Optimal Left and Right Additive Schwarz Preconditioning for Minimal Residual Methods with Euclidean and Energy Norms

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- We want to solve certain PDEs (non-selfadjoint or indefinit elliptic) discretized by FEM (or divided differences)
- Use GMRES (or other Krylov subspace method)
- Precondition with Additive Schwarz (with coarse grid correction)
- Schwarz methods optimality (energy norm)
  and Minimal Residuals Methods (usually minimizing the 2-norm)
- Comments on left vs. right preconditioning

Examples

- Helmholtz equation $-\Delta u + cu = f$
- Advection diffusion equation $-\Delta u + b.\nabla u + cu = f$
- zero Dirichlet b.c.

General Problem Statement

Solve

$$Bx = f$$

B non-Hermitian, discretization of $b(u,v) = f(v)$

$$\begin{aligned} b(u,v) &= a(u,v) + s(u,v) + c(u,v), \\ a(u,v) &= \int_\Omega \nabla u \cdot \nabla v \, dx, \end{aligned}$$

$$s(u,v) = \int_\Omega (b \cdot \nabla u) v \quad + \quad (\nabla \cdot bu) v \, dx, \qquad b \in \mathbb{R}^d,$$

$$c(u,v) = \int_\Omega c\, uv \, dx, \qquad \text{and} \qquad f(v) = \int_\Omega f\, v \, dx.$$

General Problem Statement (cont.)

$$
\begin{aligned}
b(u,v) &= a(u,v) + s(u,v) + c(u,v), \\
a(u,v) &= \int_\Omega \nabla u \cdot \nabla v \, dx,
\end{aligned}
$$

- Let A be SPD, the discretization of $a(\cdot,\cdot)$.

- So, we have $B = A + R$.

- Here $A = A^T \succ O$ is the discretization of Δu

Standard Finite Element Setting

Let $\Omega \subset \mathbb{R}^d$, with triangulation $\mathcal{T}_h(\Omega)$. Let V be the traditional finite element space formed by piecewise linear and continuous functions vanishing on the boundary of Ω. $V \subset \mathcal{H}_0^1(\Omega)$.

One-to-one correspondence between functions in finite element space and nodal values.
We abuse the notation and do not distinguish between them.
Let $\|v\|_a = a(v, v)$, and $\|v\|_A = (v^T Av)^{1/2}$ be the corresponding norms in V and in $\mathbb{R}^n$, respectively.

Problem Statement (cont.)

- Use Krylov subspace iterative methods (e.g., GMRES)
- Left preconditioning: $M^{-1}Bx = M^{-1}f$
- Right preconditioning: $BM^{-1}(Mu) = f$

Krylov subspace methods: GMRES

$$\mathcal{K}_m(B, r_0) = \text{span}\{r_0, Br_0, B^2 r_0, \ldots, B^{m-1} r_0\}.$$

Given x_0, $r_0 = f - Bx_0$, find approximation

$$x_m \in x_0 + \mathcal{K}_m(B, r_0), \quad x_m = x_0 + p(B) r_0$$

($p(t)$ polynomial degree $m-1$) satisfying

$$x_m = \text{arg min}\{\|f - Bx\|_2\}, \quad x \in x_0 + \mathcal{K}_m(B, r_0)$$

Krylov subspace methods: GMRES (cont.)
Preconditioned case, Implementation

Let $v_1, v_2, \ldots, v_m$ be an orthonormal basis of $\mathcal{K}_m(M^{-1}B, r_0) =$ span$\{r_0, M^{-1}Br_0, (M^{-1}B)^2 r_0, \ldots, (M^{-1}B)^{m-1} r_0\}$.
$x_m =$ arg min$\{\|f - M^{-1}Bx\|_2\}, \quad x \in x_0 + \mathcal{K}_m(M^{-1}B, r_0)$

- With $V_m = [v_1, v_2, \ldots, v_m]$, obtain Arnoldi relation:

$$M^{-1}BV_m = V_{m+1}H_{m+1,m}$$

  $H_{m+1,m}$ is $(m+1) \times m$ upper Hessenberg

- Element in $\mathcal{K}_m(M^{-1}B, v_1)$ is a linear combination of $v_1, v_2, \ldots, v_m$, i.e., of the form $V_m y$, $y \in \mathbb{R}^m$

GMRES (cont.)
Preconditioned case, Implementation

- Find $y = y_m$ and we have $x_m = x_0 + V_m y_m$

$$\begin{aligned} \|M^{-1}f - M^{-1}Bx\|_2 &= \|M^{-1}r_0 - M^{-1}BV_m y\|_2 = \\ = \|V_{m+1}\beta e_1 - V_{m+1}\bar{H}_m y\|_2 &= \|\beta e_1 - \bar{H}_m y\|_2 \end{aligned}$$

find y using QR factorization of $\bar{H}_m$.

One convergence bound for GMRES [Elman 1982]

(unpreconditioned version)

$$\|r_m\| = \|f - Bx_m\| \leq \left(1 - \frac{c^2}{C^2}\right)^{m/2} \|r_0\| \ ,$$

where

$$c = \min_{x \neq 0} \frac{(x, Bx)}{(x, x)} \quad \text{and} \quad C = \max_{x \neq 0} \frac{\|Bx\|}{\|x\|} \ .$$

Schwarz Preconditioning

Class of Preconditioners based on Domain Decomposition

Decomposition of V into a sum of $N+1$ subspaces $R_i^T V_i \subset V$, and

$$V = R_0^T V_0 + R_1^T V_1 + \cdots + R_N^T V_N.$$

$R_i^T : V_i \to V$ extension operator from V_i to V. This decomposition usually NOT a direct sum.

Subspaces $R_i^T V_i$, $i = 1, \ldots, N$ are related to a decomposition of the domain Ω into overlapping subregions Ω_i^δ of size $O(H)$ covering Ω. The subspace $R_0^T V_0$ is the coarse space.

Schwarz Preconditioning (cont.)

For $u_i, v_i \in V_i$ define

$$b_i(u_i, v_i) = b(R_i^T u_i, R_i^T v_i), \qquad a_i(u_i, v_i) = a(R_i^T u_i, R_i^T v_i).$$

Let

$$B_i = R_i B R_i^T, \qquad A_i = R_i A R_i^T$$

be the matrix representations of these local bilinear forms, i.e., the local problems.

R_i is a restriction operator, R_i^T is a prolongation (embeding) operator

Two versions of Additive Schwarz Preconditioning here

$M^{-1} = R_0^T B_0^{-1} R_0 + \sum_{i=1}^p R_i^T B_i^{-1} R_i,$

or $M^{-1} = R_0^T B_0^{-1} R_0 + \sum_{i=1}^p R_i^T A_i^{-1} R_i,$

where $B_i = R_i B R_i^T$ and $A_i = R_i A R_i^T$ (local problems)

R_i restriction, R_i^T prolongation with overlap δ

B_0 coarse problem, size $O(H)$, discretization $O(h)$.

Let $P = M^{-1}B$, be the preconditioned problem.

Theorem. [Cai and Widlund, 1993]

There exist constants $H_0 > 0$, $c(H_0) > 0$, and $C(H_0) > 0$, such that if $H \leq H_0$, then for $i = 1, 2$, and $u \in V$,

$$\frac{a(u, Pu)}{a(u, u)} \geq c_p,$$

and

$$\|Pu\|_a \leq C_p \|u\|_a,$$

where $C_p = C(H_0)$ and $c_p = C_0^{-2} c(H_0)$.

Two-level Schwarz preconditioners are optimal in the sense that bounds for $M^{-1}B$ (or BM^{-1}) are independent of the mesh size and the number of subdomains, or slowly varying with them.

In our PDEs, we have optimal bounds:

$$\frac{(x, M^{-1}Bx)_A}{(x, x)_A} \geq c_p \quad \text{and} \quad \|M^{-1}Bx\|_A \leq C_p \|x\|_A.$$

Cai and Zou [NLAA, 2002] observed:

Schwarz bounds use energy norms, while GMRES minimizes l_2 norms.

Optimality may be lost!

In fact, Cai and Zou [NLAA, 2002] showed is that

for Additive Schwarz $M^{-1}B$ is NOT positive real,

i.e., there is no $c > 0$ for which

$$\frac{(x, M^{-1}Bx)}{(x, x)} \geq c.$$

Thus, this GMRES bound cannot be used in this case.

We may not have the optimality.

Krylov Subspace Methods with Energy Norms

Proposed solution: Use GMRES minimizing the A-norm of the residual.

[Note: many authors mention this, e.g., Ashby-Manteuffel-Saylor, Essai, Greenbaum, Gutknecht, Weiss, ...]

In this case, we have that $M^{-1}B$ is positive real with respect to the A-inner product since

$$\frac{(x, M^{-1}Bx)_A}{(x,x)_A} \geq c_p \quad \text{and} \quad \|M^{-1}Bx\|_A \leq C_p \|x\|_A.$$

Rework convergence bound for GMRES [Elman 1982]

(preconditioned version)

$$\|r_m\|_A = \|M^{-1}f - M^{-1}Bx_m\|_A \le \left(1 - \frac{c^2}{C^2}\right)^{m/2} \|M^{-1}r_0\|_A \ ,$$

where

$$c = \min_{x \neq 0} \frac{(x, M^{-1}Bx)_A}{(x,x)_A} \quad \text{and} \quad C = \max_{x \neq 0} \frac{\|Bx\|_A}{\|x\|_A} \ .$$

Implementation:

Replace each inner product (x, y) with $(x, y)_A = x^T Ay$.

Only one matvec with A needed. Basis vectors are A-orthonormal.

Arnoldi relation: $M^{-1}BV_m = V_{m+1}H_{m+1}$.

$$\begin{aligned} \|M^{-1}b - M^{-1}Bx\|_A &= \|M^{-1}r_0 - M^{-1}BV_m y\|_A = \\ = \|V_{m+1}\beta e_1 - V_{m+1}\bar{H}_m y\|_A &= \|\beta e_1 - \bar{H}_m y\|_2 \end{aligned}$$

Same QR factorization of $\bar{H}_m$, same code for the minimization.

We use this for analysis, but sometimes also valid for computations.

Left vs. Right preconditioner

For right preconditioner $BM^{-1}u = f$, $M^{-1}u = x$.

$(x,x)_A = (M^{-1}u, M^{-1}u)_A = (u,u)_G, \qquad G = M^{-T}AM^{-1}.$

- Every left preconditioned system $M^{-1}Bx = M^{-1}f$ with the A norm is completely equivalent to a right preconditioned system with the $M^{-T}AM^{-1}$-norm.

$$\begin{aligned} \|r_0 - BM^{-1}Z_m y\|_{M^{-T}AM^{-1}} &= \|M^{-1}r_0 - M^{-1}BM^{-1}Z_m y\|_A \\ = \|\beta z_1 - M^{-1}BV_m y\|_A &= \|\beta e_1 - \bar{H}_m y\|_2 \ . \end{aligned}$$

Z_m has the G-orthogonal basis of $\mathcal{K}_m(r_0, BM^{-1})$

Left vs. Right preconditioner

- Converse also holds: for every right preconditioner M with S-norm, this is equivalent to left preconditioning with M using the $M^T SM$-norm. (True in particular for $S = I$)
- When using the same inner product (norm), left and right preconditioning produce different upper Hessenberg matrices H_m.
- When using A-inner product for left preconditioning and $M^{-T}AM^{-1}$-inner product for right preconditioning, we have the same upper Hessenberg matrices H_m.
- The experiments we show with left preconditioning and A-norm minimization are the same as with right preconditioning with G-norm minimization, $G = M^{-T}AM^{-1}$.

Energy Norms vs. ℓ_2 Norm

Now, we "have" the optimality with energy norms.

What can we say about the ℓ_2 norm? Use equivalence of norms:

$$\|x\|_2 \le \frac{1}{\sqrt{\lambda_{\min}(A)}}\|x\|_A, \qquad \|x\|_A \le \sqrt{\lambda_{\max}(A)}\|x\|_2$$

$$\begin{aligned}
\|M^{-1}r_m^L\|_2 \quad &\le \quad \|M^{-1}r_m^A\|_2 \le \frac{1}{\sqrt{\lambda_{\min}(A)}}\|M^{-1}r_m^A\|_A \\
&\le \quad \frac{1}{\sqrt{\lambda_{\min}(A)}}\left(1-\frac{c^2}{C^2}\right)^{m/2}\|M^{-1}r_0\|_A \\
&\le \quad \frac{\sqrt{\lambda_{\max}(A)}}{\sqrt{\lambda_{\min}(A)}}\left(1-\frac{c^2}{C^2}\right)^{m/2}\|M^{-1}r_0\|_2 \\
&= \quad \sqrt{\kappa(A)}\left(1-\frac{c^2}{C^2}\right)^{m/2}\|M^{-1}r_0\|_2
\end{aligned}$$

“Asymptotic” Optimality of ℓ_2 Norm

$$\|M^{-1}r_m^L\|_2 \quad \leq \quad \sqrt{\kappa(A)}\left(1-\frac{c^2}{C^2}\right)^{m/2}\|M^{-1}r_0\|_2$$

For a fixed mesh size h, Additive Schwarz preconditioned GMRES (2-norm) has a bound that goes to zero at the same speed as the optimal bound (energy norm), except for a factor $\sqrt{\kappa(A)}$ (which of course depends on h)

Numerical Experiments

- Helmholtz equation $-\Delta u + cu = f$, $c = -5$ or $c = -120$.
- Advection diffusion equation $-\Delta u + b.\nabla u + cu = f$
  $b^T = [10, 20]$, $c = 1$, upwind finite differences
- both on unit square, zero Dirichlet b.c., $f \equiv 1$
- Discretization: 64×64 $(n = 3969)$,
  128×128 $(n = 16129)$, or 256×256 $(n = 65025)$ nodes
  $p = 4 \times 4$ or $p = 8 \times 8$ subdomains
  Overlap: $0, 1, 2$ (1,3 or 5 lines of nodes)
- Tolerance $\varepsilon = 10^{-8}$

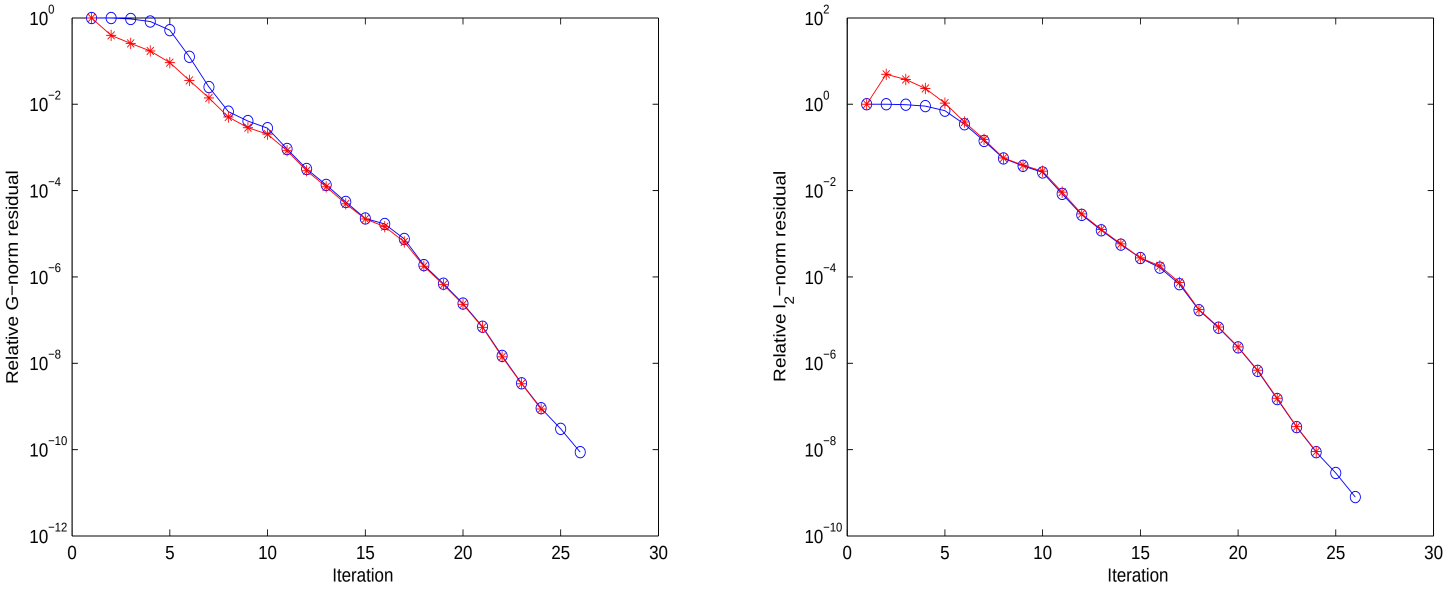

Figure 1: Helmholtz equation $c = -5$. GMRES minimizing the ℓ_2 norm (o), and the G-norm (*). 64×64 grids, 4×4 subdomains. $\delta = 0$ Left: G-norm of both residuals. Right: ℓ_2 norm of both residuals.

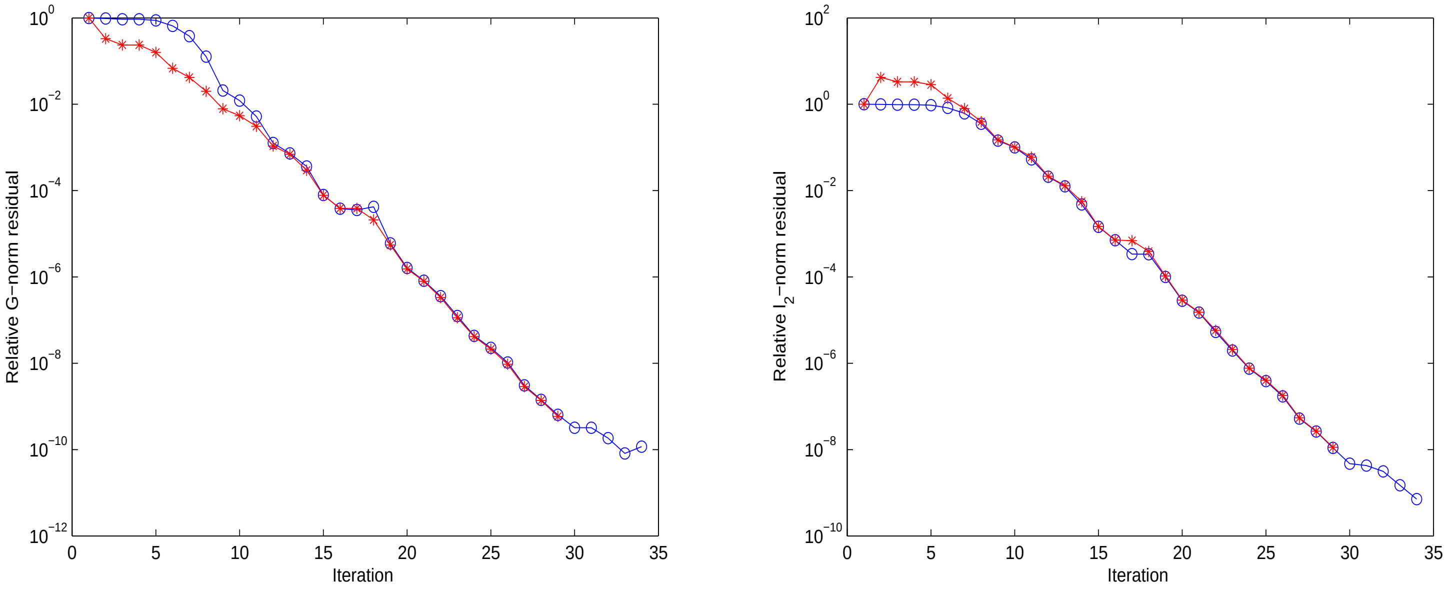

Figure 2: Helmholtz equation $c = -120$. GMRES minimizing the ℓ_2 norm (o), and the G-norm (*). 128×128 grids, 8×8 subdomains. $\delta = 1$ Left: G-norm of both residuals. Right: ℓ_2 norm of both residuals.

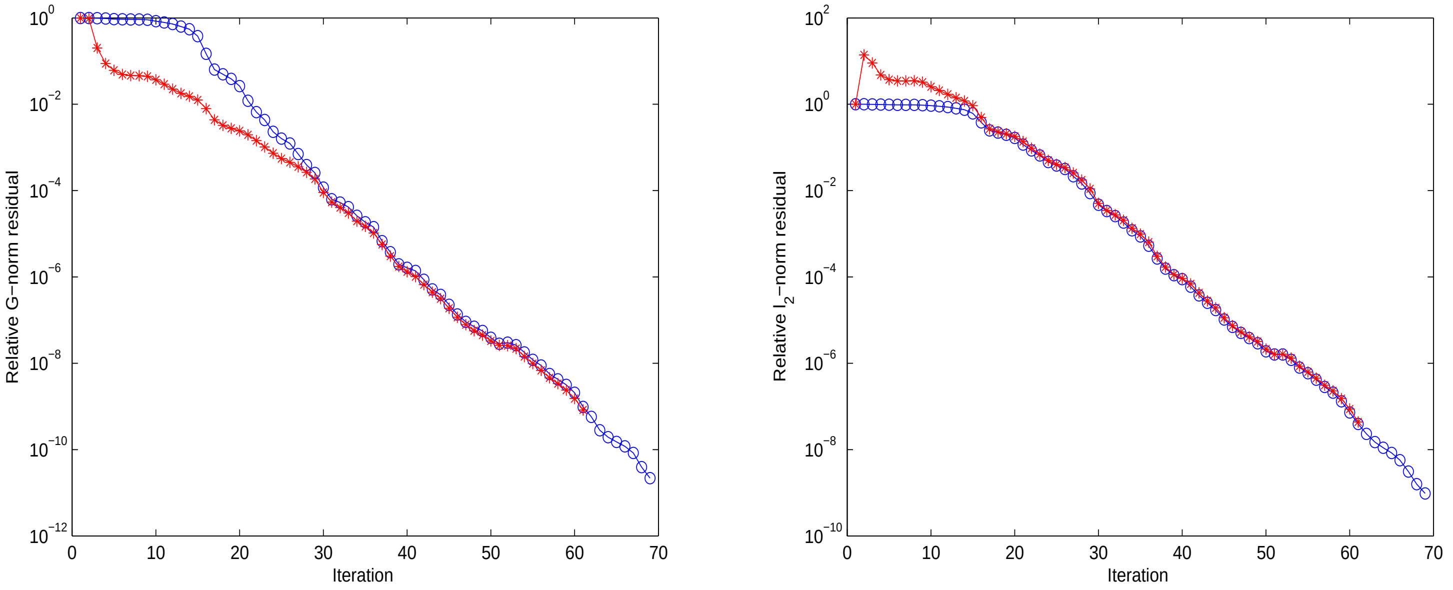

Figure 3: Helmholtz equation $c = -120$. GMRES minimizing the ℓ_2 norm (o), and the G-norm (*). 256×256 grids, 8×8 subdomains. $\delta = 0$ Left: G-norm of both residuals. Right: ℓ_2 norm of both residuals.

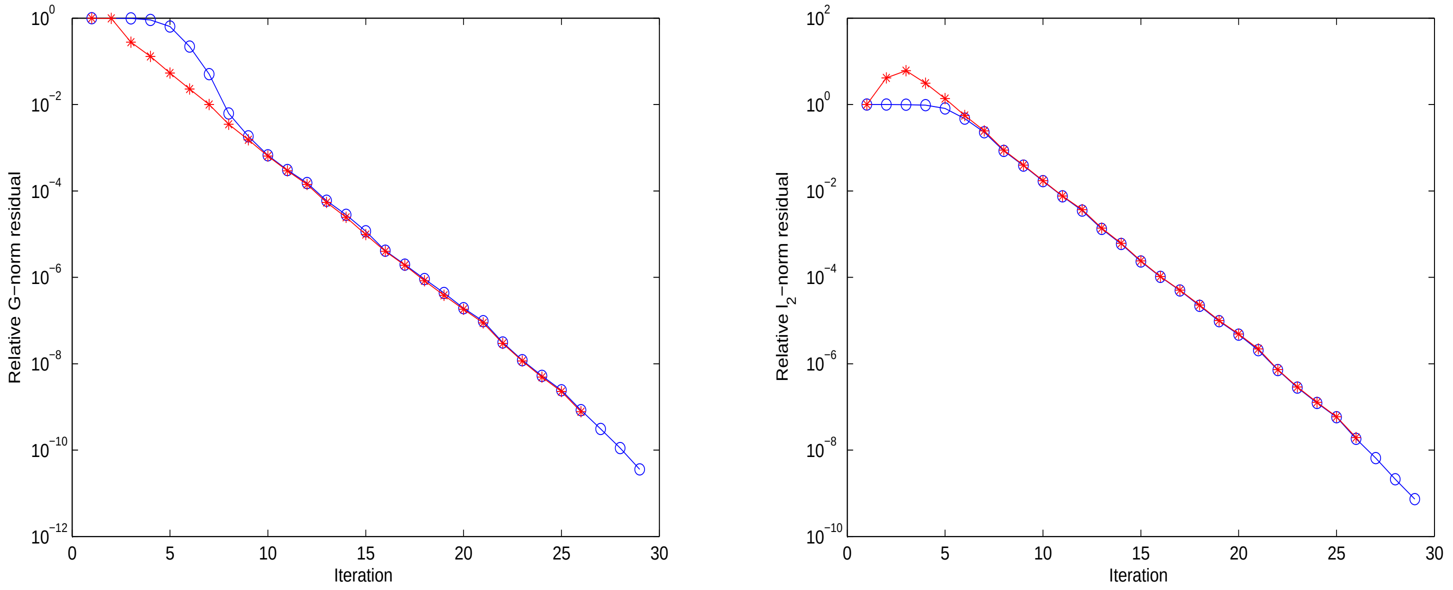

Figure 4: Advection-diffusion equation. GMRES minimizing the ℓ_2 norm (o), and the G-norm (*). 128×128 grids, 4×4 subdomains. $\delta = 2$ Left: G-norm of both residuals. Right: ℓ_2 norm of both residuals.

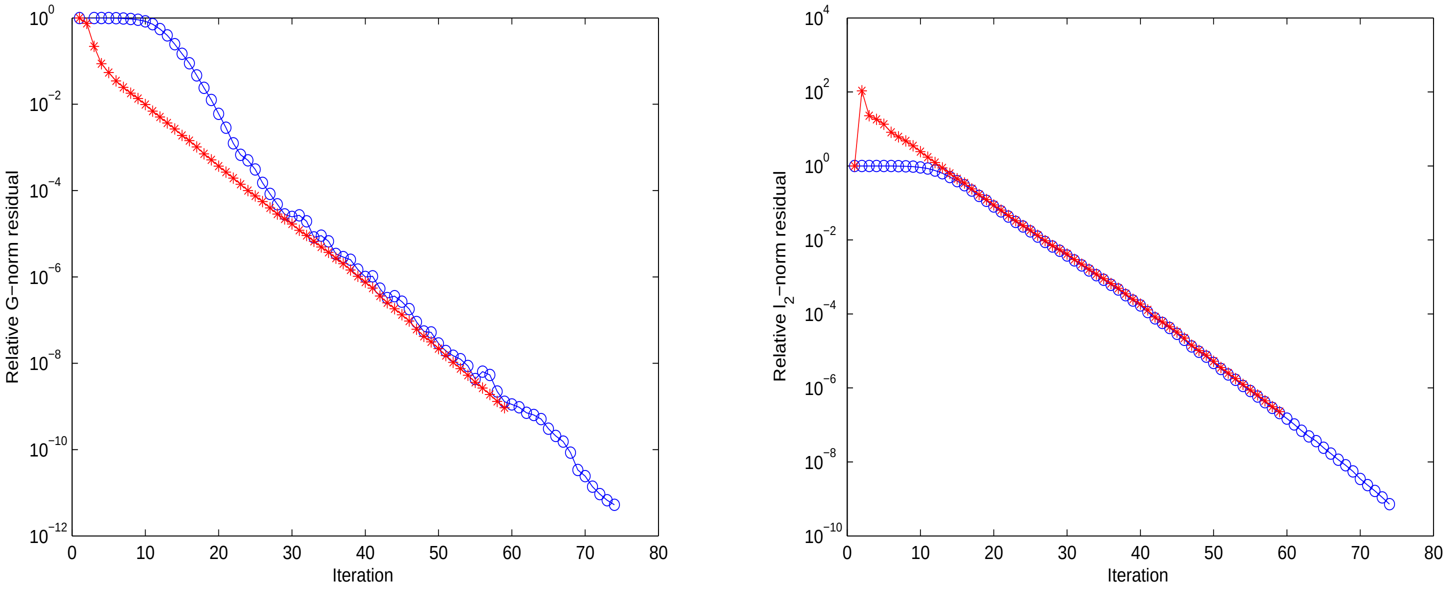

Figure 5: Advection-diffusion equation. GMRES minimizing the ℓ_2 norm (o), and the G-norm (*). 256×256 grids, 8×8 subdomains. $\delta = 0$ Left: G-norm of both residuals. Right: ℓ_2 norm of both residuals.

Conclusions

- GMRES in energy norm maintains optimality
- GMRES in ℓ_2 norm achieves "asymptotic" optimality
- Observations on left vs. right preconditioning
- Numerical experiments illustrate this

Paper to appear in *CMAME*

available at `http://www.math.temple.edu/szyld`