A walk through quantum charged particle in a (constant) magnetic field

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1 Introduction

Welcome to this brief, but I hope complete, tour of the mathematics that lies behind the motion of a quantum charged particle in a magnetic field.

It may seem that choosing a constant field may simplify too much the problem hiding most of the properties that should come out from the general case. It will be clear that the bigger part of the properties arise, also, from the constant case.

All the study will be centered on gauge transforms, unitary groups of operators and assonances with the harmonic oscillator.

But let me now introduce the basic concepts, some of them already seen in the course "Introduzione alle Meccaniche Superiori", that will be used for my talk.

1.1 Preliminaries

One of the key that will bring us properties and simplifications lie behind this first definition. To lighten notations and computations, let me assume (only for a while) $\hbar = 1$.

Definition 1.1. *A one-parameter unitary group is a function $t \to U(t)$ from the real numbers into the set of bounded operators on an Hilbert space $\mathcal{H}$ with the following properties:*

1. *$U(t)$ is a unitary operator $\forall\, t \in \mathbb{R}$, i.e. $\|U\psi\| = \|\psi\|\ \forall\psi \in \mathcal{H}$ and $U\mathcal{H} = \mathcal{H}$;*
2. *$U(0) = 1$, $U(t)U(s) = U(t+s) \quad \forall\, s,\, t \in \mathbb{R}$;*
3. *$\lim_{t\to 0} U(t)\psi = \psi \quad \forall\, \psi \in \mathcal{H}$.*

Definition 1.2. *Let $U(t)$ be a one-parameter unitary group in a Hilbert space $\mathcal{H}$. The infinitesimal generator H of the unitary group is the linear operator given by:*

$$H\psi = i \lim_{t\to 0} \frac{U(t) - 1}{t}\psi \tag{1}$$

on a domain consisting of all vectors ψ for which this limit exists.

The domain $\mathcal{D}(H)$ of the infinitesimal generator can be shown to be dense in $\mathcal{H}$.

The quantum time evolution leads to the definition of a linear time evolution operator $U(t)$ that satisfies the definitions given above, in particular we can show that

Theorem 1.1. *If $U(t)$ is a unitary group with generator H in the sense specified above and if $\psi(t) = U(t)\psi$, then $\forall\psi \in \mathcal{D}(H)$, ψ is a solution of the Cauchy problem*

$$\begin{cases} i\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = H\psi(t) \\ \psi(0) = \psi \end{cases} \tag{2}$$

and $\mathcal{D}(H)$ is invariant under the time evolution.

Proof. Supposing ψ in $\mathcal{D}(H)$, limit (1) exists. The unitarity of $U(t)$ permits us to interchange its action with the limit. Thus, with the unitary group properties, we obtain that

$$\begin{aligned} U(t)\lim_{h\to 0}\frac{U(h)-1}{h}\psi &= \lim_{h\to 0}U(t)\frac{U(h)-1}{h}\psi = \\ &= \lim_{h\to 0}\frac{U(t+h)-U(t)}{h}\psi = \\ &= \lim_{h\to 0}\frac{U(h)-1}{h}U(t)\psi \end{aligned}$$

so $U(t)\psi\in\mathcal{D}(H)$, in particular this means that

$$U(t)\mathcal{D}(H)\subset\mathcal{D}(H), \text{ and } HU(t)\psi-U(t)H\psi=0 \;\; \forall\psi\in\mathcal{D}(H)$$

and the invariance holds.

Defining $\psi(t)=U(t)\psi$, follows immediately that

$$\lim_{h\to 0}\frac{U(t+h)-U(t)}{h}\psi=\frac{\mathrm{d}}{\mathrm{d}t}\psi$$

and (2) is satisfied. □

The unitarity of the time evolution operator $U(t)$ states that the scalar product of two states ψ and φ is independent of time, it means that

$$<\psi(t),\varphi(t)>=<\psi,\phi> \;\; \forall t\in\mathbb{R}. \tag{3}$$

This imply that

$$0=i\frac{\mathrm{d}}{\mathrm{d}t}<\psi(t),\varphi(t)>=<\psi(t),H\varphi(t)>-<H\psi(t),\varphi(t)>$$

so, to have $U(t)$ unitary, we need that H has to be *symmetric*:

$$<\psi,H\varphi>=<H\psi,\varphi> \qquad \forall\psi,\varphi\in\mathcal{D}(H).$$

It is very easy to prove that

Theorem 1.2. *A symmetric operator has only real eigenvalues.*
If ψ_1 and ψ_2 are eigenvectors of a symmetric operator H belonging to different eigenvalues E_1 and E_2, then ψ_1 is orthogonal to ψ_2 ($<\psi_1,\psi_2>=0$).

If you know a little bit of linear operators theory, you know that the most important property we can find in an operator is the *self-adjointness.* We have seen that if $U(t)$ is a one-parameter unitary group, its generator H have to be symmetric. Stone's Theorem says more:

THEOREM 1.3 (Stone's Theorem)**.** *If $U(t)$ is a one-parameter unitary group, then the generator H is a self-adjoint operator. Conversely, if H is a self-adjoint operator, then H is the generator of a unique one-parameter unitary group $U(t)$.*

The self-adjointness of H give us the possibility to define the exponential e^{-iHt} as

$$e^{Ax} = \sum_{n=0}^{+\infty} \frac{A^n\, x^n}{n!}$$

and usually the unitary group generated by H is written as

$$U(t) = e^{-iHt}.$$

As we already know, time evolution operators are generated by the energy Hamiltonian H and, at the same time, are strictly connected with unitary groups. We can see that this notation gives a concrete meaning to the unitary time evolution operator $U(t)$:

$$i\frac{\mathrm{d}}{\mathrm{d}t}e^{-iHt}\psi = He^{-iHt}\psi \quad \forall\psi \in \mathcal{D}(H)$$

and if φ is an eigenfunction of H with eigenvalue E, then

$$e^{-iHt}\varphi = e^{-iEt}\varphi.$$

An interesting theorem connected with self-adjoints operators states that

THEOREM 1.4. *If U is a unitary operator and T is self-adjoint on $\mathcal{D}(T)$, then the operator $S = UTU^*$ is self-adjoint on $\mathcal{D}(S) = U\mathcal{D}(T)$. Moreover, the unitary groups are related by*

$$Ue^{-iTt}U^* = e^{-iSt} \tag{4}$$

To understand better the importance of this last theorem, it's a good idea to introduce translations groups and Weyl's relations, which will be very useful in the understanding of the symmetries of the system.

1.1.1 Translations

Consider a one-dimensional wave function ψ and a scalar $a \in \mathbb{R}$, then define

$$\psi_a(x) = \psi(x-a). \tag{5}$$

The *translation by a* is the mapping

$$\tau_a : \psi \longmapsto \psi_a, \tag{6}$$

it's not too difficult to show that $a \mapsto \tau_a$ is a one parameter unitary group in the sense shown.

If we consider a differentiable wave function, a short calculation gives

$$i\frac{\tau_a\psi(x)-\psi(x)}{a} = i\frac{\psi(x-a)-\psi(x)}{a} \underset{a\to 0}{\longrightarrow} -i\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)$$

that is the generator of the group. Recalling that the quantum Hamiltonian momentum operator p is usually defined $-i\frac{\mathrm{d}}{\mathrm{d}x}$, we may write

$$\tau_a = e^{-ipa}.$$

In this particular case, for infinitely differentiable ψ, is easy to see (with a Taylor series expansion around x) that

$$\begin{aligned}\tau_a\psi(x) &= \psi(x-a) = \psi(x) - a\psi'(x) + \frac{a^2}{2!}\psi''(x) - \cdots = \\ &= \sum_{0}^{+\infty}\frac{a^n}{n!}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\psi(x) = \sum_{n=0}^{\infty}\frac{(-ipa)^n}{n!}\psi(x) = e^{-ipa}\psi(x).\end{aligned}$$

Recalling the theorem (1.4), taken a real number b, we can define the operator

$$\mu_b\psi(x) = e^{ixb}\psi(x) \tag{7}$$

and, recollecting that the Fourier transform $\mathcal{F}$ is such that $p = -i\frac{\mathrm{d}}{\mathrm{d}x} = \mathcal{F}^{-1}\xi\mathcal{F}$, we can say that μ_b forms a unitary group with parameter b and generator $-x$. In addition, is not difficult to see that μ_b describes a translation in the momentum space, precisely

$$e^{ixb}\psi(x) \longleftrightarrow \hat{\psi}(\xi - b). \tag{8}$$

Remark 1.1. These last unitary groups have straightforward generalizations to higher dimensions, precisely, if a, $b \in \mathbb{R}^n$ and p, x are n-dimensional,

$$\begin{aligned}e^{-ip\cdot a} &= e^{-ip_1a_1}e^{-ip_2a_2}\cdots e^{-ip_na_n}\\ e^{-ix\cdot b} &= e^{-ix_1b_1}e^{-ix_2b_2}\cdots e^{-ix_nb_n}\end{aligned}$$

1.1.2 Weyl relations

We'll find useful some formulas coming from the calculation of the commutator $[\tau_a, \mu_b]$, namely, for all a, $b \in \mathbb{R}$,

$$\begin{aligned} \tau_a \mu_b \psi(x) &= \tau_a \left(e^{ibx} \psi(x) \right) = e^{ib(x-a)} \psi(x-a) && (9) \\ \mu_b \tau_a \psi(x) &= \mu_b \psi(x-a) = e^{ibx} \psi(x-a) && (10) \end{aligned}$$

In other word

Theorem 1.5 (First-form Weyl relation). *The unitary groups τ_a and μ_b satisfy the relations*

$$\tau_a \mu_b = e^{-iab} \mu_b \tau_a \qquad \forall\, a,\, b \in \mathbb{R},$$

that is

$$e^{-ipa} e^{ixb} = e^{-iab} e^{ixb} e^{-ipa} \qquad \forall\, a,\, b \in \mathbb{R},$$

The Weyl relation we are looking for is equivalent to this, but is expressed in a more convenient manner. To reach this useful form, we need to reason as follows.

Let me define the operator

$$W(t) = \tau_t \mu_t e^{it^2/2}. \tag{11}$$

I say that $W(t)$ form an one-parameter unitary group.

Proof. It suffices to use the first-form Weyl relation both to verify the group property

$$\begin{aligned} W(t)W(s) &= \tau_t \mu_t \tau_s \mu_s e^{i(t^2+s^2)/2} = \\ &= \tau_t e^{its} \tau_s \mu_t \mu_s e^{i(t^2+s^2)/2} = \\ &= \tau_{t+s} \mu_{t+s} e^{i(t^2+2ts+s^2)/2} = W(s+t) \end{aligned}$$

and the unitarity

$$W^*(t) = \mu_{-t} \tau_{-t} e^{-it^2/2} = e^{it^2} \tau_{-t} \mu_{-t} e^{-it^2/2} = \tau_{-t} \mu_{-t} e^{it^2/2} = W(-t).$$

□

Recalling that $i \frac{\mathrm{d}}{\mathrm{d}t} \tau_t \psi = p \tau_t \psi$ and $i \frac{\mathrm{d}}{\mathrm{d}t} \mu_t \psi = (-x) \mu_t \psi$, we can find the generator of $W(t)$:

$$\begin{aligned} i \frac{\mathrm{d}}{\mathrm{d}t} W(t)\psi &= p \tau_t \mu_t e^{it^2/2} \psi - \tau_t x \mu_t e^{it^2/2} \psi + \tau_t \mu_t i^2 t e^{it^2/2} \psi = \\ &= p \tau_t \mu_t e^{it^2/2} \psi - (x-t) \mu_t e^{it^2/2} \psi + \tau_t \mu_t (-t) e^{it^2/2} \psi = \\ &= (p-x) p \tau_t \mu_t e^{it^2/2} \psi = (p-x) W(t) \psi. \end{aligned}$$

So in a suitable domain (precisely where is self-adjoint) the generator of the unitary group $W(t)$ is the operator $(p - x)$. Thus we can write

$$e^{-i(p-x)t} = e^{-ipt}e^{ixt}e^{it^2/2} = e^{ixt}e^{-ipt}e^{it^2/2}.$$

This calculation can be slightly generalized in the

Theorem 1.6 (Second-form Weyl relation)**.**

$$e^{-i(ap-bx)} = e^{-ipa}e^{ixb}e^{iab/2} = e^{ixb}e^{-ipa}e^{iab/2}$$

1.2 The angular momentum operator

Classically speaking, considered the position coordinates $x \in \mathbb{R}^3$ and the momentum ones $p \in \mathbb{R}^3$, the angular momentum is

$$L = x \times p = (x_2p_3 - x_3p_2,\, x_3p_1 - x_1p_3,\, x_1p_2 - x_2p_1) \in \mathbb{R}^3.$$

Applying the substitution rule to L, we obtain the quantum *angular momentum operator* $L = (L_1, L_2, L_3)$ of a particle in three dimensions

$$L_j = -i\left(x_k\frac{\partial}{\partial x_l} - x_l\frac{\partial}{\partial x_k}\right) = (x_kp_l - x_lp_k) \tag{12}$$

where (j, k, l) is a cyclic permutation of $(1, 2, 3)$.

It is not difficult to see that

$$[L_j, L_k] = i\hbar L_l$$

in fact, skipping the development of the calculus, we can see that

$$\begin{aligned} [L_j, L_k] &= [(x_kp_l - x_lp_k), (x_lp_j - x_jp_l)] = \\ &= x_k[p_l, x_l]p_j + x_jp_k[x_l, p_l] = \\ &= i\hbar(-x_kp_j + x_jp_k) = \\ &= i\hbar\, L_l \end{aligned}$$

1.3 Recalling the harmonic oscillator

The final part of this introduction regards the harmonic oscillator, whose quantum Hamiltonian operator is

$$H_{ho} = H_0 + V(x) = -\frac{\hbar^2}{2m}\Delta + \frac{1}{2}m\omega^2\|x\|^2$$

Thanks to the next theorem, the spectrum $\sigma(H_{ho})$ consists only of isolated eigenvalues increasing to infinity.

Theorem 1.7. *Let $V(x)$ be a continuous function on $\mathbb{R}^d$ satisfying $V(x) \geq 0$ for all $x \in \mathbb{R}^d$ and $V(x) \to \infty$ as $\|x\| \to \infty$. Then*

1. *$H = -\Delta + V$ is self-adjoint on $L^2(\mathbb{R}^d)$;*

2. *$\sigma(H)$ consists of isolated eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ with $\lambda_n \to \infty$ as $n \to \infty$.*

We are mainly interested in the one-dimensional case. It is useful to rescale the operator through the change of variable

$$x \mapsto y = \lambda x$$

with some $\lambda > 0$. Now, given a wave function $\psi(x)$, we define the scaled function by

$$\varphi(y) = \psi(x).$$

Considering the derivative, we see

$$\frac{\mathrm{d}}{\mathrm{d}x}\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\varphi(y) = \frac{\mathrm{d}}{\mathrm{d}y}\varphi(y)\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda\frac{\mathrm{d}}{\mathrm{d}y}\varphi(y)$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) = \lambda^2\frac{\mathrm{d}^2}{\mathrm{d}y^2}\varphi(y),$$

therefore we can say that

$$x\psi(x) = \frac{y}{\lambda}\varphi(y) \quad \text{and} \quad x^2\psi(x) = \frac{y^2}{\lambda^2}\varphi(y).$$

That is the Schrödinger equation can be described as follows

$$H\psi(x) = \left(-\frac{\hbar^2}{2m}\lambda^2\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{k}{2}\frac{1}{\lambda^2}y^2\right)\varphi(y).$$

If we choose $\lambda = \sqrt{\frac{\hbar}{m\omega}}$, i.e. the value such that

$$\frac{\hbar^2}{2m}\lambda^2 = \frac{m\omega^2}{2}\frac{1}{\lambda^2} \tag{13}$$

we have reduced the problem to

$$H\psi(x) = \hbar\omega\, H_{new}\,\varphi(y) = \hbar\omega\,\frac{1}{2}\left(p_{new}^2 + y^2\right)\varphi(y) \tag{14}$$

where $p_{new} = -i\frac{\mathrm{d}}{\mathrm{d}y}$.

In quantum field theory there is a standard way to proceed that is functional also for the study of the harmonic oscillator and that is the real reason of this section. This way requires the introduction of some new operators, in primis *creation* and *annihilation* ones:

$$\begin{aligned} a^* &= \frac{1}{\sqrt{2}}(y - ip_{new}), \\ a &= \frac{1}{\sqrt{2}}(y + ip_{new}). \end{aligned}$$

The following calculation states that

$$\begin{aligned} [a, a^*] &= aa^* - a^*a = \\ &= \frac{1}{2}((y+ip)(y-ip) - (y-ip)(y+ip)) = \\ &= \frac{1}{2}(y^2 - iyp + ipy + p^2 - y^2 - iyp + ipy - p^2) = \\ &= \frac{1}{2}(-2i[p, y]) = 1 \end{aligned}$$

proving the equation

$$[a, a^*] = aa^* - a^*a = 1. \tag{15}$$

Thus we can rewrite the Hamiltonian H_{new} in terms of a and a^* as follows:

$$H_{new} = a^*a + \frac{1}{2}.$$

This expression is said in *normal form* because a^* appears to the left of a.

Defining the *number operator* as

$$N = a^*a,$$

we have $H_{new} = N + \frac{1}{2}$. The importance of N is that, using (15), it satisfies the relations

$$\begin{aligned} Na &= (aa^* - 1)a = aa^*a - a = a(N-1) \qquad (16) \\ Na^* &= a^*aa^* = a^*(a^*a + 1) = a^*(N+1) \qquad (17) \end{aligned}$$

giving us a sort of algebra of the operators.

We have reduced the problem to the search of the eigenvalues of N that has the two wonderful properties written above.

Theorem 1.8. *We have*

1. $N \geq 0$;
2. $\sigma(N) = \mathbb{Z}^+$ *and each eigenvalue has multiplicity* 1.

Proof. 1. Because a^* is the adjoint of a, we have

$$< \psi, N\psi >=< a\psi, a\psi >= \|a\psi\|^2 \geq 0$$

for all ψ.

2. From the previous point, $N\psi = 0 \iff a\psi = 0$. Note that the function

$$\psi_0 = c_0 e^{-\frac{y^2}{2}},$$

where c_0 is a constant, is the unique family of solution of

$$a\psi = \frac{1}{\sqrt{2}}\left(y + \frac{\mathrm{d}}{\mathrm{d}y}\right)\psi = 0,$$

and hence $N\psi = 0$. Thus, normalizing it with $c_0 = (2\pi)^{-\frac{1}{4}}$, ψ_0 is the ground state. The commutation relation (17) implies

$$Na^*\psi_0 = a^*(N+1)\psi_0 = a^*N\psi_0 + a^*\psi_0 = 0 + a^*\psi_0 = a^*\psi_0$$

and, generalizing by induction,

$$\begin{aligned} N(a^*)^n\psi_0 &= a^*(N+1)(a^*)^{n-1}\psi_0 = \\ &= a^*N(a^*)^{n-1}\psi_0 + (a^*)^n\psi_0 = \\ &= a^*(n-1)(a^*)^{n-1} + (a^*)^n\psi_0 = \\ &= n(a^*)^n\psi_0 \end{aligned}$$

Hence

$$\phi_n = (a^*)^n\psi_0$$

is an eigenfunction of N with eigenvalue n.

If $n = 0$ we have

$$\|\phi_0\|^2 =< \psi_0, \psi_0 >= 1 = 0!$$

and, similarly, if $n = 1$ we have

$$\begin{aligned} \|\phi_1\|^2 &= < a^*\psi_0, a^*\psi_0 >=< \psi_0, aa^*\psi_0 >= \\ &= < \psi_0, (1+N)\psi_0 >=< \psi_0, \psi_0 + N\psi_0 >= \\ &= < \psi_0, \psi_0 + 0 >= 1 = 1! \end{aligned}$$

Therefore, by induction,

$$\begin{aligned}
\|\phi_n\|^2 &= < (a^*)^n\psi_0, (a^*)^n\psi_0 >=< \psi_0, a^n(a^*)^n\psi_0 >= \\
&= < \psi_0, a^{n-1}aa^*(a^*)^{n-1}\psi_0 >= \\
&= < \psi_0, a^{n-1}(1+N)(a^*)^{n-1}\psi_0 = \\
&= < \psi_0, a^{n-1}(a^*)^{n-1}\psi_0 + a^{n-1}N(a^*)^{n-1}\psi_0 >= \\
&= < \psi_0, a^{n-1}(a^*)^{n-1}\psi_0 + (n-1)a^{n-1}(a^*)^{n-1}\psi_0 >= \\
&= n < \psi_0, a^{n-1}(a^*)^{n-1}\psi_0 >= n\|\phi_{n-1}\|^2 = \\
&= n(n-1)! = n!
\end{aligned}$$

so we can normalize ϕ_n and find the normalized eigenfunction of N with eigenvalue n

$$\psi_n = \frac{1}{\sqrt{n!}}(a^*)^n\psi_0. \tag{18}$$

It remains only to show that these are the only eigenfunctions. It follows from the commutation relations (in a way similar as the above) that if ψ is any eigenfunction of N with eigenvalue $\lambda > 0$, then

$$Na^m\psi = (\lambda - m)a^m\psi.$$

Choosing m such that $\lambda - m \leq 0$ we contradict the positivity of N unless $a^m\psi = 0$. But, recalling the initial remarks of this point's demonstration, this implies that

$$a^j\psi_0 = c\psi_0 \tag{19}$$

for some integer j and a proper constant c. Reapplying the equation (19) we get $\lambda = j$. If we finally apply $(a^*)^j$ to the equation, using the commutation relations, we can show that $\psi = c'\psi_j$ for another constant c', so we are done.

□

One direct consequence of this theorem is that

$$\sigma(H_{new}) = \left\{ E_n^{new} = n + \frac{1}{2} : n \in \mathbb{N} \right\}$$

with eigenfunctions $\varphi_n = \frac{1}{\sqrt{n!}}(a^*)^n\varphi_0$ or, more precisely,

$$\varphi_n(y) = \frac{1}{\pi^{\frac{1}{4}}\sqrt{2^n n!}} H_n(y) e^{-\frac{y^2}{2}},$$

where H_n are the *Hermite polynomials* of order n (they may be defined by $H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2}$).

Finally, coming back to the spectrum of the original harmonic oscillator Hamiltonian, we find

$$\sigma(H) = \left\{ E_n = \hbar\omega \left(n + \frac{1}{2} \right) : n \in \mathbb{N} \right\}$$

with eigenfunctions obtained rescaling the φ_n's:

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right) e^{-\frac{m\omega}{2\hbar} x^2}$$

2 Quantum charged particle in a magnetic field

2.1 First observations about quantum mechanics of magnetic fields

A simple way to introduce magnetic fields in quantum mechanics is considering the usual transition procedure from classical to quantum Hamiltonian.

Prefacing that the electric field is described by a scalar potential field and so it can be incorporated in the potential function $V(x)$, let's analyze what happens in the presence of a magnetic field.

Classically speaking, a magnetic field $\vec{B}(x)$ can be described by a vector potential $\vec{A}(x)$ such that

$$\vec{B}(x) = \nabla \times \vec{A}(x)$$

so that we have automatically $\nabla \cdot \vec{B}(x) = 0$.
The general Hamiltonian function $H(p, x)$ for a particle in a magnetic field is defined through this vector potential:

$$H(p, x) = \frac{1}{2m} \left(p - \frac{q}{c} \vec{A}(x) \right)^2$$

where q is the charge of the particle and c is the light speed.

Looking at the formal analogy used to switch from classical to quantum mechanics, we can define Hamiltonian operator sending $H(p, x) \mapsto H(-i\hbar\nabla, x)$ as

$$H = \frac{1}{2m} \left(-i\hbar\nabla - \frac{q}{c} \vec{A}(x) \right)^2 . \tag{20}$$

Careful readers, now, can argue that in this formal change the correspondence between the classical and quantum structure is lost. We can better notice it expanding the square in both cases:

$$\text{C.} \qquad \left(p - \frac{q}{c}\vec{A}(x)\right)^2 = p^2 - 2\frac{q}{c}p \cdot \vec{A}(x) + \vec{A}(x)^2 \tag{21}$$

$$\text{Q.} \qquad \left(p - \frac{q}{c}\vec{A}(x)\right)^2 = p^2 - \frac{q}{c}p \cdot \vec{A}(x) - \frac{q}{c}\vec{A}(x) \cdot p + \vec{A}(x)^2 \tag{22}$$

The two addendum of the first equation $\frac{q}{c}p \cdot \vec{A}(x) + \frac{q}{c}\vec{A}(x) \cdot p$ represent the same function while in the quantum case they may be different operators, in fact unless $\nabla \cdot \vec{A} = 0$

$$\begin{aligned} p \cdot \vec{A}(x)\psi(x) &= -i\hbar\nabla \cdot \vec{A}(x)\psi(x) = \\ &= -i\hbar(\nabla \cdot \vec{A}(x) + \vec{A}(x) \cdot p)\psi(x) \neq \vec{A}(x) \cdot p\psi(x). \end{aligned}$$

The non commutativity of two operators leads to the presence of an uncertainty principle and establish a relation between them.

2.2 Hints in the non uniqueness of the wave function and Gauge invariance

One key point of this study, regards the analysis and the comprehension of a bug (at least for certain aspects) that is connected with the given interpretation of the quantum mechanics itself, more precisely the heart of the question arise in the choice of *normalized* wave function.

In fact, if ψ is a normalized wave function, the multiplication with a phase factor in $\{e^{i\lambda} : \lambda \in \mathbb{R}\}$ doesn't change the probability density represented by $\|\psi\|^2$.

Now, look ad the choice of a vector potential for the magnetic field. A particularly important step, in mathematics, is the search for existence and uniqueness. Is the magnetic field representation unique?

One of the property of the curl is that for any differentiable function f,

$$\nabla \times \nabla f = 0.$$

Looking at the definition given, is clear that take the vector potential

$$\vec{A}' = \vec{A} + \nabla f$$

is the same that take $\vec{A}$. So, if the model is correct, the vector potentials $\vec{A}$ and $\vec{A}'$ describe the same magnetic field. This fact is commonly indicated saying that "the two vector potentials are related by a gauge transformation".

This freedom in the choice of the vector potential is often used to simplify the mathematical description of the system. Frequent, in three dimensions, is the choice of the vector potential

$$\vec{A} = \int_0^1 s\vec{B}(xs) \times x \, \mathrm{d}s \tag{23}$$

that is said to be in the *Poincaré gauge*, i.e. is characterized by

$$\vec{A}(x) \cdot x = 0.$$

Another frequently used gauge is the *Coulomb gauge*, characterized by

$$\nabla \cdot \vec{A}(x) = 0.$$

Remark that a vector potential with this property has also the property that $p \cdot \vec{A}(x) = \vec{A}(x) \cdot p$.

Clearly, it is fundamental to understand the way the gauge transformations affect the Schrödinger equation.

Suppose there is no scalar potential $V(x)$ and apply a gauge transformation ∇f where f is a real differentiable function. This transformation, somehow, may change the description. We can consider, for example, (putting $\hbar = m = 1$) the Hamiltonian operator

$$H = \frac{1}{2}\left(-i\nabla - (\nabla f)\right)^2 \tag{24}$$

that describes the free motion after a gauge transformation (in fact ∇f corresponds to a field of zero strength).
Let ψ be a solution of the free Schrödinger equation and consider

$$\varphi_t(x) = e^{if(x)}\psi_t(x). \tag{25}$$

With a simple calculation is easy to see that

$$(-i\nabla - (\nabla f(x)))^2 \varphi_t(x) = -e^{if(x)}\Delta\psi_t(x)$$

and so

$$\begin{aligned} i\frac{\partial}{\partial t}\varphi_t(x) &= ie^{if(x)}\frac{\partial}{\partial t}\psi_t(x) = \\ &= -\frac{1}{2}e^{if(x)}\Delta\psi_t(x) = \\ &= (-i\nabla - (\nabla f(x)))^2\varphi_t(x). \end{aligned}$$

It means that φ_t is a solution of the equation (24).

Summarizing, both φ and ψ describe the same physical state of the free particle but they solve different equations. Therefore, in the next analysis we must find invariant real observable to try a physical check of this model and to give a reasonable explanation of the gauge invariance. The point is that gauge invariance is a big useful properties but for a consistent model we need other properties, such as suitable observables and (in general) uniqueness of solutions...

2.3 Preparatory analysis of the magnetic field

We want to analyze the energy spectrum in a constant magnetic field. Propaedeutic for the analysis is the study of a general magnetic field with constant direction and of how we can reduce it in an useful form.

Hence, if the direction of $\vec{B}(x)$ is independent of $x \in \mathbb{R}^3$, we can choose a convenient coordinate system in such a way that

$$B\vec{(x)} = (0, 0, B(x)).$$

Obviously this imply that

$$0 = \nabla \cdot \vec{B} = \frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} = \frac{\partial B_3}{\partial x_3}$$

or, in other words, that $B(x)$ is independent from x_3! Thus, we have practically a bi-dimensional situation invariant with respect to the translations along the third axis. So, in our formalism, is sufficient to consider

$$\vec{A}(x) = (A_1(x_1, x_2), A_2(x_1, x_2), 0)$$

then

$$B(x_1, x_2) = \frac{\partial}{\partial x_1} A_2(x_1, x_2) - \frac{\partial}{\partial x_2} A_1(x_1, x_2).$$

Under this hypothesis the Schrödinger equation can be written as

$$i \frac{\partial}{\partial t} \psi_t(x) = \left(\frac{1}{2} \left(p_1 - \frac{q}{c} A_1 \right)^2 + \frac{1}{2} \left(p_2 - \frac{q}{c} A_2 \right)^2 + \frac{1}{2} p_3^2 \right) \psi_t(x)$$

and the wave function can be factorized as follows

$$\psi_t(x) = \phi_t(x_1, x_2) \varphi_t(x_3). \tag{26}$$

This lead at the conclusion that ψ is a solution if it is true that, assumed $\vec{A}'(x_1, x_2) = (A_1(x_1, x_2), A_2(x_1, x_2))$,

$$\begin{cases} i\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x_3) = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}\varphi_t(x_3) \\ i\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(x_1, x_2) = \frac{1}{2}\left(-i\nabla - \frac{q}{c}\vec{A}'(x_1, x_2)\right)^2 \phi_t(x_1, x_2) \end{cases} \tag{27}$$

Recalling the (23), we may choose the Poincaré gauge and write

$$\vec{A}(x_1, x_2) = (-x_2, x_1)\int_0^1 sB(xs)\mathrm{d}s.$$

Remark 2.1 (Influence of the vector potential). It may happen (and it happens frequently) that the vector potential is nonzero even in regions where the magnetic field strength is null. Assuming that the two-dimensional magnetic field B(x) is nonzero only in some bounded region and has a non-vanishing flux, we have

$$\int B(x)\mathrm{d}^2x \neq 0.$$

Applying the Stoke's theorem we obtain

$$\oint \vec{A}(x)\cdot \mathrm{d}\vec{s} = \int \nabla \times \vec{A}(x)\mathrm{d}^2x = \int B(x)\mathrm{d}^2x,$$

where the circulation is taken along a large circle outside the support of B. Thus the vector potential cannot vanish everywhere on the circle, no matter which gauge we choose.
The really interesting thing is that the vector potential influence the wave function also in regions that are far away from the support of B.

Keeping all in mind, let us assume the constance of the magnetic field. As already said, we can reduce the problem in two dimensions. Suppose that for all $x \in \mathbb{R}^2$ the constant field strength is $B(x) \equiv \frac{c}{q}B \in \mathbb{R}$, then choosing the Poincaré gauge we have

$$\vec{A}(x) = \frac{\frac{c}{q}B}{2}(-x_2, x_1) \tag{28}$$

that, in this case, coincide with the Coulomb gauge.

Remark 2.2. Thanks to gauge invariance, we may choose a lot of vector potential for the same field, for example, $(-\frac{c}{q}Bx_2, 0)$ or $(0, \frac{c}{q}Bx_1)$.

Recalling the notation used in the equation (26), is known how to find φ_t, then we concentrate our attention on the ϕ_t and on the equation (27).

2.4 Energy spectrum of a charged quantum particle in a constant magnetic field

We want to analyze the Schrödinger equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(x) = \left(\frac{1}{2}\left(p_1 + \frac{B}{2}x_2\right)^2 + \frac{1}{2}\left(p_2 - \frac{B}{2}x_1\right)^2\right)\phi_t(x). \tag{29}$$

Introducing the *velocity operator* $v = (v_1, v_2)$ where

$$v_1 = p_1 + \frac{B}{2}x_2, \tag{30}$$

$$v_2 = p_2 - \frac{B}{2}x_1. \tag{31}$$

we can rewrite the Hamiltonian operator for a charged particle in a bi-dimensional constant magnetic field obtaining

$$H = \frac{1}{2}\left(v_1^2 + v_2^2\right).$$

If we define, as usual, the time-dependent position operator as $x(t) = e^{iHt}xe^{-iHt}$ we find

$$\frac{\mathrm{d}}{\mathrm{d}t}x_i(t) = e^{iHt}i[H, x_i]e^{-iHt} = v_i(t) \tag{32}$$

in fact, noting that $j \neq i \Rightarrow [v_j^2, x_i] = 0$, we have

$$\begin{aligned} i[H, x_i] &= \frac{i}{2}[v_i^2, x_i] = \\ &= \frac{i}{2}(v_i[v_i, x_i] + [v_i, x_i]v_i) = \\ &= \frac{i}{2}(v_i[p_i, x_i] + [p_i, x_i]v_i) = \\ &= \frac{i}{2}(v_i(-i) + (-i)v_i) = v_i. \end{aligned}$$

Very interesting is the fact that under the presence of the magnetic field the velocity operator's components don't commute but are canonically conjugate variables, like position and momentum operators:

$$\begin{aligned}
[v_1, v_2] &= \left[p_1 + \frac{B}{2}x_2, p_2 - \frac{B}{2}x_1\right] = \\
&= -\frac{B}{2}[p_1, x_1] + \frac{B}{2}[x_2, p_2] = iB;
\end{aligned}$$

notice also that

$$\begin{aligned}
[H, v_1] &= \left[\frac{1}{2}(v_1^2 + v_2^2), v_1\right] = \frac{1}{2}([v_1^2, v_1] + [v_2^2, v_1]) = \\
&= \frac{1}{2}(v_2^2 v_1 - v_1 v_2^2) = \frac{1}{2}(v_2(v_1 v_2 - iB) - (iB + v_2 v_1)v_2) = \\
&= -iBv_2,
\end{aligned}$$

$$\begin{aligned}
[H, v_2] &= \left[\frac{1}{2}(v_1^2 + v_2^2), v_2\right] = \frac{1}{2}([v_1^2, v_2] + [v_2^2, v_2]) = \\
&= \frac{1}{2}(v_1(iB + v_2 v_1) - (v_2 v_1 - iB)v_1) = \\
&= iBv_1.
\end{aligned}$$

Now we are at the turning point, and we will soon understand why in the first section we have recalled the quantum harmonic oscillator. Let us write in parallel two systems

$$\begin{cases} H = \frac{1}{2}(v_1^2 + v_2^2) \\ [v_1, v_2] = iB \end{cases} \qquad\qquad \begin{cases} H_{ho} = \frac{1}{2}(p^2 + x^2) \\ [x, p] = i \end{cases}$$

After this parallelism we must await that the bi-dimensional magnetic operator and the one-dimensional harmonic oscillator operator have essentially the same spectrum of eigenvalues. Not only, we can try to adapt the procedure used for the harmonic oscillator to our system.

First of all, define a sort of annihilation and creation operators

$$A = \sqrt{\frac{1}{2|B|}}(v_1 + iv_2)$$

$$A^* = \sqrt{\frac{1}{2|B|}}(v_1 - iv_2)$$

Looking at the commutator

$$\begin{aligned}
A^*A &= \frac{1}{2|B|}(v_1 + iv_2)(v_1 - iv_2) = \\
&= \frac{1}{2|B|}(v_1^2 + v_2^2 + i(v_1 v_2 - v_2 v_1)) = \\
&= \frac{1}{2|B|}(v_1^2 + v_2^2 - B) \\
AA^* &= \frac{1}{2|B|}(v_1 - iv_2)(v_1 + iv_2) = \\
&= \frac{1}{2|B|}(v_1^2 + v_2^2 + B)
\end{aligned}$$

we find

$$[A, A^*] = 1.$$

Assuming B>0, the Hamiltonian can be written as

$$H = B\left(A^*A + \frac{1}{2}\right) \tag{33}$$

and, exactly with the same argument used for the harmonic oscillator, we can conclude that H has the eigenvalues

$$E_n = B\left(n + \frac{1}{2}\right), \quad n \in \mathbb{N}. \tag{34}$$

Thus, we can try to search the nontrivial low-energy solution ψ_0 such that

$$A\psi_0 = 0.$$

To find them we have to solve the first-order differential equation

$$\left(-i\frac{\partial}{\partial x_1} + \frac{B}{2}x_2 + \frac{\partial}{\partial x_2} - i\frac{B}{2}x_1\right)\psi_0(x) = 0, \tag{35}$$

but is evident that de variables are separated, so we can take

$$\psi_0(x) = \gamma_1(x_1)\gamma_2(x_2)$$

and factorize the equation obtaining the conditions

$$\begin{cases} -i\left(\frac{\partial}{\partial x_1} + \frac{B}{2}x_1\right)\gamma_1(x_1) = 0 \\ -i\left(\frac{\partial}{\partial x_2} + \frac{B}{2}x_2\right)\gamma_2(x_2) = 0 \end{cases} \tag{36}$$

that are identical and can be compared with the same for the ground state of the harmonic oscillator giving

$$\gamma_1(x) = \gamma_2(x) = e^{-\frac{B}{4}x^2}$$

and, therefore,

$$\psi_0(x) = e^{-\frac{B}{4}\left(x_1^2+x_2^2\right)} \tag{37}$$

so that we have

$$H\psi_0 = \frac{B}{2}\psi_0.$$

Now, computing

$$\int_{\mathbb{R}^2} \psi_0(x)\mathrm{d}^2x,$$

we can normalize the function obtaining

$$\psi_0(x) = \sqrt{\frac{B}{2\pi}}e^{-\frac{B}{4}\|x\|^2}. \tag{38}$$

For $B < 0$, just exchange A with A^* and repeat the same reasoning. We have proved that

Theorem 2.1. *The two-dimensional constant magnetic field Hamiltonian operator*

$$H = \frac{1}{2}\left(p_1 + \frac{B}{2}x_2\right)^2 + \frac{1}{2}\left(p_1 + \frac{B}{2}x_2\right)^2$$

has eigenvalues

$$E_n = |B|\left(n + \frac{1}{2}\right), \quad n \in \mathbb{N}.$$

A normalized ground state, with energy $E_0 = \frac{B}{2}$, is given by

$$\psi_0(x) = \sqrt{\frac{|B|}{2\pi}}e^{-\frac{|B|}{4}\|x\|^2}.$$

2.5 Symmetries and invariance. Final considerations

On a formal level, the harmonic oscillator and the constant magnetic field look very similar. There is a simple correspondence between the components of the magnetic field velocity operator and the position and momentum operators of the harmonic oscillator.

Nevertheless, from a physical point of view, the two systems are quite different. The particle in a constant magnetic fields (as somehow we have just said but we will delve in a moment) has properties which the harmonic oscillator in phase space does not. In fact, the harmonic oscillator force distinguishes the coordinate origin as an equilibrium point, while in a magnetic field all points are the same: despite the origin appears to be distinguished by the properties $\vec{A}(0) = (0, 0)$, this is only due to our particular choice of $\vec{A}$ and does not correspond to a physical property of the system. For every $a = (a_1, a_2) \in \mathbb{R}^2$, taking

$$\alpha = -\frac{B}{2}(a_2 x_1 - a_1 x_2)$$

as a gauge transformation, we will obtain

$$\vec{A}_\alpha = \vec{A} + \nabla\alpha = \frac{B}{2}(-x_2 + a_2, x_1 - a_1)$$

that vanishes in the point a. Because every gauge transformation leads to a physically equivalent description, every point is the same as the origin.

REMARK 2.3. This reasoning can be applied also to the free particle. No matter where the particle is, the evolution remains the same!

This translational symmetry will brings to the conclusion that all the eigenvalues E_n in the constant magnetic field are *infinitely degenerate*. So let us analyze this symmetry (and perhaps compare it with the rotational symmetry of the system...).

2.5.1 Remarks on classical constant magnetic field motion

To analyze the translational symmetry is useful a comparison with the correspondent classical motion. The classical Hamiltonian equations

$$\begin{cases} \dot{x}_1(t) = \frac{\partial}{\partial p_1} H(x,p) = p_1(t) + \frac{B}{2} x_2(t) \\ \dot{x}_2(t) = \frac{\partial}{\partial p_2} H(x,p) = p_2(t) - \frac{B}{2} x_1(t) \\ \dot{p}_1(t) = -\frac{\partial}{\partial x_1} H(x,p) = \frac{B}{2}\left(p_2(t) - \frac{B}{2} x_1\right) \\ \dot{p}_2(t) = -\frac{\partial}{\partial x_2} H(x,p) = \frac{B}{2}\left(p_1(t) + \frac{B}{2} x_2\right) \end{cases}$$

can be rewritten as the (evidently) translationally invariant system

$$\begin{cases} \ddot{x}_1(t) = B\dot{x}_2(t) \\ \ddot{x}_2(t) = -B\dot{x}_1(t) \end{cases}$$

From the conservation of energy, realizing that

$$H = \frac{1}{2}\left(\dot{x}_1^2 + B\dot{x}_2^2\right),$$

we can see that the absolute value of the velocity is constant and, solving the system, we find a circular motion with constant angular velocity. Namely, if the initial velocity is $(\dot{x}_1(0), \dot{x}_2(0)) = (v_1, v_2)$,

$$\begin{cases} \dot{x}_1(t) = v_1 \cos(Bt) + v_2 \sin(Bt) \\ \dot{x}_2(t) = v_2 \cos(Bt) - v_1 \sin(Bt) \end{cases}$$

then for the position we have

$$\begin{cases} x_1(t) = \bar{x}_1 - \frac{1}{B}\dot{x}_2(t) \\ x_2(t) = \bar{x}_2 + \frac{1}{B}\dot{x}_1(t) \end{cases} \tag{39}$$

Thus, the classical orbit of the particle is a circle with center $(\bar{x}_1, \bar{x}_2)$ and radius $\frac{\|v\|}{|B|}$.

2.5.2 Symmetries and eigenvalues

Looking at the equation (39) we can deduce the quantum operator that correspond to the center of the classical orbit:

$$\begin{cases} \bar{x}_1 = x_1 + \frac{1}{B}v_2 = \frac{1}{2}x_1 + \frac{1}{B}p_2 \\ \bar{x}_2 = x_2 - \frac{1}{B}v_1 = \frac{1}{2}x_2 + \frac{1}{B}p_1 \end{cases}$$

Is interesting that

$$[v_1, \bar{x}_1] = \left[p_1 + \tfrac{B}{2}x_2, \tfrac{1}{2}x_1 + \tfrac{1}{B}p_2\right] = \tfrac{1}{2}([p_1, x_1] + [x_2, p_2]) = 0$$

$$[v_1, \bar{x}_2] = \left[p_1 + \tfrac{B}{2}x_2, \tfrac{1}{2}x_2 - \tfrac{1}{B}p_1\right] = p_1 x_2 = 0$$

and more precisely that, if $i, j \in \{1, 2\}$,

$$[v_i, \bar{x}_j] = 0 \text{ and so } [H, \bar{x}_j] = 0.$$

For the Quantum Nöther theorem, this implies that $\bar{x}_j$ are conserved quantities.

On the other hand we have a canonical commutation relation:

$$[\bar{x}_1, \bar{x}_2] = [\frac{1}{2}x_1 + \frac{1}{B}p_2, \frac{1}{2}x_2 - \frac{1}{B}p_1] = \frac{1}{2B}([p_1, x_1] + [p_2, x_2]) = -i\frac{1}{B}.$$

This fact holds a very deep significance for the symmetry. In order to understand it let us calculate the action of the transformations generated by $\bar{x}_1$ on a general state $\psi(x_1, x_2)$. The operator $\bar{x}_i$ generate the unitary transformation $e^{-ia\bar{x}_i}$. Let us consider $\bar{x}_1$, recalling the section §1.1.1, we have

$$\begin{aligned} e^{-ia\bar{x}_1}\psi(x_1, x_2) &= e^{-i\frac{a}{2}x_1 - i\frac{a}{B}p_2}\psi(x_1, x_2) = \\ &= e^{-i\frac{a}{2}x_1}e^{-i\frac{a}{B}p_2}\psi(x_1, x_2) = \\ &= e^{-i\frac{a}{2}x_1}\psi\left(x_1, x_2 - \frac{a}{B}\right). \end{aligned}$$

Hence the transformation generated by $\bar{x}_1$ is made up a shift in the variable x_2 and a multiplication by a phase factor that change the momentum p_1 of the particle but non its state. In particular, because of the commutation between v_1 and $\bar{x}_1$, the value of the velocity in the x_1 direction remains unchanged.

REMARK 2.4. A gauge transformation that shifts the vector potential to the point $(0, \frac{a}{B})$ cancels the phase factor caused by this translation and consequently we have the invariance under translations.

The transformation $e^{-ib\bar{x}_2}$, in the same way, translate the wave function by $-\frac{b}{2}$ in the direction of x_1 and multiply a phase factor $e^{-i\frac{b}{2}x_2}$.

We can finally say in what sense the operator $\bar{x} = (\bar{x}_1, \bar{x}_2)$ generate translations.

DEFINITION 2.1. *For an arbitrary $w = (w_1, w_2) \in \mathbb{R}^2$ we can define the unitary operator*

$$e^{-iB\bar{x}\times w} = e^{-iB(\bar{x}_1 w_2 - \bar{x}_2 w_1)}.$$

This operator shifts a wave packet by w in position space and at the same time adds the vector $\vec{A}(w) = \frac{B}{2}(-w_2, w_1)$ to the momentum.

In fact, because of the commutativity of $x \times w$ and $p \cdot w$, we have

$$\begin{aligned} e^{-iB\bar{x}\times w}\psi(x) &= e^{-i\frac{B}{2}x\times w} e^{\frac{1}{B}p\cdot w}\psi(x) = \\ &= e^{-i\frac{B}{2}x\times w}\psi(x-w) = \\ &= e^{-ix\cdot\vec{A}(w)}\psi(x-w) \end{aligned}$$

Like above, we can say that the translation we have just described changes the wave packet but doesn't change the velocity (because of the commutation between $\bar{x}$ and v). In other words the shift in position space changes the velocity from $p - \vec{A}(x_0)$ to $p - \vec{A}(x_0 + w)$, that in the constant magnetic field is the same as $\vec{A}(x_0) + \vec{A}(w)$, but the simultaneous shift in the momentum space cancels precisely that change preserving the velocity constant. At the same time, the transformation that we are doing indirectly on the vector potential is essentially a gauge, so the magnetic field remains the same.

The vector potential is not translationally invariant, there is only invariance up to gauge transformation. Fortunately all the physical measurable quantities do not depend on the choice of the gauge and in particular the energy of the state is the same that of a shifted state.

Let ψ be an eigenfunction of H, then for some n

$$H\psi = E_n\psi.$$

Because of the commutativity between H and $\bar{x}$ we have

$$He^{-ia\bar{x}_j}\psi = e^{-ia\bar{x}_j}H\psi = E_n e^{-ia\bar{x}_j}\psi,$$

so the shifted eigenfunction is again an eigenfunction with the same energy.

Starting from the ground state eigenfunction, we have infinitely many translated eigenfunction with the same energy and the same holds for every eigenfunction for every energy level. Therefore *the ground state and all the other eigenvalues have infinite multiplicity.*

The last point apropos of the symmetries regard velocity operator. Like $\bar{x}$, also v_1 and v_2 generate unitary transformations:

$$\begin{aligned} e^{-iav_1}\psi(x_1, x_2) &= e^{-iap_1 - i\frac{aB}{2}x_2}\psi(x_1, x_2) = \\ &= e^{-i\frac{aB}{2}x_2}\psi(x_1 - a, x_2) \end{aligned}$$

and similarly

$$e^{-iav_2}\psi(x_1, x_2) = e^{-i\frac{aB}{2}x_1}\psi(x_1, x_2 - a).$$

Hence they generate translations in the direction of the coordinate x_i. Now, because of the non commutativity between Hamiltonian operator and v_i, there is a change in the state of the particle. The canonical conjugation shows that the transformation generated by v_1 change the values of v_2, on the other hands the fact that v_i and $\bar{x}_j$ commutes causes that the classical center $\bar{x}$ remain unchanged under this translations!

Recall for a while the equation (29), expanding the squares we arrive at the following expression

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\left(\frac{B}{2}\right)^2 (x_1^2 + x_2^2) - \frac{B}{2}(x_1 p_2 - x_2 p_1).$$

In the last term we can recognize the momentum operator L_3. The other summands represent the Hamiltonian of a two-dimensional harmonic oscillator with frequency $\omega = \frac{B}{2}$:

$$H_{2ho}(w) = \frac{1}{2}(p^2 + \omega^2 x^2).$$

Hence the Hamiltonian operator in a constant bi-dimensional magnetic field can be written as

$$H = H_{2ho}\left(\frac{B}{2}\right) - \frac{B}{2}L_3.$$

We know from the theory that $[H_{2ho}, L_3] = 0$ and so

$$[H, L_3] = 0.$$

As a consequence the canonical angular momentum L_3 is a constant of motion. This is a very important issue because it is involved in the spherical symmetry of the constant magnetic field Hamiltonian, property that is central for the possibility of finding the solution of the equations in the case of the constant magnetic field letting us move in a way similar to the hydrogen atom and the spherical harmonic oscillator.

Moreover, the vector potential $\vec{A}(x) = \frac{B}{2}(-x_2, x_1)$ satisfies the relation

$$p \cdot \vec{A}(x) = \frac{B}{2} L$$

showing that also $p \cdot \vec{A}$ is a conserved quantity. It means, for example, that for a particle located at time 0 at the origin, the canonical angular momentum is 0, so the canonical momentum has to be always orthogonal to the vector potential.

3 Hints on the time evolution in a constant magnetic field

Notice that the solutions of equation (39)

$$\begin{cases} x_1(t) = \bar{x}_1 + \frac{1}{B} v_1 \sin(Bt) - \frac{1}{B} v_2 \cos(Bt) \\ x_2(t) = \bar{x}_2 + \frac{1}{B} v_1 \cos(Bt) + \frac{1}{B} v_2 \sin(Bt) \end{cases}$$

can be seen, interpreting $\bar{x}_j$ and v_j as quantum observables, as the solutions of the quantum evolution equation for the observables

$$x_j(t) = e^{iHt} x_j e^{-iHt} \tag{40}$$

given by

$$\frac{\mathrm{d}}{\mathrm{d}t} x_j(t) = i[H, x_j(t)]. \tag{41}$$

Proof. It comes directly from the comparison between the time-derivative of x_j

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_1(t) = v_1 \cos(Bt) + v_2 \sin(Bt) \\ \frac{\mathrm{d}}{\mathrm{d}t} x_2(t) = -v_1 \sin(Bt) + v_2 \cos(Bt) \end{cases} \tag{42}$$

and the commutators

$$\begin{aligned} i[H, x_1(t)] &= \frac{1}{B} i[H, v_1] \sin(Bt) - \frac{1}{B} i[H, v_2] \cos(Bt) = \\ &= \frac{1}{B} B v_2 \sin(Bt) - \frac{1}{B}(-B v_1) \cos(Bt) = \\ &= v_1 \cos(Bt) + v_2 \sin(Bt) \end{aligned}$$

$$\begin{aligned} i[H, x_2(t)] &= \frac{1}{B} i[H, v_1] \cos(Bt) + \frac{1}{B} i[H, v_2] \sin(Bt) = \\ &= \frac{1}{B} B v_2 \cos(Bt) + \frac{1}{B}(-B v_1) \sin(Bt) = \\ &= -v_1 \sin(Bt) + v_2 \cos(Bt) \end{aligned}$$

□

We can hence write the time-evolution of the velocity observables $v_j(t) = e^{iHt} v_j e^{-iHt}$ as follow

$$\begin{cases} v_1(t) = v_1 \cos(Bt) + v_2 \sin(Bt) \\ v_2(t) = -v_1 \sin(Bt) + v_2 \cos(Bt) \end{cases} \tag{43}$$

in agreement with the classical ones and in complete analogy with the $x(t)$ and $p(t)$ operators in the harmonic oscillator dynamics.

Only one last curiosity regards the analysis of the time-evolution of a state shifted by e^{-iav_1} (we consider only v_1 for simplicity). This group of operators, as we have already said, generates translations in the x_2 direction leaving invariant the center of motion $(\bar{x}_1, \bar{x}_2)$.
If $\psi(x_1, x_2)$ is an arbitrary initial state, its time-evolution is given by

$$\psi_t(x_1, x_2) = e^{-iHt} \psi(x_1, x_2).$$

Shifting ψ, we obtain

$$\begin{aligned} e^{-iHt} e^{-iav_1} \psi(x_1, x_2) &= e^{-iHt} e^{-iav_1} e^{iHt}\, e^{-iHt} \psi(x_1, x_2) = \\ &= e^{-iav_1(-t)} \psi_t(x_1, x_2) = \\ &= e^{-ia\cos(Bt) v_1 + ia \sin(Bt) v_2} \psi_t(x_1, x_2) \end{aligned}$$

and, using the Weyl relation for the canonically conjugate operators v_1 and v_2, namely

$$e^{ia_1 v_1 + ia_2 v_2} = e^{ia_1 a_2 \frac{B}{2}} e^{ia_1 v_1} e^{ia_2 v_2},$$

it becomes

$$
\begin{aligned}
& e^{-ia\cos(Bt)v_1+ia\sin(Bt)v_2}\psi_t(x_1,x_2) = \\
= \; & e^{-i\frac{a^2B}{2}\cos(Bt)\sin(Bt)}e^{-ia\cos(Bt)v_1}e^{ia\sin(Bt)v_2}\psi_t(x_1,x_2) = \\
= \; & e^{-i\frac{a^2B}{2}\cos(Bt)\sin(Bt)}e^{-i\frac{aB}{2}\cos(Bt)x_2}e^{-ia\cos(Bt)p_1}e^{-i\frac{aB}{2}\sin(Bt)x_1}e^{ia\sin(Bt)p_2}\psi_t(x_1,x_2) = \\
= \; & e^{-i\frac{aB}{2}\cos(Bt)x_2}e^{-i\frac{aB}{2}\sin(Bt)x_1}e^{-ia\cos(Bt)p_1}e^{ia\sin(Bt)p_2}\psi_t(x_1,x_2) = \\
= \; & e^{-i\frac{aB}{2}(\sin(Bt)x_1+\cos(Bt)x_2)}\psi_t(x_1-a\cos(Bt),x_2+a\sin(Bt)).
\end{aligned}
$$

If we consider as initial state the centered ground state

$$\psi_t(x) = e^{-i\frac{B}{2}t}e^{-\frac{B}{4}x^2},$$

its shifting $\varphi_t(x) = e^{-iav_2}\psi_t(x)$ is a Gaussian function centered at $x_0 = (a,0)$. Classically speaking we have a particle initially at rest at the origin that gets shifted toward x_0 in a way that leave the center of the orbit invariant. Therefore the shifted particle must have the initial velocity $v = (0,-aB)$ and hence it performs the circular motion $x_t = a(\cos(Bt),-\sin(Bt))$ with velocity $\dot{x}_t = -aB(\sin(Bt),\cos(Bt))$.
According to the classical equations of motion, the canonical momentum of the particle is

$$
\begin{aligned}
p_t &= \dot{x}_t + \vec{A}(x_t) = \frac{B}{2}(-\bar{x}_2,\bar{x}_1) + \frac{1}{2}\dot{x}_t = \\
&= \frac{1}{2}\dot{x}_t = -a\frac{B}{2}(\sin(Bt),\cos(Bt)).
\end{aligned}
$$

We have a similar behavior in the quantum mechanical solution:

$$\varphi_t(x) = e^{i\frac{B}{2}t}e^{ip_t\cdot x-\frac{B}{4}(x-x_t)^2} = e^{(ip_t\cdot x)}\psi_t(x-x_t),$$

that is, a Gaussian function centered at x_t with average momentum p_t.

With the translation operators e^{iax_i} and e^{iav_i} we can prepare an initial state with an arbitrary velocity at an arbitrary position if we start with a centered initial state with average velocity 0.

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