Jacobi method for small matrices

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OUTLINE

- Why small matrices?
- Jacobi method and pivot strategies
- Parallel strategies on 4×4 matrices
- Conclusion

JACOBI METHOD

- Goal: diagonal matrix.
- One step:

$$A^{(k+1)} = U_k^T A^{(k)} U_k, \qquad k \geq 0, \quad A^{(0)} = A,$$

where $U_k = R(i(k), j(k), \phi(k))$ are orthogonal,

$$R(i,j,\phi) = \begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & \cos\phi & & & & -\sin\phi & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & \sin\phi & & & & \cos\phi & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{matrix} .$$

PIVOT STRATEGIES

Pivot strategy is the function $k \mapsto (i(k), j(k))$.

- Cyclic strategies
- Serial strategies (row and column)

- Strategies with permutations inside columns / rows
- Examples:

$$\begin{bmatrix} * & 0 & 2 & 4 & 6 \\ & * & 1 & 5 & 8 \\ & & * & 3 & 7 \\ & & & * & 9 \\ & & & & * \end{bmatrix} \in \mathcal{C}_1, \quad \begin{bmatrix} * & 6 & 9 & 7 & 8 \\ & * & 3 & 5 & 4 \\ & & * & 2 & 1 \\ & & & * & 0 \\ & & & & * \end{bmatrix} \in \mathcal{C}_2,$$

$$\begin{bmatrix} * & 9 & 8 & 4 & 2 \\ & * & 7 & 6 & 1 \\ & & * & 5 & 0 \\ & & & * & 3 \\ & & & & * \end{bmatrix} \in \mathcal{C}_3, \quad \begin{bmatrix} * & 3 & 1 & 0 & 2 \\ & * & 6 & 5 & 4 \\ & & * & 8 & 7 \\ & & & * & 9 \\ & & & & * \end{bmatrix} \in \mathcal{C}_4.$$

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Pivot strategies:

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PARALLEL STRATEGIES ON 4×4 MATRICES

We restrict our attention to those cyclic pivot strategies (16 strategies) that enable full parallelization of the method.

It will be sufficient to study pivot strategy

$$\begin{bmatrix} * & 4 & 0 & 2 \\ 4 & * & 3 & 1 \\ 0 & 3 & * & 5 \\ 2 & 1 & 5 & * \end{bmatrix},$$

which corresponds to parallel strategy

$$\begin{bmatrix} * & 2 & 0 & 1 \\ 2 & * & 1 & 0 \\ 0 & 1 & * & 2 \\ 1 & 0 & 2 & * \end{bmatrix}.$$

CONVERGENCE

- Off-norm of a symmetric matrix

$$S(A) = \frac{\sqrt{2}}{2}\mathrm{off}(A) = \frac{\sqrt{2}}{2}\|A - \mathrm{diag}(A)\|_F = \sqrt{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_{ij}^2}, \quad A = A^T.$$

We find the constant γ, $0 \le \gamma < 1$, such that

$$S^2(A^{[3]}) \le \gamma S^2(A),$$

where $A^{[3]}$ is obtained from A after three full cycles.

SLOW CONVERGENCE - EXAMPLE

$$A = A^{(0)} = \begin{bmatrix} d + p_1 + p_2 & 0 & \epsilon + p_1 & -a + p_1 \\ 0 & d + p_2 & a & -\epsilon \\ \epsilon + p_1 & a & d + p_1 & 0 \\ -a + p_1 & \epsilon & 0 & d \end{bmatrix},$$

where $a = 1$, $\epsilon = 10^{-k-10}$, $d = 1 + \epsilon$, $p_2 = 10^{-1.5k}$, $p_3 = 10^{-1.5k-10}$ and

$$S(A) = 1.4142135623730950488016887242096980785696718753769\text{4}.$$

SLOW CONVERGENCE - EXAMPLE

k	$(i(k), j(k))$	$S(A^{(k)})$
1	$(1, 3)$	1.41421356237309504880168872420969807856967187537694
2	$(2, 4)$	1.41421356237309504880168872420969807856967187537694
3	$(1, 4)$	1.41421356237309504880168872420969807856967187537694
4	$(2, 3)$	1.41421356237309504880168872420969807856967187537694
5	$(1, 2)$	1.41421356237309504880168872420969807856967187537694
6	$(3, 4)$	1.41421356237309504880168872420969807856967187537694
7	$(1, 3)$	1.41421356237309504880168872420969807856967187537694
8	$(2, 4)$	1.41421356237309504880168872420969807856967187537694
9	$(1, 4)$	0.99999999999999999999999999999999999999999999999999
10	$(2, 3)$	$0.14142135623730950488016887242096980785696718753769e{-89}$

SLOW CONVERGENCE - EXAMPLE

As long as the rotation angle is "close enough" to $\pm\frac{\pi}{4}$, the off-norm will not change in the next two steps.

k	(i,j)	$\sin\phi(i,j)$
1	$(1,3)$	$-0.70710678118654752440084434442717950962116550533689$
2	$(2,4)$	$0.70710678118654752440084434442717950962114782766736$
3	$(1,4)$	$0.70710678118654752440084436210484903221376812582298$
4	$(2,3)$	$-0.70710678118654752440084436210484903221376812582299$
5	$(1,2)$	$0.70710678118654752440084436210484903928483593768847$
6	$(3,4)$	$0.70710678118654752440084436210484903928481826001894$
7	$(1,3)$	$0.24999999999999987500000000000003906250000000000390e{-}15$
8	$(2,4)$	$-0.24999999999999987500000000000003906250000000000390e{-}15$

CHANGING THE APPROACH

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Pivot strategy $\mathcal{O} = (1,3),(2,4),(1,4),(2,3),(1,2),(3,4)$ on matrix

$$A = \begin{bmatrix} a_{11} & 0 & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & 0 \\ a_{14} & a_{24} & 0 & a_{44} \end{bmatrix}.$$

Let

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad Q^T X Q = \begin{bmatrix} x_{11} & x_{13} & x_{14} & -x_{12} \\ x_{31} & x_{33} & x_{34} & -x_{32} \\ x_{41} & x_{43} & x_{44} & -x_{42} \\ -x_{21} & -x_{23} & -x_{24} & x_{22} \end{bmatrix}.$$

OPERATOR $\mathcal{T}$

Definition

For $A \in \mathcal{S}_4$ we define

$$\mathcal{T} = \big(R(1,3,\phi)R(2,4,\psi)Q\big)^T A \big(R(1,3,\phi)R(2,4,\psi)Q\big),$$

where $R(1,3,\phi)$ and $R(2,4,\psi)$ are Jacobi rotations which annihilate the elements a_{13} and a_{24} of A, respectively, and $\phi, \psi \in [-\frac{\pi}{4}, \frac{\pi}{4}]$.

Properties of $\mathcal{T}$:

- $\mathcal{T}^k(A) = (Q^k)^T A^{(2k)} Q^k, \quad k \geq 0,$
- $S(\mathcal{T}^k(A)) = S(A^{(2k)}), \quad k \geq 0.$

MAIN THEOREM

Proposition

Let $A \in \mathcal{S}_4$ *be such that* $a_{12} = 0$, $a_{34} = 0$ *and* $\epsilon = 10^{-5}$. *Then*

$$S(\mathcal{T}^6(A)) \leq (1 - \epsilon) S(A).$$

Theorem

Let $A \in \mathcal{S}_4$ *be such that* $a_{12} = 0$, $a_{34} = 0$ *and let* $A^{(12)}$ *be obtained by applying* 12 *steps of the Jacobi method under the strategy* $\mathcal{O} = (1,3), (2,4), (1,4), (2,3), (1,2), (3,4)$ *to* A. *Then*

$$S(A^{(12)}) \leq (1 - 10^{-5}) S(A).$$

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CONCLUSION

Theorem

Let $A \in \mathcal{S}_4$ *and let* $A^{[3]}$ *be obtained by applying* 3 *full cycles of the Jacobi method under* **any** *cyclic strategy on* A*. Then there is a constant* γ *such that*

$$S^2(A^{[3]}) \leq \gamma S^2(A), \quad 0 \leq \gamma < 1.$$

CONCLUSION

Theorem

Let $A \in \mathcal{S}_4$ *and let* $A^{[3]}$ *be obtained by applying* 3 *full cycles of the Jacobi method under* **any** *cyclic strategy on* A. *Then there is a constant* γ *such that*

$$S^2(A^{[3]}) \leq \gamma S^2(A), \quad 0 \leq \gamma < 1.$$

Thank you!