty to prove rigorous results about nonlinear waves is the fact that the water waves equations are quasi-linear/fully nonlinear.

In this course we shall present novel long time existence results of the initial value problem in Sobolev spaces for initial small amplitude waves. For example we shall prove that given a periodic, even in space, initial datum u_0 of small size ε, the solution of the gravity-capillary water waves equations is almost globally defined in time on Sobolev spaces, i.e. it exists on a time interval of length of magnitude ε^{-N} for any N, as soon as u_0 is smooth enough, and the gravity-capillarity parameters are taken outside an exceptional subset of zero measure, see M. Berti, J-M. Delort, "*Almost global existence of solutions for capillarity-gravity water waves equations with periodic spatial boundary conditions*".
The proof is based on a novel Birkhoff normal form approach for quasi-linear PDEs, relying on a paradifferential reduction of the equations to constant coefficients, up to smoothing remainders. In the course we shall develop the required techniques of para-differential calculus (i.e. the nonlinear analogue of pseudo-differential calculus).