# Locking phenomena in Computational Mechanics: nearly incompressible materials and plate problems

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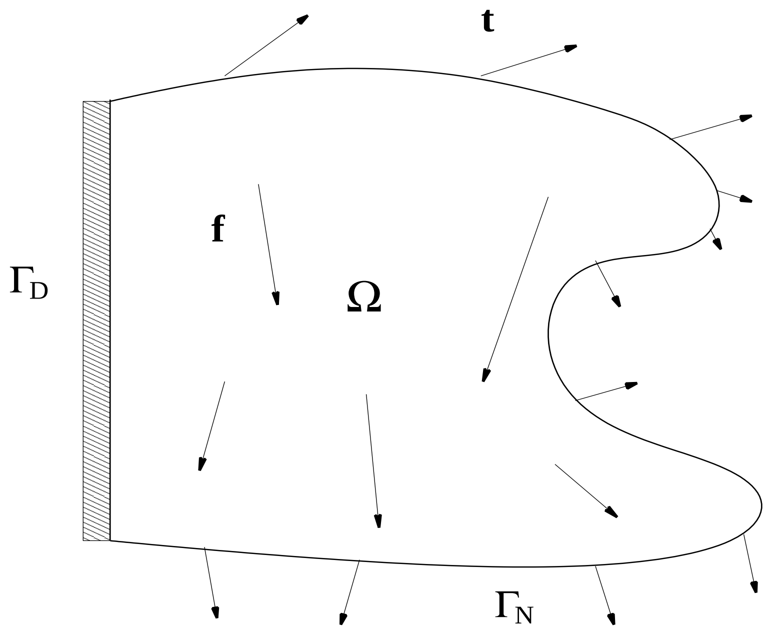
# Aim of the talk

- Discuss the troubles of problems with small (or large) parameters.
  1. Understand the roots of the difficulties.
  2. Suggest possible cures.
- Examples: Finite Element approximation of nearly incompressible elasticity and plate problems.

# THE LINEAR ELASTICITY PROBLEM

- Plane linear elasticity problem in the framework of the infinitesimal theory, and plane strain case.
- Isotropic and Homogeneous material.



Solve the problem: Find $\mathbf{u}$ such that

$$\begin{cases} -\operatorname{div}\Big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda(\operatorname{div}\mathbf{u})\boldsymbol{\delta}\Big) = \mathbf{f} & \text{in } \Omega \\ \mathbf{u} = \mathbf{u}_0 & \text{on } \Gamma_D \\ \big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda(\operatorname{div}\mathbf{u})\boldsymbol{\delta}\big)\mathbf{n} = \mathbf{t} & \text{on } \Gamma_N \end{cases}$$

1. $\mathbf{u}$ , $\mathbf{f}$ : $\Omega \to \mathbf{R}^2$ are the displacement field and the loading term;
2. $\boldsymbol{\varepsilon}(\cdot)$ is the symmetric gradient operator;
3. $\boldsymbol{\delta}$ is the second-order identity tensor.

$\mu$ and $\lambda$ are the Lamé coefficients

$0 < \mu_0 \leq \mu \leq \mu_1 < +\infty$ and $0 < \lambda_0 \leq \lambda \leq +\infty$

## Equivalent formulation

Find $\mathbf{u}$ which minimizes the elastic energy

$$E(\mathbf{v}) = \mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{v}) + \frac{\lambda}{2} \int_\Omega |\operatorname{div} \mathbf{v}|^2 - \int_\Omega \mathbf{f} \cdot \mathbf{v} - \int_{\Gamma_N} \mathbf{t} \cdot \mathbf{v}$$

over a suitable set of admissible functions $\mathbf{V}$.

Remark. For homogeneous Dirichlet boundary conditions, as in the sequel:

$$\mathbf{V} = \left\{ \mathbf{v} \in \mathbf{H}^1(\Omega) \ : \ \mathbf{v}|_{\Gamma_D} = \mathbf{0} \right\}$$

# Variational formulation

Find $\mathbf{u} \in \mathbf{V}$ such that:

$$2\mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) + \lambda \int_\Omega \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} = F(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}$$

where

$$F(\mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \mathbf{t} \cdot \mathbf{v}$$

- The bilinear form $2\mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) + \lambda \int_\Omega \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}$ is continuous, symmetric and coercive; $F$ is linear and continuous.

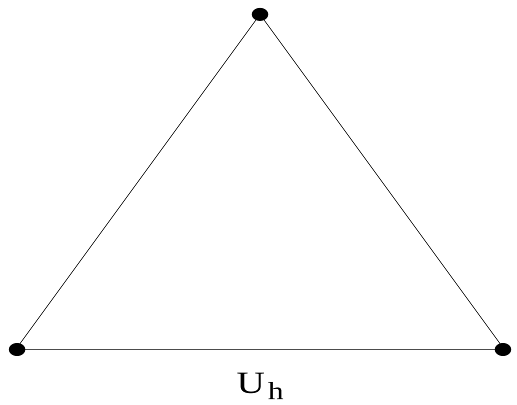
Consequence

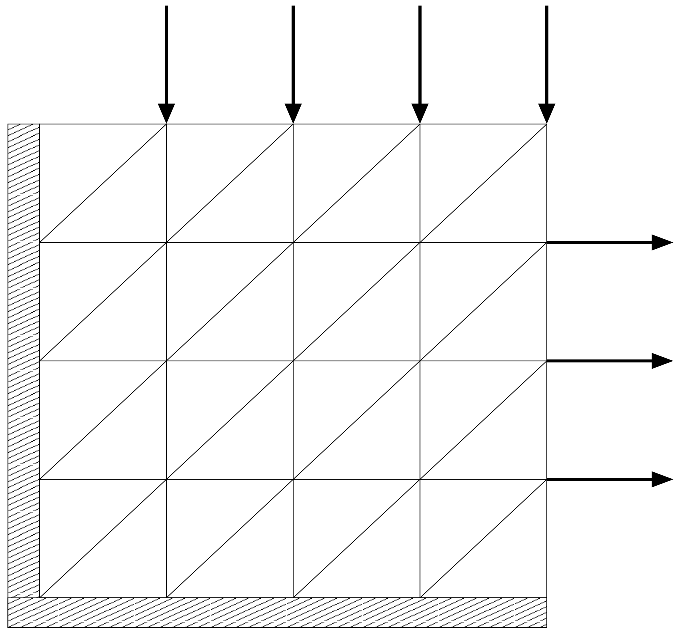
The problem has a unique solution and there is stability (continuous dependence on the data).

# Standard Finite Elements

Choose $\mathbf{V}_h$ finite element space. Find $\mathbf{u}_h \in \mathbf{V}_h$ such that:

$$2\mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{u}_h) + \lambda \int_\Omega \operatorname{div} \mathbf{u}_h \operatorname{div} \mathbf{v}_h = F(\mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h$$

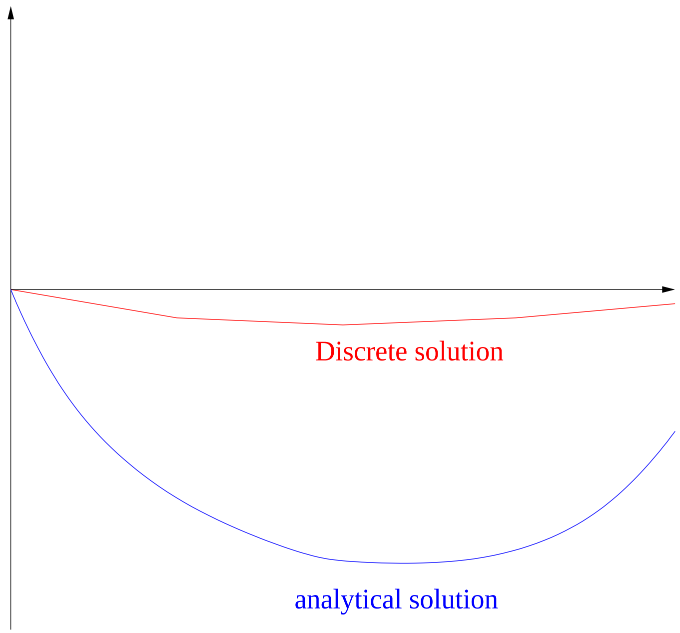




- Test problem using the above mesh and data (given tractions on Neumann boundary)
- Rubber material (nearly incompressible): $\lambda/\mu >> 1$

A "good" description of the deformation is expected!!

HOWEVER...




- blue: the analytical solution.
- red: the discrete solution.

The method HEAVILY underestimates the solution

# WHY??

- We need to recall the energy functional:

$$E(\mathbf{v}) = \mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{v}) + \frac{\lambda}{2} \int_\Omega |\operatorname{div} \mathbf{v}|^2 - F(\mathbf{v})$$

which has to be minimized over $\mathbf{V}$.

$\lambda/\mu >> 1$ "corresponds" to

$$\mu \sim 1 \qquad , \qquad \lambda \to +\infty$$

Therefore, for $\lambda/\mu >> 1$, the minimiser $\mathbf{u} \in \mathbf{V}$ satisfies:

$$\operatorname{div} \mathbf{u} \sim 0$$

# The limit problem

It can be shown that $\mathbf{u} \to \mathbf{u}^0$ as $\lambda \to +\infty$, where $\mathbf{u}^0$ solves the the limit problem

Find $\mathbf{u}^0 \in \mathbf{K}$ which minimizes

$$E^0(\mathbf{v}) = \mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{v}) - F(\mathbf{v}) \qquad \text{in } \mathbf{K}$$

$$\mathbf{K} = \{\mathbf{v} \in \mathbf{V} \; : \; \operatorname{div} \mathbf{v} = 0\}$$

Remark: The above problem is well-posed and reasonable. It is the elasticity problem for incompressible materials.

## Finite Element & Minimization

- Standard Finite Elements corresponds to

  Find $\mathbf{u}_h \in \mathbf{V}_h$ which minimizes

$$E(\mathbf{v}_h) = \mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{v}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h) + \frac{\lambda}{2} \int_\Omega |\operatorname{div} \mathbf{v}_h|^2 - F(\mathbf{v}_h) \qquad \text{in } \mathbf{V}_h$$

Remark: The SAME energy, but different admissible functions.

# The Finite Element limit problem

It can be shown that $\mathbf{u}_h \to \mathbf{u}_h^0$ as $\lambda \to +\infty$, where $\mathbf{u}_h{}^0$ solves the the limit problem

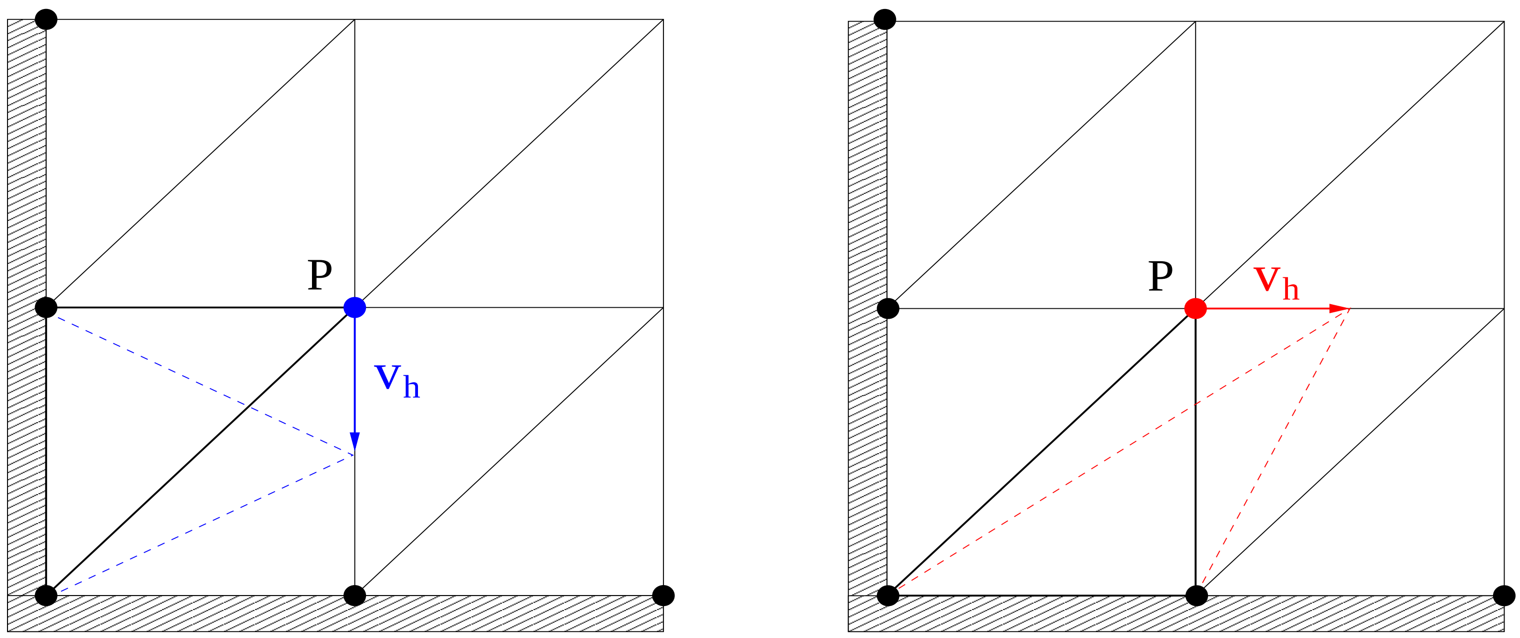
Find $\mathbf{u}_h{}^0 \in \mathbf{K}_h$ which minimizes

$$E(\mathbf{v}_h) = \mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{v}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h) - F(\mathbf{v}_h) \qquad \text{in } \mathbf{K}_h$$

$$\mathbf{K}_h = \{\mathbf{v}_h \in \mathbf{V}_h \ : \ \operatorname{div} \mathbf{v}_h = 0\} = \mathbf{K} \cap \mathbf{V}_h$$

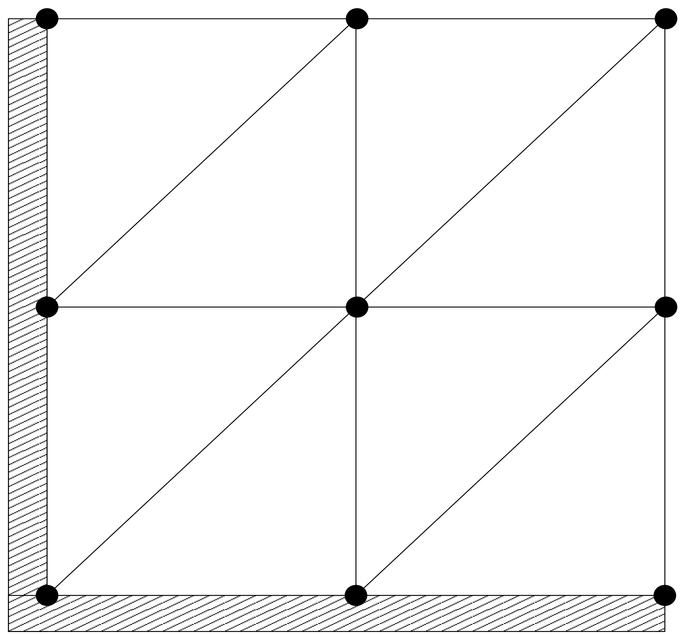
Question: Is it a "good" limit problem?

$$\mathbf{v}_h \in \mathbf{K}_h \Longrightarrow \left\{ \mathbf{v}_h \text{ is continuous, piecewise linear, and } \operatorname{div} \mathbf{v}_h = 0 \right\}$$




Therefore, it holds: $\mathbf{v}_h(P) = \mathbf{0}$

Repeating the argument: $\mathbf{v}_h \in \mathbf{K}_h \Longrightarrow \mathbf{v}_h \equiv \mathbf{0}$



Therefore, $\mathbf{K}_h = (\mathbf{0})$

# The Finite Element limit problem revisited

Find $\mathbf{u}_h^0 \in \mathbf{K}_h$ which minimizes

$$E^0(\mathbf{v}_h) = \mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{v}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h) - F(\mathbf{v}_h) \qquad \text{in } \mathbf{K}_h$$

$$\mathbf{K}_h = (\mathbf{0})$$

- The divergence-free constraint is too severe for finite element functions: Volumetric Locking!

# A possible cure

The enemy: the term $\frac{\lambda}{2}\int_\Omega |\operatorname{div}\mathbf{v}_h|^2$ for $\lambda \to +\infty$

Idea

Take $\frac{\lambda}{2}\int_\Omega |P_h(\operatorname{div}\mathbf{v}_h)|^2$

$P_h$ suitable "reduction operator" (typically a projection operator)

# New Finite Element problem

Find $\mathbf{u}_h \in \mathbf{V}_h$ which minimizes

$$E_h(\mathbf{v}_h) = \mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{v}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h) + \frac{\lambda}{2} \int_\Omega |P_h(\operatorname{div} \mathbf{v}_h)|^2 - F(\mathbf{v}_h) \qquad \text{in } \mathbf{V}_h$$

Limit Problem

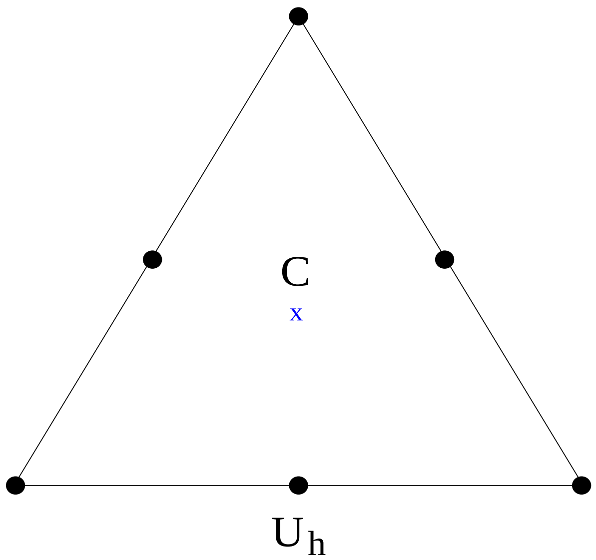
Find $\mathbf{u}_h^0 \in \mathbf{K}_h$ which minimizes

$$E_h^0(\mathbf{v}_h) = \mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{v}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h) - F(\mathbf{v}_h) \qquad \text{in } \mathbf{K}_h$$

$$\mathbf{K}_h := \{\mathbf{v}_h \in \mathbf{V}_h \ : \ P_h(\operatorname{div} \mathbf{v}_h) = 0\,\}$$

Remark: $P_h(\operatorname{div} \mathbf{v}_h) = 0$ may be weaker than $\operatorname{div} \mathbf{v}_h = 0$.

# Example: pw. quadratic functions (not recommended)




$P_h$ : Projection operator on piecewise constant functions

$$\mathbf{v}_h \in \mathbf{K}_h \Longrightarrow \operatorname{div} \mathbf{v}_h(C) = 0$$

# Classical Approaches

- Galerkin Least–Squares Methods: Hughes–Franca (1985-1986)
- Enhanced Strain Methods: Simo–Rifai (1990), Pantuso–Bathe (1995)...
- MINI element: Arnold, Brezzi, Fortin (1984).
- Taylor–Hood Elements (1973).
- Non-conforming methods: Crouziex-Raviart Element (1973).

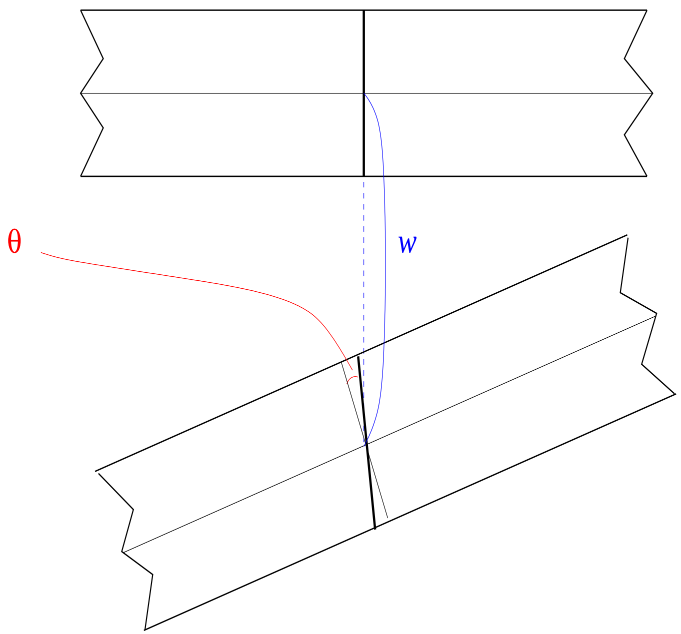
Remark: Most of them are Mixed Methods.

## THE PLATE PROBLEM

We consider a plate subjected to a transversal load (with respect to its mid-plane)

The 2D Reissner-Mindlin model

- Undeformed plate: made up by fibers, which are rectilinear and perpendicular to the mid-plane $\Omega$.
- Due to deformation, the fibers remain rectilinear, but they are not perpendicular to the mid-plane anymore.




## Problem unknowns

- The vectorial field $\boldsymbol{\theta} = \boldsymbol{\theta}(x, y)$ (fiber rotations).
- The scalar field $w = w(x, y)$ (vertical displacements).

## The equations (clamped plate)

Find $(\boldsymbol{\theta}, w)$ s.t.

$$-\operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}) - \lambda t^{-2}(\boldsymbol{\nabla} w - \boldsymbol{\theta}) = 0 \quad \text{in } \Omega,$$

$$-\operatorname{div} \left(\lambda t^{-2}(\boldsymbol{\nabla} w - \boldsymbol{\theta})\right) = g \quad \text{in } \Omega,$$

$$\boldsymbol{\theta} = 0,\ w = 0 \text{ on } \partial\Omega.$$

- $\mathbf{C}$ and $\lambda$: material parameters;
- $t$: plate thickness ($t << \text{diam}(\Omega)$);
- $g$: transversal load.

# Energy minimization

Find $(\boldsymbol{\theta}, w)$ which minimizes

$$E(\boldsymbol{\eta}, v) = \frac{1}{2}\int_\Omega \mathbf{C}\,\varepsilon(\boldsymbol{\eta}) : \varepsilon(\boldsymbol{\eta}) + \frac{\lambda t^{-2}}{2}\int_\Omega |\boldsymbol{\nabla} v - \boldsymbol{\eta}|^2 - \int_\Omega g v$$

over the admissible space $\boldsymbol{\Theta} \times W$

$$\boldsymbol{\Theta} = (H_0^1(\Omega))^2\ ,\ W = H_0^1(\Omega)$$

# Variational formulation

Find $(\boldsymbol{\theta}, w) \in \boldsymbol{\Theta} \times W$ such that

$$\int_\Omega \mathbf{C}\,\varepsilon(\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\eta}) + \lambda t^{-2} \int_\Omega (\boldsymbol{\nabla} w - \boldsymbol{\theta}) \cdot (\boldsymbol{\nabla} v - \boldsymbol{\eta}) - \int_\Omega g v$$

for every $(\boldsymbol{\eta}, v) \in \boldsymbol{\Theta} \times W$.

- The bilinear form $\int_\Omega \mathbf{C}\,\varepsilon(\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\eta}) + \lambda t^{-2} \int_\Omega (\boldsymbol{\nabla} w - \boldsymbol{\theta}) \cdot (\boldsymbol{\nabla} v - \boldsymbol{\eta})$ is continuous, symmetric and coercive; $\int_\Omega g v$ is linear and continuous.

Consequence

The problem has a unique solution and there is stability (continuous dependence on the data).

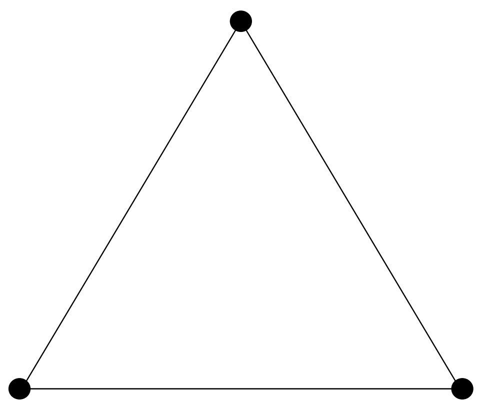
# Standard Finite Elements

Choose $\mathbf{\Theta}_h \subset \mathbf{\Theta}$ and $W_h \subset W$ finite element spaces.

Find $(\boldsymbol{\theta}_h, w_h) \in \mathbf{\Theta}_h \times W_h$ s.t.

$$\int_\Omega \mathbf{C}\,\varepsilon(\boldsymbol{\theta}_h) : \varepsilon(\boldsymbol{\eta}_h) + \lambda t^{-2} \int_\Omega (\boldsymbol{\nabla} w_h - \boldsymbol{\theta}_h) \cdot (\boldsymbol{\nabla} v_h - \boldsymbol{\eta}_h) - \int_\Omega g v_h$$

for every $(\boldsymbol{\eta}_h, v_h) \in \mathbf{\Theta}_h \times W_h$.



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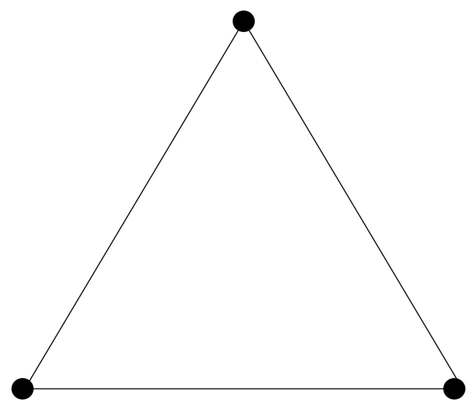

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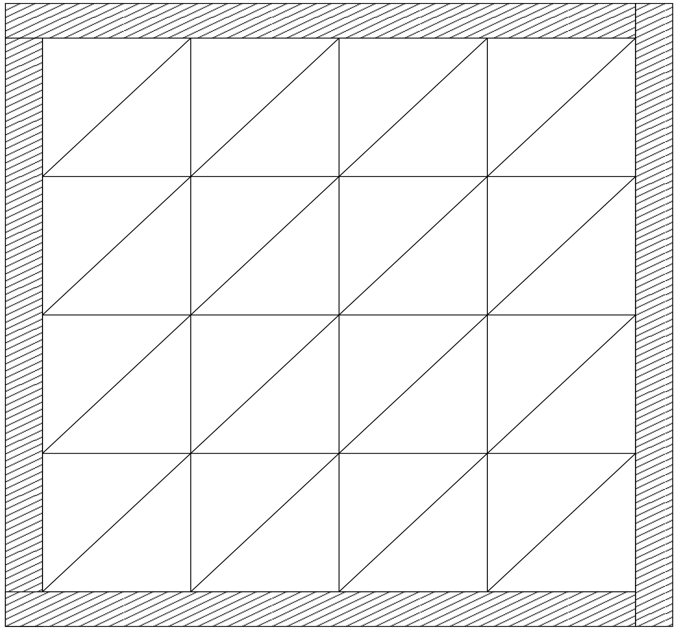
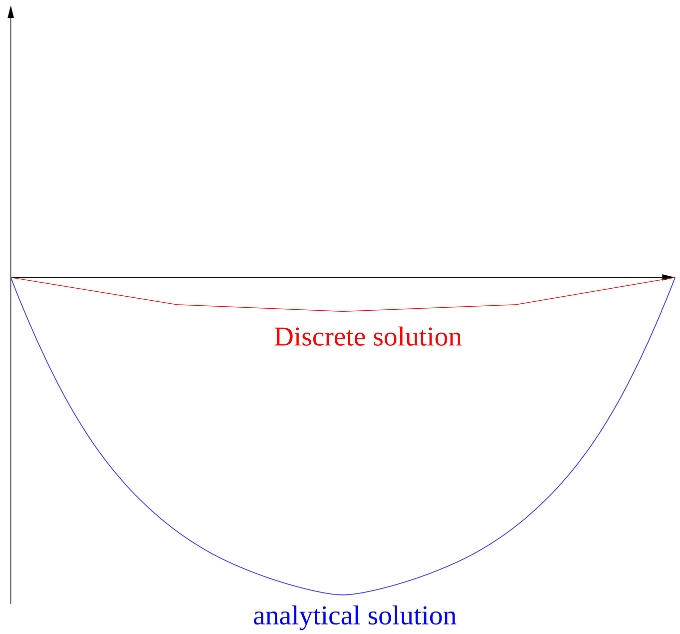


- Test problem using the above mesh and uniform constant load
- Thin plate: $\mathrm{diam}(\Omega)/t >> 1$

A "good" description of the deformation is expected!!

HOWEVER...




- blue profile: the analytical solution.
- red profile: the discrete solution.

The method HEAVILY underestimates the solution

# Energy minimization

The minimization problem $\boldsymbol{\Theta} \times W$ for:

$$E(\boldsymbol{\eta}, v) = \frac{1}{2}\int_\Omega \mathbf{C}\,\varepsilon(\boldsymbol{\eta}) : \varepsilon(\boldsymbol{\eta}) + \frac{\lambda t^{-2}}{2}\int_\Omega |\boldsymbol{\nabla} v - \boldsymbol{\eta}|^2 - \int_\Omega g v$$

converges to the limit problem:

Find $(\boldsymbol{\theta}^0, w^0) \in \mathbf{K}$ which minimizes

$$E^0(\boldsymbol{\eta}, v) = \frac{1}{2}\int_\Omega \mathbf{C}\,\varepsilon(\boldsymbol{\eta}) : \varepsilon(\boldsymbol{\eta}) - \int_\Omega g v \qquad (\boldsymbol{\eta}, v) \in \mathbf{K}$$

$$\mathbf{K} = \{(\boldsymbol{\eta}, v) \in \boldsymbol{\Theta} \times W \;:\; \boldsymbol{\nabla} v = \boldsymbol{\eta}\}$$

Remark It is a coercive problem in $\mathbf{K}$.

# Finite Elements

The minimization problem in $\boldsymbol{\Theta}_h \times W_h$ for:

$$E(\boldsymbol{\eta}_h, v_h) = \frac{1}{2}\int_\Omega \mathbf{C}\,\varepsilon(\boldsymbol{\eta}_h) : \varepsilon(\boldsymbol{\eta}_h) + \frac{\lambda t^{-2}}{2}\int_\Omega |\boldsymbol{\nabla} v_h - \boldsymbol{\eta}_h|^2 - \int_\Omega g v_h$$

converges to the limit problem:

Find $(\boldsymbol{\theta}_h^0, w_h^0) \in \mathbf{K}_h$ which minimizes

$$E^0(\boldsymbol{\eta}_h, v_h) = \frac{1}{2}\int_\Omega \mathbf{C}\,\varepsilon(\boldsymbol{\eta}_h) : \varepsilon(\boldsymbol{\eta}_h) - \int_\Omega g v_h \qquad (\boldsymbol{\eta}_h, v_h) \in \mathbf{K}_h$$

$$\mathbf{K}_h = \{(\boldsymbol{\eta}_h, v_h) \in \boldsymbol{\Theta}_h \times W_h \ : \ \boldsymbol{\nabla} v_h = \boldsymbol{\eta}_h\}$$

## Structure of $\mathbf{K}_h$

$$(\boldsymbol{\eta}_h, v_h) \in \mathbf{K}_h \Longrightarrow \boldsymbol{\nabla} v_h = \boldsymbol{\eta}_h \in C^0(\Omega) \Longrightarrow v_h \in C^1(\Omega)$$

But

$$\{v_h \in C^1(\Omega) \text{ and piecewise linear}\} \Longrightarrow \; v_h \text{ is globally linear}$$

It follows

$$\{v_h \text{ is globally linear and } v_h = 0 \text{ on the boundary}\} \Longrightarrow \; v_h \equiv 0 \Longrightarrow \boldsymbol{\eta}_h = \mathbf{0}$$

Hence

$$\mathbf{K}_h = (\mathbf{0}) \qquad \text{Shear Locking!!}$$

# A possible cure

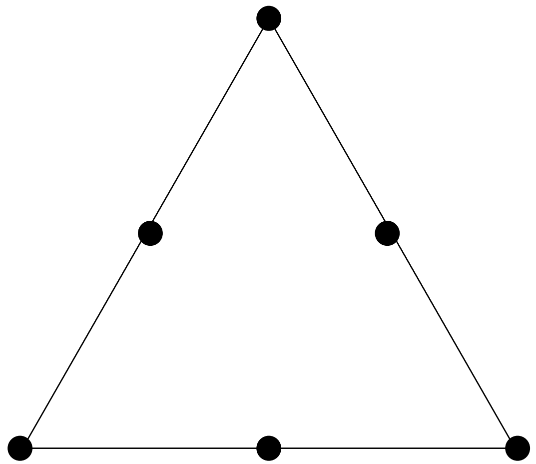
The enemy: the term $\frac{\lambda t^{-2}}{2} \int_\Omega |\boldsymbol{\nabla} v_h - \boldsymbol{\eta}_h|^2$ for $t \to 0$

Idea

Take $\frac{\lambda t^{-2}}{2} \int_\Omega |R_h(\boldsymbol{\nabla} v_h - \boldsymbol{\eta}_h)|^2$

$R_h$ suitable "reduction operator"

# Example: pw. quadratic functions



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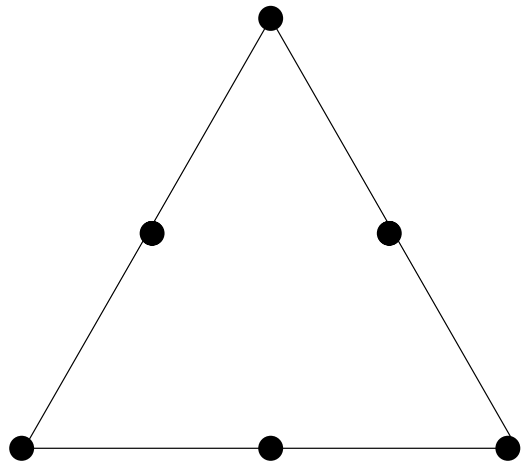

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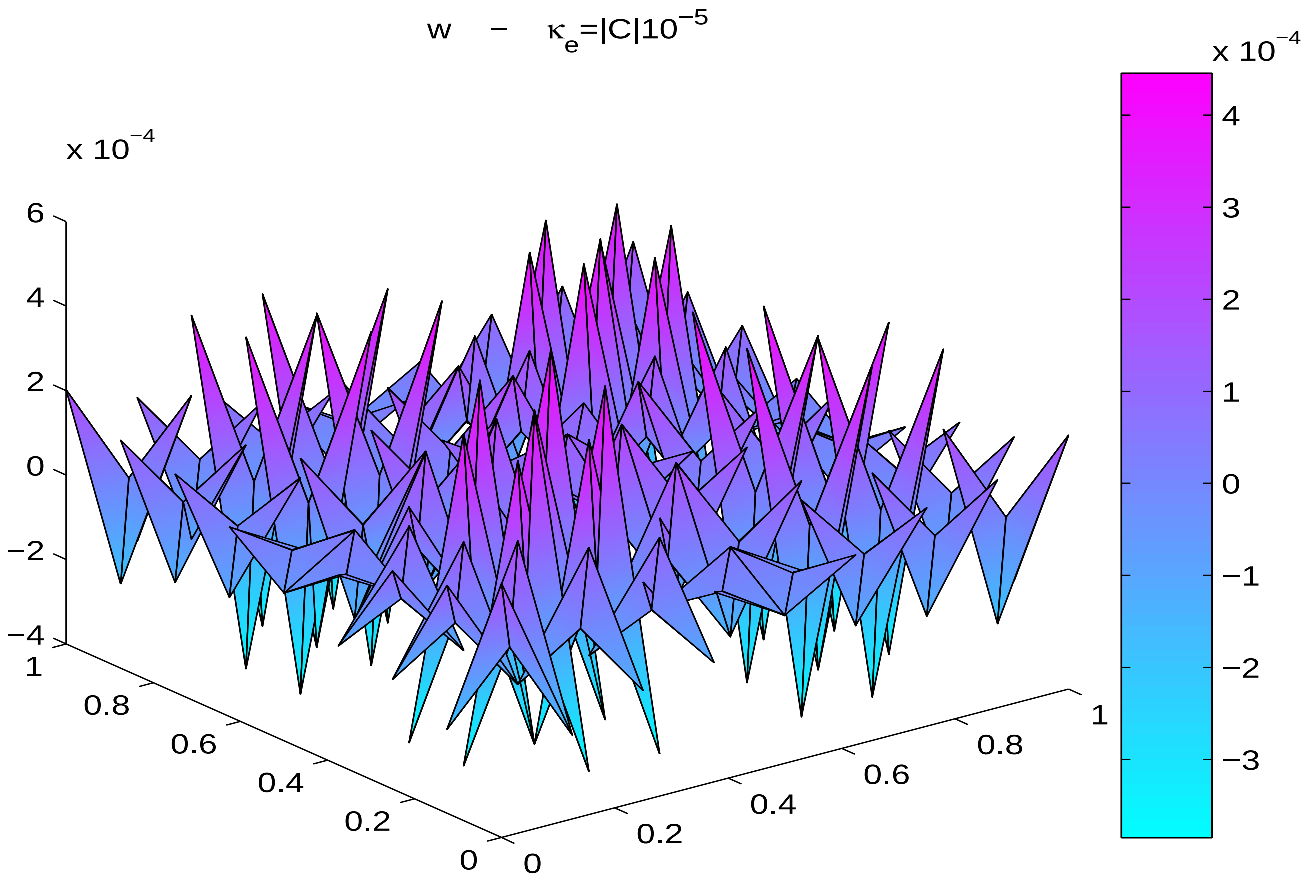
$R_h$ : Projection operator on piecewise constant functions

# Computed vertical displacements



# Modified Finite Elements & Minimization

Minimization problem in $\mathbf{\Theta}_h \times W_h$ for the new energy:

$$E_h(\boldsymbol{\eta}_h, v_h) = \frac{1}{2}\int_\Omega \mathbf{C}\,\varepsilon(\boldsymbol{\eta}_h) : \varepsilon(\boldsymbol{\eta}_h) + \frac{\lambda\, t^{-2}}{2}\int_\Omega |R_h(\boldsymbol{\nabla} v_h - \boldsymbol{\eta}_h)|^2 - \int_\Omega g v_h$$

- If $R_h(\boldsymbol{\nabla} v_h) \neq \boldsymbol{\nabla} v_h$ we risk. It may happen:

$$\boldsymbol{\nabla} v_h \neq \mathbf{0} \quad \text{but} \quad R_h(\boldsymbol{\nabla} v_h) = \mathbf{0}$$

The energy on $(\mathbf{0}, \alpha\, v_h)$:

$$E_h(\mathbf{0}, \alpha\, v_h) = \frac{\lambda\, t^{-2}}{2}\int_\Omega |\alpha\, R_h(\boldsymbol{\nabla} v_h)|^2 - \alpha \int_\Omega g v_h = -\alpha \int_\Omega g v_h$$

LINEAR functional along the direction $v_h$ !!!

# Finite Elements for plates

## We need a reduction operator $R_h$

- If $R_h$ reduces “too much”: Spurious modes.
- If $R_h$ do not reduce “enough”: Shear Locking

Remarks

- Balancing $R_h$ is not trivial.
- Other difficulties arise: boundary layer effects.

## Possible Approaches

- Arnold–Falk Element (1989).
- *MITC* Elements: Brezzi, Bathe, Fortin, Stenberg ... ('80–'90).
- Linking Technique: Auricchio, Taylor, L. ... ('90).
- Stabilized Elements: Chapelle, Hughes, Stenberg ... ('90).
- Non-conforming and DG Elements: Arnold, Brezzi, L., Marini ('04–)

Remark: Most of them are Mixed Methods.

## Conclusions

- In Computational Mechanics: often problems with "small" or "large" parameter.
- Finite element discretization of such problems requires care.
- Different situations may arise,with different peculiarities.
- Shell Problems fall into this structure, but MUCH MORE DIFFICULT.