Technical report

The finite element method as a solution of the eikonal equation applied to cardiac wavefronts diffusion.

Model

A mathematical model that studies the wavefronts propagation in the myocardium (cardiac muscle) tissue can be represented by the following eikonal problem.

$$\begin{cases} -\nabla(M \cdot \nabla u) + c_0 \cdot \sqrt{\nabla u \cdot M \cdot \nabla u} = t_m \quad su \quad H \\ n \cdot M \cdot \nabla u = 0 \quad su \quad \partial H \end{cases}$$

where c_0 and t_m are speed and time constants.

H is the problem domain or else, in this case, the myocardium of a single ventricle.

The system has a proper unique solution only if the signal's sources are defined by the essential conditions, like $u = g_a$ on a region of H.

The matrix $M(x, \nabla u)$ abstracts the bioelectrical features of the myocardium and is a non linear function of $\nabla u(x)$ and of the position x.

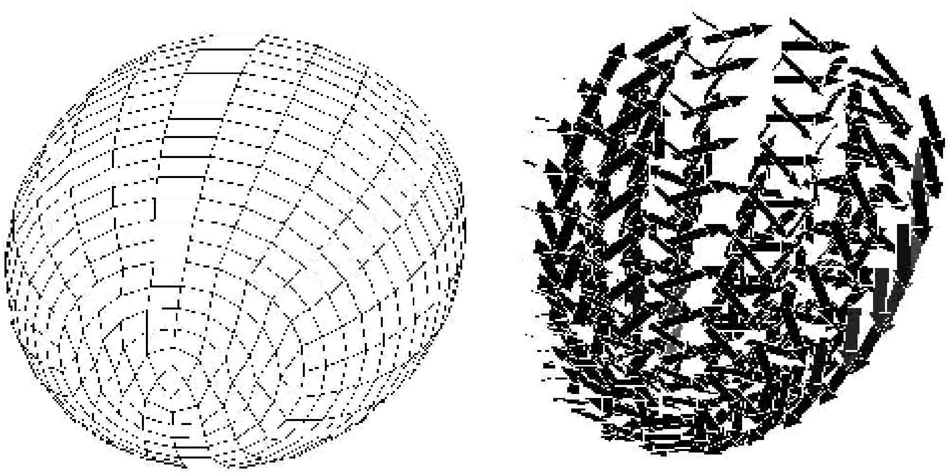
Such features are essentially two.

1. Bidomain. In the space H coexist two different conducting media: the intracellular medium, compound by the myocardial cells and their inter junctions, and the extracellular medium, compound by the interstitial space. These two media are electrically defined by two conductivity tensors M_i^* and M_e^*.
2. Fiber architecture. In order to simulate the fiber structure of the myocardium it is possible to associate to each point x a unit vector a_l that defines the local direction of the fiber. It is then possible, using this versor a_l, to define a local orthogonal system and subsequently a matrix A.

Hence, since $M_i = A \cdot M_i^* \cdot A^T$, $M_e = A \cdot M_e^* \cdot A^T$, then $M(x, \nabla u)$ is a non linear function of M_i e M_e. In this sense, this matrix abstracts the bioelectrical features of the cardiac tissue.

The function $u(x)$ is named activation time, or else, the time at which, in each position, the myocardium electrical potential changes from a resting state to an activated state.

Obviously, the electric dynamic of the heart is more complex and it expresses a multimodal behaviour similar to neuronal action potentials, but this approximation allows to have a better qualitative vision of the propagation of cardiac stimuli. Indeed, the isosurfaces of $u(x)$ identify the activation wavefronts travelling.



Mesh and fiber architecture

Numerical approximation

This problem can be solved with the method of the finite elements with the Galerkin method. The elements are linear lagrangian hexahedra.

The eikonal equation, as above seen, is elliptic, so it is convenient to add a temporal derivative term of $u(x)$ transforming the equation into an hyperbolic one: so is more straightforward to look for an asymptotic solution.

The searched approximation of the solution has then the form $u(x) = \lim_{t\to\infty} w(t) = \sum_{j=1}^{Nr} w_j(t) \cdot f_j + \sum_{s=Nr+1}^{Nr+Na} g_a^s \cdot f_s$ where $\{f_j\}$ are the shape functions and the coefficients in the second sum are the corresponding nodal values of the Dirichlet conditions.

It is possible consequently to write the system of non linear ordinary differential equations of the first order that has, as solution, $w(t) = (w_1(t), w_2(t), ..., w_{Nr}(t))^T$:

$$\begin{cases} C\dfrac{dw}{dt} + A(w)w + m(w) = f \\ w(0) = w_0 \end{cases}$$, where,

$$C = \left\{ c_{i,j} = \int_{Hr} f_i \cdot f_j dx; \quad i, j = 1, ..., N_r \right\},$$

$$f = \left\{ \int_{Hr} t_m \cdot f_i \cdot dx + \frac{1}{c_m}\int_{Sn} g_n \cdot f_i ds - \int_{Hr} \sum_{s=Nr+1}^{Nr+Na} g_a^s (\nabla f_s)^T M(x, \nabla w_h) \nabla f_i dx; i = 1, ..., N_r \right\},$$ where the second integral is equal to zero since $g_n = 0$,

$$A(w) = \left\{ a_{i,j} = \int_{Hr} (\nabla f_j)^T M(x, \nabla w_h) \cdot \nabla f_i \cdot dx; \quad i, j = 1, ..., N_r \right\},$$

$$m(w) = \left\{ c_0 \int_{Hr} \sqrt{(\nabla w_h)^T M(x, \nabla w_h) \cdot \nabla w_h} \cdot f_i \cdot dx; \quad i = 1, ..., N_r \right\}.$$

In this implementation, it has been preferred the following completely explicit way to discretize:

$$C \cdot w^{k+1} = C \cdot w^k + t \cdot \left(f - A(w^k) \cdot w^k - m(w^k)\right).$$

Algorithm

The algorithm solves the problem through a sequence of steps that now will be listed with the aim of summarize the implementation applied in this work.

1. The mesh has been built and has been numerated coherently with the topology of the hexahedral reference element.
2. The sources of the signal are given, and so the essential conditions are defined.
3. Given the initial hypothesis of the solution, the iterative process, that leads to the searched approximation, starts.
4. The gradient of the temporary solution is estimated using a finite elements interpolation.
5. At the boundaries, the gradient is modified so to assure the boundary conditions.
6. The upwind gradients are calculated and applied to the non linear transport part $m(w)$.
7. The integrals are built with the finite elements method and, through them, are defined the coefficient of the linear system.
8. The linear system is solved.
9. The temporary solution is compared with the precedent temporary solution and so the relative residual error of the linear system: if the reached approximation is satisfying the algorithm continues or else it iterates back to step 4.
10. The files, that are used to produce the pictures of the solution, are written as output data of the algorithm.

Upwind

An upwind technique [3] has been applied to the term $m(w)$ in order to stabilize it during the iterations.

$\Gamma(x,\nabla u) = c_0 \cdot \sqrt{\nabla u \cdot M \cdot \nabla u}$, $\Gamma(x,\nabla u) = |\nabla u| \cdot \Gamma(x,n)$ where $n = \nabla u / |\nabla u|$.

$$|\nabla u|(x_i) \cong |\nabla_{up} u|(x_i) = \left(\sum_{i=1}^{3} \left[\left(\max(D_i^- u,0)\right)^2 + \left(\min(D_i^+ u,0)\right)^2 \right] \right)^{1/2},$$

where $D_i^+ u$ and $D_i^- u$ for $i = 1,2,3$ are the forward and backward finite difference that approximate the derivatives in the three cartesian directions, these are determined using the values of w at each iteration.

Gradients' interpolation

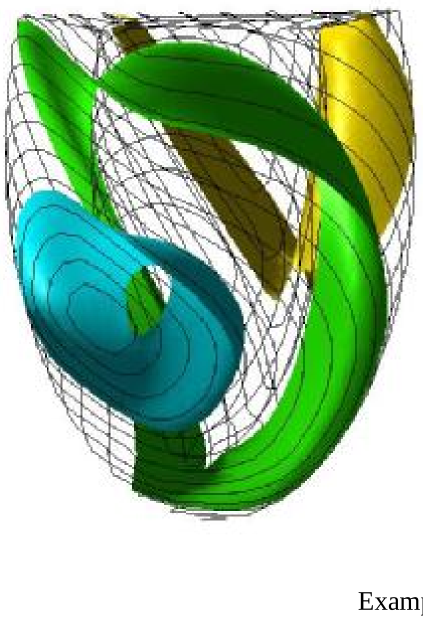
The interpolation of the gradients has been implemented using the gradients of the shape functions as it follows: $\nabla w = \sum_{j \in N} w_j \cdot \nabla j_j$.

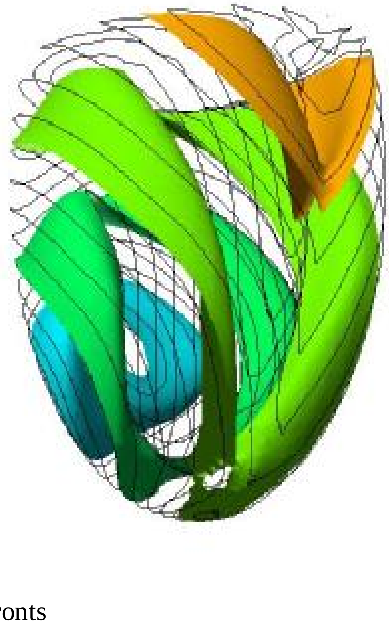
This approach provokes discontinuities at the boundaries between the elements. At the domain boundaries of this interpolation, the natural boundary conditions have to be forced. These conditions have to be estimated at each iteration because $M(x,\nabla u)$ is non linear. Hence, at each iteration, the boundary condition problem can be seen as $f(x) = 0$, so is possible to understand that this problem can be conceived as the problem of iteratively solve a non linear equation.

Essential references

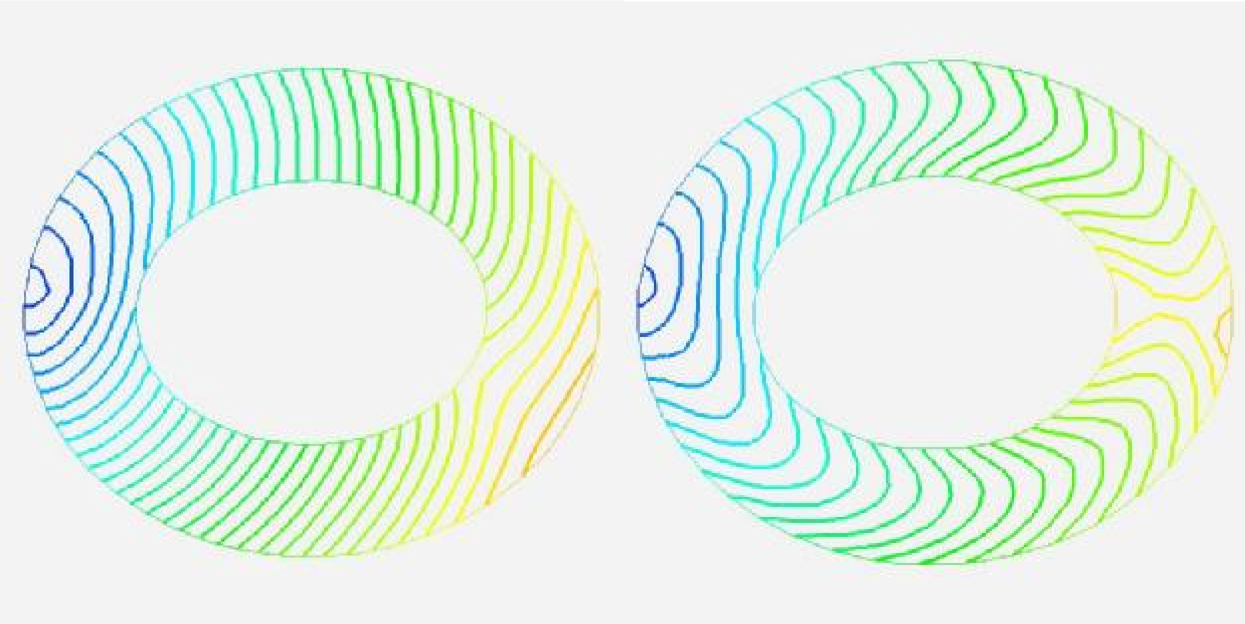
1. **P. Colli Franzone, L. Guerri, M. Pennacchio e B. Traccardi**, 'Spreading of Excitation in 3-D models of the anisotropic cardiac tissue. II. Effect of fiber architecture and ventricular geometry.', Math. Biosci. 147: 131-171 (1998).
2. **P. Colli Franzone e L. Guerri**, 'Spreading of excitation in 3-D models of the anisotropic cardiac tissue. I. Validation of the eikonal model.', Math. Biosci., 113: 145-209 (1993).
3. **S. Oshier e J.A. Sethian**, 'Front propragating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations.' J. Comp. Phys. 79: 12-49 (1988).

Pictures





Examples of wavefronts



Horizontal sections without or with the fiber intramural rotation

Vertical sections without or with the fiber intramural rotation